Semidefinite Programming Relaxations of the Traveling Salesman Problem

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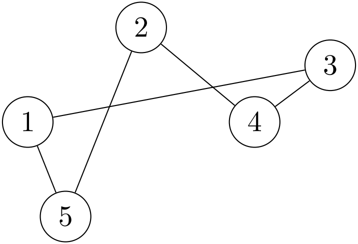
Outline

1. Introduction: Three TSP Relaxations
2. Proof Sketch: An SDP with Unbounded Integrality Gap
3. Corollaries and Open Questions

Based on *The Unbounded Integrality Gap of a Semidefinite Relaxation of the Traveling Salesman Problem*, G. and Williamson, SIAM Journal on Optimization, 2018, Vol. 28, No. 3

The (Symmetric, Metric) TSP

- Complete undirected graph K_n
- Edge costs c_{ij} for distinct $i, j \in [n] = \{1, 2, ..., n\}$ with $c_{ij} = c_{ji}$ and $c_{ij} \leq c_{ik} + c_{kj}$ for all distinct i, j, k



Goal

Find a *minimum-cost Hamiltonian cycle:* the cheapest cycle visiting every city exactly once.

The Subtour Elimination LP Relaxation (1950s)

Let $\delta(S) := \{e = \{i,j\} : |\{i,j\} \cap S| = 1\}$ be the set of edges with exactly one endpoint in S, and let $\delta(v) := \delta(\{v\})$.

$$\begin{array}{lll} \min & \sum_{e \in E} c_e x_e & \\ \text{subject to} & \sum_{e \in \delta(v)} x_e = 2, & v = 1, \ldots, n \\ & \sum_{e \in \delta(S)} x_e \geq 2, & S \subset V : S \neq \emptyset, S \neq V \\ & 0 \leq x_e \leq 1, & e = 1, \ldots, n. \end{array}$$

Theorem (Wolsey 1980, Shmoys and Williamson 1990)

The *integrality gap* of this relaxation is at most by $\frac{3}{2}$. That is, for any, for any set of metric and symmetric edge costs,

$$\frac{\text{Optimal TSP Solution}}{\text{Optimal LP Solution}} \leq \frac{3}{2}.$$

A First SDP Relaxation (1999)

Let $C = (c_{ij})_{i,j=1}^n$ be the matrix of edge costs.

Let $A \succeq 0$ denote that A is a positive semidefinite matrix, J denote the all-ones matrix, and e denote the all-ones vector.

$$
\begin{array}{lll}
\min & \frac{1}{2}\operatorname{trace}(CX) = \frac{1}{2}\sum_{i,j=1}^n C_{ij}X_{ij} & \\
\text{subject to} & Xe = 2e & \\
 & X_{ii} = 0, & i = 1, ..., n \\
 & 0 \le X_{ij} \le 1, & i, j = 1, ..., n \\
 & 2I - X + \left(2 - 2\cos\left(\frac{2\pi}{n}\right)\right)(J - I) \succeq 0 & \\
 & X \text{ a real, symmetric } n \times n \text{ matrix.} &
\end{array}
$$

Theorem (Cvetković, Čangalović, and Kovačević-Vujčić 1999)

This semidefinite program is a relaxation of the TSP: the adjacency matrix of any Hamiltonian cycle is feasible and has the appropriate objective value.

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& X \text{ a real, symmetric } n \times n \text{ matrix.} &
\end{array}
$$

X is a fractional adjacency matrix of K_n :

for $e = \{i, j\}, X_{ij} = X_{ji}$ is the proportion of edge e used.

