Robust Multicast Beamforming in Cognitive Radio Networks: Semidefinite Relaxation and Approximation Analysis

Anthony Man-Cho So
(Joint Work with Xiaoxiao Wu, Man-Chung Yue, and Wing-Kin Ma)

Department of Systems Engineering & Engineering Management
The Chinese University of Hong Kong (CUHK), Hong Kong

BIRS Distributionally Robust Optimization Workshop

8 March 2018

Outline of Talk

- Robust multicast beamforming in cognitive radio networks
- Robust quadratically constrained quadratic optimization (QCQO) formulation
- Semidefinite relaxation of the robust QCQO formulation
- Approximation analysis of the semidefinite relaxation

Introduction

- The demand for wireless services has been ever increasing.
  - live mobile TV
  - multi-party video conferencing
  - multimedia streaming for groups of paid users

From http://money.cnn.com

From http://www.telepresenceoptions.com

From http://www.slashgear.com

- This creates a great demand for spectrum-based communication links.

- However, there is a shortage of available frequencies, as most have been allocated to licensed users.

Introduction

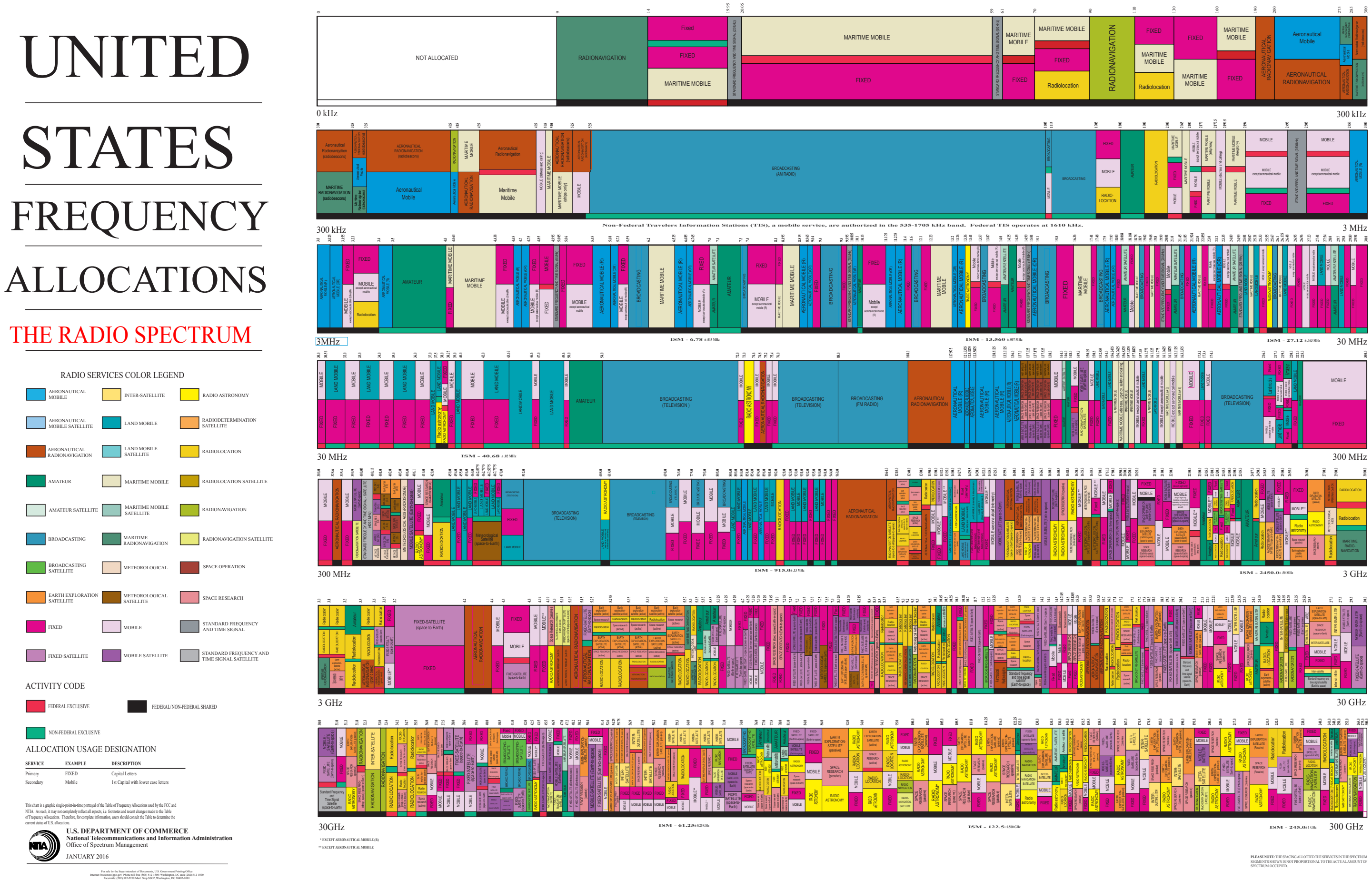


Figure 1: US Frequency Allocation Chart, as of January 2016. (Source: National Telecommunications and Information Administration, US Department of Commerce)

