Non-Euclidean elasticity

Cy Maor
University of Toronto

Topics in Calculus of Variations
BIRS, May 2018

"Euclidean" Elasticity

- **The (simplest?) elastostatic problem:**
minimize, over all valid configurations $f{:}\boldsymbol{\Omega}\to\boldsymbol{R}^n$,
the energy
$$E_\Omega(f) = \int_\Omega W(x, df_x)\, dx$$

- Energy density $W(x, df_x) \geq 0$ measures how much df_x is far from an isometry:

- $W(A) = 0$ iff $A \in \mathrm{SO}(n)$

- Typically, $W(x, A) \geq C \mathrm{dist}^2(A, \mathrm{SO}(n))$

"Euclidean" Elasticity

- The inclusion map $f_0{:}\boldsymbol{\Omega}\to\boldsymbol{R}^n$, $f_0(x)=x$, satisfies $E_\Omega(f_0) = 0$.
- ***Reference configuration*** of the body (=stress-free)
- The existence of a reference configuration is an underlying assumption in most elastic models.
- However, many elastic bodies *do not* have a reference configuration!

Pre-stressed bodies

- **Pre-stressed bodies** – Elastic bodies w/o a reference configuration.
- Exhibit stress (positive elastic energy) even in the absence of boundary conditions / external forces.
- **Incompatible/Non-Euclidean elasticity** – elastic theory for pre-stressed bodies.

A stressed (non-Euclidean) carrot

Non-Euclidean elasticity

Non-Euclidean elasticity

- **Elastic body**: Riemannian manifold (M^n,g)

Non-Euclidean elasticity

- **Elastic body**: Riemannian manifold (M^n,g)
- **Configuration**: $f:M^n \to \boldsymbol{R}^n$ (same dimension!)

Non-Euclidean elasticity

- **Elastic body**: Riemannian manifold (M^n, g)
- **Configuration**: $f: M^n \to \boldsymbol{R}^n$ (same dimension!)
- Energy density $W(df_x) = 0$ iff $df_x : T_x M \longrightarrow \boldsymbol{R}^n$ is an <u>orientation-preserving isometry</u>: $df_x \in \mathrm{SO}(g_x)$.

Non-Euclidean elasticity

- **Elastic body**: Riemannian manifold (M^n,g)
- **Configuration**: $f{:}M^n \to \boldsymbol{R}^n$ (same dimension!)
- Energy density $W(df_x) = 0$ iff $df_x : T_xM \longrightarrow \boldsymbol{R}^n$ is an <u>orientation-preserving isometry</u>: $df_x \in \mathrm{SO}(g_x)$.
- Simplest elastic energy:

Non-Euclidean elasticity

- **Elastic body**: Riemannian manifold (M^n,g)
- **Configuration**: $f:M^n\to\boldsymbol{R}^n$ (same dimension!)
- Energy density $W(df_x) = 0$ iff $df_x : T_xM\longrightarrow\boldsymbol{R}^n$ is an orientation-preserving isometry: $df_x \in \mathrm{SO}(g_x)$.
- Simplest elastic energy:

$$E_M(f) := \int_M \mathrm{dist}^2(df_x, \mathrm{SO}(g_x))\mathrm{dVol}_g$$

Non-Euclidean elasticity

- **Elastic body**: Riemannian manifold (M^n,g)
  Ambient space: Riemannian manifold (N^n,h)
- **Configuration**: $f{:}M^n \to N^n$
- Energy density $W(df_x) = 0$ iff $df_x : T_xM \longrightarrow T_{f(x)}N$ is an orientation-preserving isometry: $df_x \in \mathrm{SO}(g_x,h_{f(x)})$.
- Prototypical elastic energy:

$$E_{M,N}(f) := \int_M \mathrm{dist}^2(df_x, \mathrm{SO}(g_x, h_{f(x)}))\mathrm{dVol}_g$$

Two notions of incompatibility

$$E_{M,N}(f) := \int_M \mathrm{dist}^2(df_x, \mathrm{SO}(g_x, h_{f(x)}))\mathrm{dVol}_g$$