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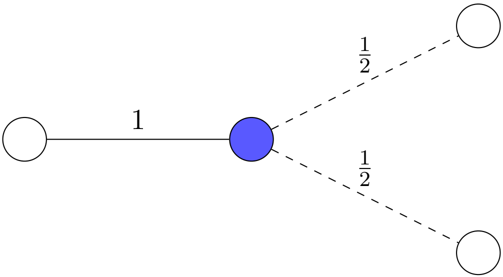

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$$

The weighted graph corresponding to X (as a weighted adjacency matrix) is at least as connected as a cycle graph, with respect to algebraic connectivity

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$$

Theorem (Goemans and Rendl, 2000)

This SDP is weaker than the subtour elimination LP: any feasible solution for the subtour LP is also feasible for this SDP.

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$$

Theorem (G. and Williamson, 2017)

This SDP has an unbounded integrality gap

A Second SDP Relaxation (2008)

Let $C = (c_{ij})_{i,j=1}^n$ be the matrix of edge costs and S^n be the set of real, symmetric $n \times n$ matrices. Also let $d = \lfloor \frac{n}{2} \rfloor$.

$$
\begin{array}{lll}
\min & \frac{1}{2}\operatorname{trace}\left(CX^{(1)}\right) & \\
\text{subject to} & X^{(k)} \geq 0, & k = 1, \ldots, d \\
& \sum_{j=1}^{d} X^{(j)} = J - I, & \\
& I + \sum_{j=1}^{d} \cos\left(\frac{2\pi jk}{n}\right) X^{(j)} \succeq 0, & k = 1, \ldots, d \\
& X^{(k)} \in S^n, & k = 1, \ldots, d.
\end{array}
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Theorem (de Klerk, Pasechnik, and Sotirov 2008)

This semidefinite program is a relaxation of the TSP.

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\end{array}
$$

Theorem (de Klerk, Pasechnik, and Sotirov 2008)

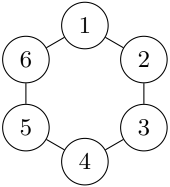
This semidefinite program is a relaxation of the TSP. Moreover, it is incomparable with the subtour elimination LP and dominates the SDP of Cvetković et. al.

A Second SDP Relaxation (2008)

Idea

Let $\mathcal{C}$ be a Hamiltonian cycle. For $i = 1, ..., d = \lfloor \frac{n}{2} \rfloor$, let $X^{(i)}$ be the ith distance matrix of $\mathcal{C}$:

$$X_{jk}^{(i)} = \begin{cases} 1, & j \text{ and } k \text{ are distance } i \text{ apart in } \mathcal{C} \\ 0, & \text{otherwise.} \end{cases}$$



$$X^{(1)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

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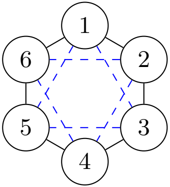

$$X^{(2)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

A Second SDP Relaxation (2008)

$$\begin{array}{lll} \min & \frac{1}{2}\text{trace}\left(CX^{(1)}\right) & \\ \text{subject to} & X^{(k)} \geq 0, & k = 1, \ldots, d \\ & \sum_{j=1}^{d} X^{(j)} = J - I, & \\ & I + \sum_{j=1}^{d} \cos\left(\frac{2\pi jk}{n}\right) X^{(j)} \succeq 0, & k = 1, \ldots, d \\ & X^{(k)} \in S^n, & k = 1, \ldots, d. \end{array}$$

For $i = 1, ..., d = \lfloor \frac{n}{2} \rfloor$, these quickly follow from

$$X_{jk}^{(i)} = \begin{cases} 1, & j \text{ and } k \text{ are distance } i \text{ apart in } \mathcal{C} \\ 0, & \text{otherwise.} \end{cases}$$

A Second SDP Relaxation (2008)

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\end{array}
$$

- The distance matrices of a cycle form an *association scheme.*
- This is an application of a more general statement about association schemes.

(See de Klerk, Filho, Pasechnik 2012)

- The distance matrices of a cycle are *circulant matrices.*
- Linear combinations of circulant matrices are circulant.
- Circulant matrices have well-understood eigenvalues.