Introduction

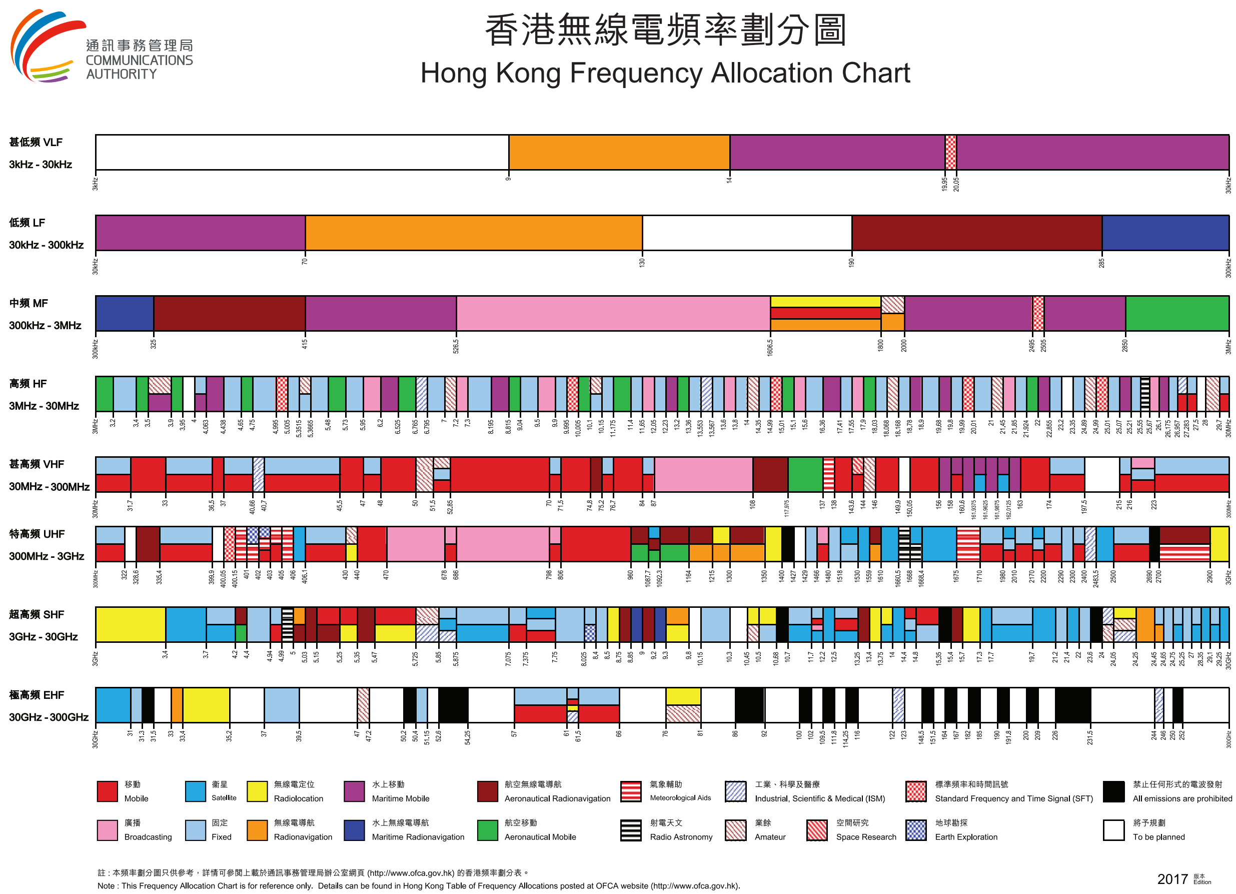


Figure 2: Hong Kong Frequency Allocation Chart, as of 2017. (Source: Office of the Communications Authority, The Government of the HKSAR)

Cognitive Radio Systems

- Cognitive radio (CR) technologies, which can adapt a radio's use of spectrum to real-time conditions of its operating environment, has emerged as a promising technology for improving spectrum utilization and bandwidth efficiency.

- In a CR network, secondary (unlicensed) users (SUs) are allowed to operate at the same frequency bands as the primary (licensed) users (PUs).