Two notions of incompatibility

$$E_{M,N}(f) := \int_M \mathrm{dist}^2(df_x, \mathrm{SO}(g_x, h_{f(x)}))\mathrm{dVol}_g$$

- **Geometric incompatibility**:
  there is no isometric immersion $f\colon M^n \to N^n$
  (no reference configuration)

Two notions of incompatibility

$$E_{M,N}(f) := \int_M \mathrm{dist}^2(df_x, \mathrm{SO}(g_x, h_{f(x)}))\mathrm{dVol}_g$$

- **Geometric incompatibility**:
  there is no isometric immersion $f{:}M^n \to N^n$
  (no reference configuration)
- **Energetic incompatibility**: $\inf E_{M,N} > 0$.

Two notions of incompatibility

$$E_{M,N}(f) := \int_M \mathrm{dist}^2(df_x, \mathrm{SO}(g_x, h_{f(x)}))\mathrm{dVol}_g$$

- **Geometric incompatibility**:
  there is no isometric immersion $f\colon M^n \to N^n$
  (no reference configuration)

- **Energetic incompatibility**: $\inf E_{M,N} > 0$.

- Are they equivalent?

Two notions of incompatibility

$$E_{M,N}(f) := \int_M \mathrm{dist}^2(df_x, \mathrm{SO}(g_x, h_{f(x)}))\mathrm{dVol}_g$$

Two notions of incompatibility

$$E_{M,N}(f) := \int_M \mathrm{dist}^2(df_x, \mathrm{SO}(g_x, h_{f(x)}))\mathrm{dVol}_g$$

Theorem (Lewicka-Pakzad 2011):

$\inf E_M > 0$ iff g is not flat.

Two notions of incompatibility

$$E_{M,N}(f) := \int_M \mathrm{dist}^2(df_x, \mathrm{SO}(g_x, h_{f(x)}))\mathrm{dVol}_g$$

Theorem (Lewicka-Pakzad 2011):

inf $E_M > 0$ iff g is not flat.

Theorem (Kupferman-M.-Shachar ≥2018):

inf $E_{M,N} > 0$ iff M is not isometrically immersible N.

Two notions of incompatibility

$$E_{M,N}(f) := \int_M \mathrm{dist}^2(df_x, \mathrm{SO}(g_x, h_{f(x)}))\mathrm{dVol}_g$$

Theorem (Kupferman-M.-Shachar ≥2018):

$\inf E_{M,N} > 0$ iff M is not isometrically immersible N.

Two notions of incompatibility

$$E_{M,N}(f) := \int_M \mathrm{dist}^2(df_x, \mathrm{SO}(g_x, h_{f(x)}))\mathrm{dVol}_g$$

Theorem (Kupferman-M.-Shachar ≥2018):

inf $E_{M,N} > 0$ iff M is not isometrically immersible N.

Slightly stronger formulation:

Theorem (Kupferman-M.-Shachar ≥2018):

If $f_k{:}M^n \to N^n$ satisfy dist$(df_k,$ SO$(g,h)) \longrightarrow 0$ in L_p,

then $f_k \longrightarrow f$ in $W^{1,p}$, and f is a smooth iso. immersion.

Rigidity

Theorem (Kupferman-M.-Shachar ≥2018):

If $f_k: M^n \to N^n$ satisfy $\mathrm{dist}(df_k, \mathrm{SO}(g,h)) \to 0$ in L_p,
then $f_k \to f$ in $W^{1,p}$, and f is a smooth iso. immersion.

Rigidity

Theorem (Kupferman-M.-Shachar ≥2018):

If $f_k\!:M^n\to N^n$ satisfy $\mathrm{dist}(df_k,\ \mathrm{SO}(g,h))\longrightarrow 0$ in L_p,
then $f_k\longrightarrow f$ in $W^{1,p}$, and f is a smooth iso. immersion.