(see G. and Willamson 17)

A Second SDP Relaxation (2008)

$$
\begin{array}{lll}
\text{min} & \frac{1}{2}\text{trace}\left(CX^{(1)}\right) & \\
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\end{array}
$$

$$
\begin{pmatrix}
m_0 & m_1 & m_2 & \cdots & m_{n-1} \\
m_{n-1} & m_0 & m_1 & \cdots & m_{n-2} \\
m_{n-2} & m_{n-1} & m_0 & \ddots & m_{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
m_1 & m_2 & m_3 & \cdots & m_0
\end{pmatrix}
$$

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A Second SDP Relaxation (2008)

Goal

For $X_{st}^{(j)} = \mathbb{1}_{\{s \text{ and } t \text{ are distance } j \text{ apart in } \mathcal{C}\}}$,

$$I + \sum_{j=1}^{d} \cos\left(\frac{2\pi jk}{n}\right) X^{(j)} \succeq 0, \qquad k = 1, \ldots, d.$$

$$\begin{pmatrix} m_0 & m_1 & m_2 & \cdots & m_{n-1} \\ m_{n-1} & m_0 & m_1 & \cdots & m_{n-2} \\ m_{n-2} & m_{n-1} & m_0 & \ddots & m_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_1 & m_2 & m_3 & \cdots & m_0 \end{pmatrix}$$

$$\lambda_t(M) = \sum_{s=0}^{n-1} m_s e^{-\frac{2\pi st\sqrt{-1}}{n}}, \quad t = 1, ..., n-1, \quad \lambda_n(M) = \sum_{s=0}^{n-1} m_s.$$

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Goal

For $X^{(j)}_{st} = \mathbb{1}_{\{s \text{ and } t \text{ are distance } j \text{ apart in } \mathcal{C}\}}$,

$$I + \sum_{j=1}^{d} \cos\left(\frac{2\pi jk}{n}\right) X^{(j)} \succeq 0, \qquad k = 1, \ldots, d.$$

$$\begin{pmatrix} 1 & \cos(2\pi k/n) & \cos(2\pi 2k/n) & \cdots & \cos(2\pi 2k/n) & \cos(2\pi k/n) \\ \cos(2\pi k/n) & 1 & \cos(2\pi k/n) & \cdots & \cos(2\pi 3k/n) & \cos(2\pi 2k/n) \\ \cos(2\pi 2k/n) & \cos(2\pi k/n) & 1 & \ddots & \cos(2\pi 4k/n) & \cos(2\pi 3k/n) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \cos(2\pi k/n) & \cos(2\pi 2k/n) & \cos(2\pi 3k/n) & \cdots & \cos(2\pi k/n) & 1 \end{pmatrix}$$

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A Second SDP Relaxation (2008)

Goal

For $X^{(j)}_{st} = \mathbb{1}_{\{s \text{ and } t \text{ are distance } j \text{ apart in } \mathcal{C}\}}$,

$$I + \sum_{j=1}^{d} \cos\left(\frac{2\pi jk}{n}\right) X^{(j)} \succeq 0, \qquad k = 1, \ldots, d.$$

For $t \leq n$,

$$\begin{aligned}
\lambda_t(M) &= \sum_{s=0}^{n-1} m_s e^{-\frac{2\pi st\sqrt{-1}}{n}} \\
&= 1 + \cos\left(\frac{2\pi kd}{n}\right) e^{-\frac{2\pi dt\sqrt{-1}}{n}} + \sum_{s=1}^{d-1} \cos\left(\frac{2\pi sk}{n}\right)\left(e^{-\frac{2\pi st\sqrt{-1}}{n}} + e^{-\frac{2\pi (n-s)t\sqrt{-1}}{n}}\right) \\
&= \cdots \\
&= \begin{cases} 2d, & \text{if } k = t = d \\ d, & \text{if } k \neq d, t \in \{k, n-k\} \\ 0, & \text{else.} \end{cases}
\end{aligned}$$

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Let $C = (c_{ij})_{i,j=1}^n$ be the matrix of edge costs and S^n be the set of real, symmetric $n \times n$ matrices. Also let $d = \lfloor \frac{n}{2} \rfloor$.