This is a Riemannian version of Reshetnyak's rigidity thm:

Theorem (Reshetnyak 1967):

If $f_k\!:\Omega\to \boldsymbol{R}^n$ satisfy $\mathrm{dist}(df_k,\ \mathrm{SO}(n))\longrightarrow 0$ in L_p,
then $f_k\longrightarrow f = Ax + b,\ \ A\in\mathrm{SO}(n)$.

Rigidity

Theorem (Kupferman-M.-Shachar ≥2018):

If $f_k\colon M^n \to N^n$ satisfy $\mathrm{dist}(df_k, \mathrm{SO}(g,h)) \to 0$ in L_p,
then $f_k \to f$ in $W^{1,p}$, and f is a smooth iso. immersion.

This is a Riemannian version of Reshetnyak's rigidity thm:

Theorem (Reshetnyak 1967):

If $f_k\colon \Omega \to \mathbf{R}^n$ satisfy $\mathrm{dist}(df_k, \mathrm{SO}(n)) \to 0$ in L_p,
then $f_k \to f = Ax + b$, $A \in \mathrm{SO}(n)$.

Note that we do not assume a-priori that $M^n \hookrightarrow N^n$ iso.

Technical difference: $W^{1,p}(M;N)$ is more complicated than $W^{1,p}(\Omega;\mathbf{R}^n)$.

Sketch of proof

Theorem (Kupferman-M.-Shachar 2017):

If $f_k : M^n \to N^n$ satisfy $\mathrm{dist}(df_k, \mathrm{SO}(g,h)) \to 0$ in L_p,
then $f_k \to f$ in $W^{1,p}$, and f is a smooth iso. immersion.

Sketch of proof

Theorem (Kupferman-M.-Shachar 2017):

If $f_k\colon M^n\to N^n$ satisfy $\operatorname{dist}(df_k, \mathrm{SO}(g,h))\longrightarrow 0$ in L_p,
then $f_k\longrightarrow f$ in $W^{1,p}$, and f is a smooth iso. immersion.

- $A\in\mathrm{SO}(n)$ iff $\det(A)=1$ and $\operatorname{cof}(A)=A$.

Sketch of proof

Theorem (Kupferman-M.-Shachar 2017):

If $f_k\!:M^n\to N^n$ satisfy $\mathrm{dist}(df_k, \mathrm{SO}(g,h))\longrightarrow 0$ in L_p, then $f_k\longrightarrow f$ in $W^{1,p}$, and f is a smooth iso. immersion.

- $A\in\mathrm{SO}(n)$ iff $\det(A)=1$ and $\mathrm{cof}(A)=A$.
- Similarly, $df\in\mathrm{SO}(g,h)$ iff $\det(df)=1$ and $\mathrm{cof}_{g,h}(df)=df$.

Sketch of proof

Theorem (Kupferman-M.-Shachar 2017):

If $f_k : M^n \to N^n$ satisfy $\mathrm{dist}(df_k, \mathrm{SO}(g,h)) \to 0$ in L_p,
then $f_k \to f$ in $W^{1,p}$, and f is a smooth iso. immersion.

- $A \in \mathrm{SO}(n)$ iff $\det(A)=1$ and $\mathrm{cof}(A)=A$.
- Similarly, $df \in \mathrm{SO}(g,h)$ iff $\det(df)=1$ and $\mathrm{cof}_{g,h}(df)=df$.
- Allows us to use Young measures to prove:
  $f_k \to f, \quad df \in \mathrm{SO}(g,h)$ a.e.

Sketch of proof

Theorem (Kupferman-M.-Shachar 2017):

If $f_k{:}M^n\to N^n$ satisfy $\mathrm{dist}(df_k,\ \mathrm{SO}(g,h))\to 0$ in L_p,
then $f_k\to f$ in $W^{1,p}$, and f is a smooth iso. immersion.

- $A\in\mathrm{SO}(n)$ iff $\det(A)=1$ and $\mathrm{cof}(A)=A$.

- Similarly, $df\in\mathrm{SO}(g,h)$ iff $\det(df)=1$ and $\mathrm{cof}_{g,h}(df)=df$.

- Allows us to use Young measures to prove:
  $f_k\to f,\quad df\in\mathrm{SO}(g,h)$ a.e.

- $df\in\mathrm{SO}(g,h)$ a.e.
  $\Rightarrow f$ weakly-harmonic and continuous

  $\Rightarrow f$ is smooth $\Rightarrow f$ is an isometric immersion.

Sketch of proof

Sketch of proof

$df \in \mathrm{SO}(g,h)$ a.e. $\Rightarrow f$ weakly-harmonic and continuous

Sketch of proof

$df \in \mathrm{SO}(g,h)$ a.e. $\Rightarrow f$ weakly-harmonic and continuous

- In Euclidean settings:
  $\mathrm{div}(\mathrm{cof}(df))=0$, $\mathrm{cof}(df)=df$ a.e. $\Rightarrow \mathrm{div}(df)=0$

Sketch of proof

$df \in \mathrm{SO}(g,h)$ a.e. $\Rightarrow f$ weakly-harmonic and continuous

- In Euclidean settings:
  $\mathrm{div}(\mathrm{cof}(df))=0$, $\mathrm{cof}(df)=df$ a.e. $\Rightarrow \mathrm{div}(df)=0$

- In general:
  $f \in W^{1,p}(M;N)$, $p>2(n-1)$, $\quad i:N\to\boldsymbol{R}^D$ iso. embedding

Sketch of proof

$df \in \mathrm{SO}(g,h)$ a.e. ⇒ f weakly-harmonic and continuous

- In Euclidean settings:
  $\mathrm{div}(\mathrm{cof}(df))=0$, $\mathrm{cof}(df)=df$ a.e. ⇒ $\mathrm{div}(df)=0$

- In general:
  $f \in W^{1,p}(M;N)$, $p>2(n-1)$, $i:N\to\boldsymbol{R}^D$ iso. embedding

$$\mathrm{div}_g(di \circ \mathrm{cof}_{g,h} df) + \mathrm{tr}_g \mathrm{II}_i(\mathrm{cof}_{g,h} df, df) = 0$$

Sketch of proof

$df \in \mathrm{SO}(g,h)$ a.e. $\Rightarrow f$ weakly-harmonic and continuous

- In Euclidean settings:
  $\mathrm{div}(\mathrm{cof}(df))=0$, $\mathrm{cof}(df)=df$ a.e. $\Rightarrow$ $\mathrm{div}(df)=0$

- In general:
  $f \in W^{1,p}(M;N)$, $p>2(n-1)$, $\quad i:N\to\boldsymbol{R}^D$ iso. embedding

$$\mathrm{div}_g(di \circ \mathrm{cof}_{g,h} df) + \mathrm{tr}_g \mathrm{II}_i(\mathrm{cof}_{g,h} df, df) = 0$$

- If $\mathrm{cof}_{g,h}(df)=df$ a.e. this equation implies that f is weakly harmonic, and continuous, hence smooth.

I am out of time,
but let me just mention…

I am out of time, but let me just mention…

- Curvature-based lower-bounds for the elastic energy

I am out of time,
but let me just mention…

- Curvature-based lower-bounds for the elastic energy
- Dimension reduction in non-Euclidean elasticity (Lewicka et al., Kupferman-Solomon, Kohn-O'Brien, M.-Shachar, …)

I am out of time, but let me just mention…

- Curvature-based lower-bounds for the elastic energy
- Dimension reduction in non-Euclidean elasticity (Lewicka et al., Kupferman-Solomon, Kohn-O'Brien, M.-Shachar, …)
- Dislocation theory

I am out of time, but let me just mention…

- Curvature-based lower-bounds for the elastic energy
- Dimension reduction in non-Euclidean elasticity (Lewicka et al., Kupferman-Solomon, Kohn-O'Brien, M.-Shachar, …)
- Dislocation theory

Happy Victoria Day!