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\begin{array}{lll}
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& X^{(k)} \in S^n, & k = 1, \ldots, d.
\end{array}
$$

Theorem (G. and Williamson, 2017)

This SDP has an unbounded integrality gap. That is, there exists no constant $\alpha > 0$ such that

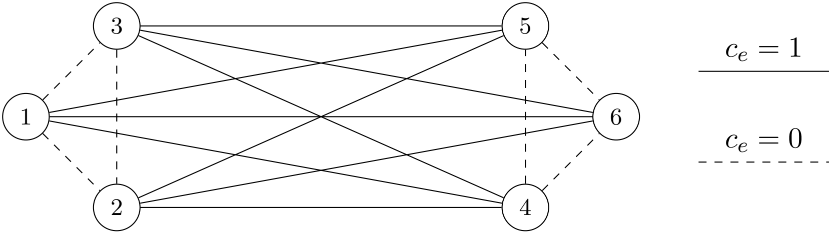
$$\frac{\mathrm{OPT}_{\mathrm{TSP}}(C)}{\mathrm{OPT}_{\mathrm{SDP}}(C)} \leq \alpha$$

for all cost matrices C with metric, symmetric edge costs.

Our Main Theorem: Proof Sketch

Let n be even and consider the cost matrix

$$\hat{C} := \begin{pmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & 1 \\ 1 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes J_d.$$



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$\hat{C}$ corresponds to:

- a *cut semimetric*: costs where, for some $S \subset V$, $c_{ij} = 1$ if $\{i, j\} \in \delta(S)$ and $c_{ij} = 0$ otherwise.
- an instance of *Euclidean TSP*: vertices $1, ..., \frac{n}{2}$ are at $0 \in \mathbb{R}^1$ and vertices $\frac{n}{2} + 1, ..., n$ are at $1 \in \mathbb{R}^1$. Costs are given by the Euclidean distance between corresponding vertices.

Our Main Theorem: Proof Sketch

Theorem (G. and Williamson, 2017)

For $\hat{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes J_d$, we have $\mathrm{OPT}_{\mathrm{SDP}}(\hat{C}) \leq \frac{\pi^2}{n} = \frac{\pi^2}{2n}\mathrm{OPT}_{\mathrm{TSP}}(\hat{C})$.

Corollary

There exists no constant $\alpha > 0$ such that

$$\frac{\mathrm{OPT}_{\mathrm{TSP}}(C)}{\mathrm{OPT}_{\mathrm{SDP}}(C)} \leq \alpha$$

for all cost matrices C with metric, symmetric edge costs.

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Strategy:

1. Look within a class of feasible solutions that respect the symmetry of $\hat{C}$.
2. Exploit the structure of such solutions by reducing the SDP to an LP *for solutions in that class.*
3. Find a feasible solution to the LP achieving the desired cost.

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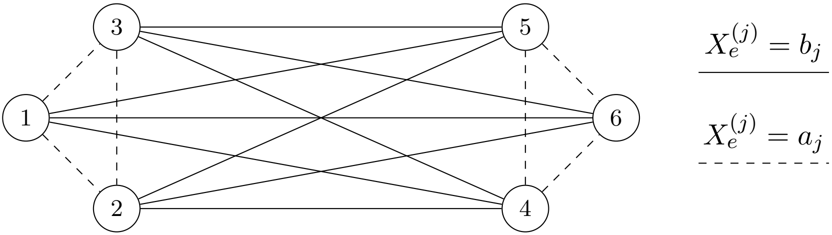
Theorem (G. and Williamson, 2017)

For $\hat{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes J_d$, we have $\text{OPT}_{\text{SDP}}(\hat{C}) \leq \frac{\pi^2}{n} = \frac{\pi^2}{2n}\text{OPT}_{\text{TSP}}(\hat{C})$.

Candidate solutions:

$$X^{(j)} = \left(\begin{pmatrix} a_j & b_j \\ b_j & a_j \end{pmatrix} \otimes J_d\right) - a_j I_n, \qquad b_j = \begin{cases} \frac{4}{n} - \left(1 - \frac{2}{n}\right) a_j, & j \leq d-1 \\ \frac{2}{n} - \left(1 - \frac{2}{n}\right) a_j, & j = d. \end{cases}$$



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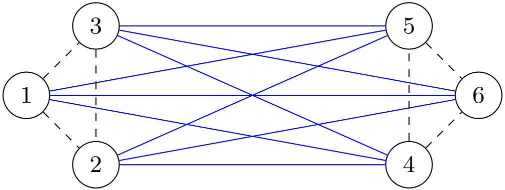


$X_e^{(j)} = b_j$, cost 1

$X_e^{(j)} = a_j$, cost 0

TSP Solutions

$$\text{OPT}_{\text{TSP}}(\hat{C}) = 2$$

SDP Solutions

$$\text{OPT}_{\text{SDP}}(\hat{C}) = \frac{1}{2}\text{trace}\left(CX^{(1)}\right)$$
$$= 0 \times 2\binom{n/2}{2}a_1 + 1 \times \left(\frac{n}{2}\right)^2 b_1$$

Our Main Theorem: Proof Sketch

Theorem (G. and Williamson, 2017)

For $\hat{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes J_d$, we have $\mathrm{OPT}_{\mathrm{SDP}}(\hat{C}) \leq \frac{\pi^2}{n} = \frac{\pi^2}{2n}\mathrm{OPT}_{\mathrm{TSP}}(\hat{C})$.

Let

$$X^{(j)} = \left(\begin{pmatrix} a_j & b_j \\ b_j & a_j \end{pmatrix} \otimes J_d\right) - a_j I_n, \qquad b_j = \begin{cases} \frac{4}{n} - \left(1 - \frac{2}{n}\right) a_j, & j \leq d-1 \\ \frac{2}{n} - \left(1 - \frac{2}{n}\right) a_j, & j = d. \end{cases}$$

The SDP constraint $I + \sum_{j=1}^{d} \cos\left(\frac{2\pi jk}{n}\right) X^{(j)} \succeq 0$ becomes

$$\left(\begin{pmatrix} a^{(k)} & b^{(k)} \\ b^{(k)} & a^{(k)} \end{pmatrix} \otimes J_d\right) + (1 - a^{(k)}) I_n \succeq 0,$$

where $a^{(k)}$ and $b^{(k)}$ are linear combinations of $a_1, \ldots, a_d$. The eigenvalues of this matrix are linear combinations of $a_1, \ldots, a_d$.

Our Main Theorem: Proof Sketch

Theorem (G. and Williamson, 2017)

For $\hat{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes J_d$, we have $\mathrm{OPT}_{\mathrm{SDP}}(\hat{C}) \leq \frac{\pi^2}{n} = \frac{\pi^2}{2n}\mathrm{OPT}_{\mathrm{TSP}}(\hat{C})$.

Intermediate step: finding a minimum-cost feasible solution of this form is equivalent to solving the following linear program:

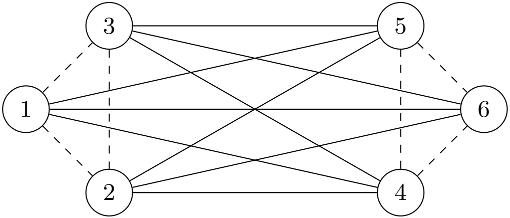
$$
\begin{array}{llll}
\max & a_1 & & \\
\text{subject to} & \sum_{i=1}^d \cos\left(\frac{2\pi ik}{n}\right) a_i & \geq -\frac{2}{n-2}, & k = 1, \dots, d \\
& \sum_{i=1}^d \cos\left(\frac{2\pi ik}{n}\right) a_i & \leq 1, & k = 1, \dots, d \\
& \sum_{i=1}^d a_i & = 1 & \\
& a_i & \leq \frac{4}{n-2}, & i = 1, \dots, d-1 \\
& a_d & \leq \frac{2}{n-2} & \\
& a_i & \geq 0, & i = 1, \dots, d.
\end{array}
$$

Our Main Theorem: Proof Sketch

Theorem (G. and Williamson, 2017)

For $\hat{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes J_d$, we have $\mathrm{OPT}_{\mathrm{SDP}}(\hat{C}) \leq \frac{\pi^2}{n} = \frac{\pi^2}{2n}\mathrm{OPT}_{\mathrm{TSP}}(\hat{C})$.

The punch-line: We find solutions where



— $b_1 = \frac{1-\cos\left(\frac{\pi}{d}\right)}{n} \sim \frac{1}{n^3}$

- - - $a_1 = \frac{2\cos\left(\frac{\pi}{d}\right)+2}{n-2}$

$$\mathrm{OPT}_{\mathrm{SDP}}(\hat{C}) \leq \frac{n^2}{4} b_1 \sim \frac{1}{n}.$$

Corollaries of Our Theorem

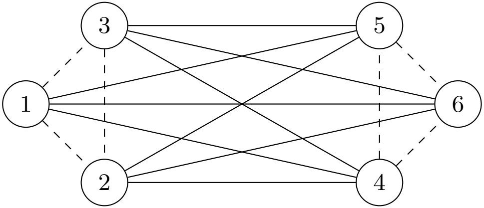
Theorem (G. and Williamson, 2017)

The SDP has an unbounded integrality gap.

Corollary

The SDP is non-monotonic, unlike the TSP and subtour elimination LP.

We've found SDP solutions costing $\frac{n^2}{4}b_1 \approx \frac{1}{n}$, which become arbitrarily small with n



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Corollaries of Our Theorem

Theorem (G. and Williamson, 2017)

The SDP has an unbounded integrality gap.

Corollary

The earlier SDP of Cvetković, Čangalović, and Kovačević-Vujčić has an unbounded integrality gap: the same $X^{(1)}$ we found is feasible (and has exactly the same algebraic connectivity as a cycle).

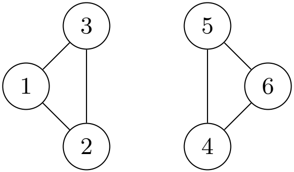
Corollaries of Our Theorem

Theorem (G. and Williamson, 2017)

The SDP has an unbounded integrality gap.

Corollary

A related SDP from de Klerk, de Oliveira Filho, and Pasechnik 20012 for the $k-$cycle cover problem also has an unbounded integrality gap.



Open Questions

1. How does this SDP perform on special cases of the TSP?

- We've shown that the integrality gap is unbounded on the general metric and symmetric TSP, as well as on Euclidean TSP.
- On *graphic* TSP (where edge costs correspond to shortest paths in a connected input graph), the integrality gap is at most 2. Is it strictly better?

Open Questions

1. How does this SDP perform on special cases of the TSP?
2. If you combine both this SDP and the subtour LP, can you guarentee an integrality gap of $1.5 - \epsilon$ for any $\epsilon > 0$?

Open Questions

1. How does this SDP perform on special cases of the TSP?
2. If you combine both this SDP and the subtour LP, can you guarentee an integrality gap of $1.5 - \epsilon$ for any $\epsilon > 0$?
3. De Klerk and Sotirov introduced a stronger SDP in 2012. Does this SDP have a bounded integrality gap?

Thanks!

Acknowledgments

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