- We consider an underlay-CR network, in which both the PUs and SUs can use the frequency bands simultaneously.

- Naturally, the SUs should not cause excessive interference to the PUs.

- Current standards supporting CR: TV-broadcast bands (IEEE 802.22), LTE Advanced.

System Model

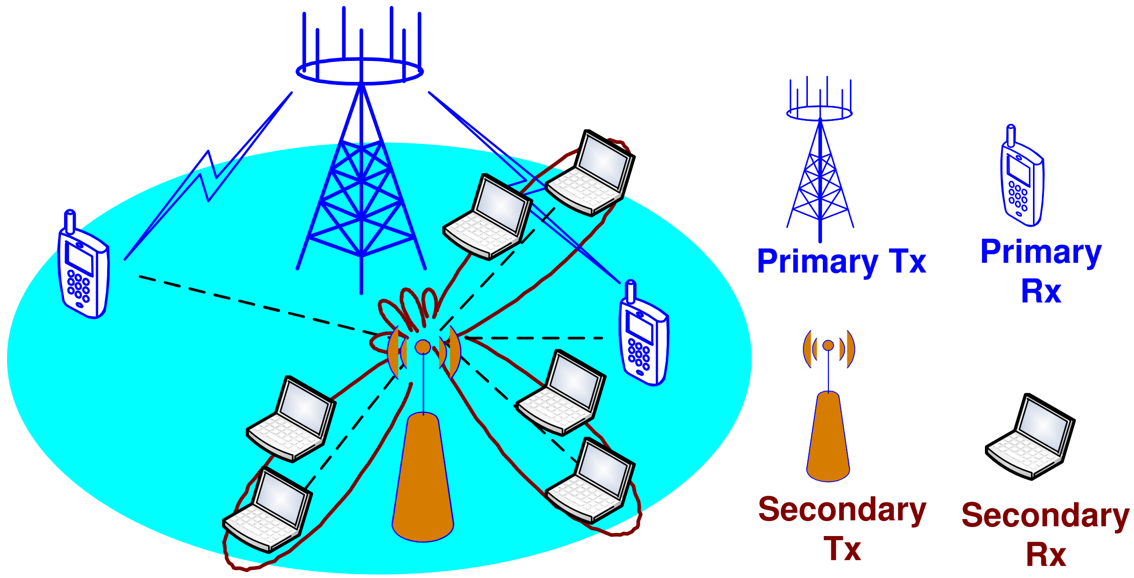


Figure 3: The multi-group multicast cognitive radio system model.

- Cognitive radio (CR) and multi-group multicast (MM) delivery play a significant role in supporting resource- and spectral-efficient data services in future communication systems.
- In general, one is interested in transmit designs in the MM-CR network.
- For simplicity, we focus on the scenario where there is only one group of SUs.

Problem Formulation

- Consider a secondary transmitter (ST) with N transmit antennae sending common information (i.e., multicasting) to M single-antenna SUs in the presence of J single-antenna PUs.

- By adopting the so-called transmit beamforming scheme **[Sidiropoulos-Davidson-Luo'06]**, the received signal for user i ($i = 1, \ldots, M$) is modeled as

$$y_i = \boldsymbol{h}_i^H \boldsymbol{w} s + n_i,$$

  where

  - $\boldsymbol{h}_i \in \mathbb{C}^N$: channel between the ST and user i;
  - $\boldsymbol{w} \in \mathbb{C}^N$: ST's beamforming vector;
  - $s \in \mathbb{C}$: unit power data stream;
  - $n_i \sim \mathcal{CN}(0, \sigma_i^2)$: additive complex Gaussian noise at user i.

- The quality of service received by user i is then measured by the signal-to-noise ratio (SNR):

$$\mathsf{SNR}_i(\boldsymbol{w}) = \frac{\left|\boldsymbol{h}_i^H \boldsymbol{w}\right|^2}{\sigma_i^2}.$$

Problem Formulation

- On the other hand, the interference power at the j-th PU is

$$\mathsf{INT}_j(\boldsymbol{w}) = \left|\boldsymbol{g}_j^H \boldsymbol{w}\right|^2,$$

  where

  – $\boldsymbol{g}_j \in \mathbb{C}^N$: channel between the ST and PU j.

- Our goal is to design the beamforming vector $\boldsymbol{w} \in \mathbb{C}^N$ under the max-min-fair (MMF) criterion, subject to a power constraint at the ST and so-called interference temperature constraints at the PUs:

$$\begin{array}{ll} \displaystyle\max_{\boldsymbol{w}\in\mathbb{C}^N} & \displaystyle\min_{i=1,\ldots,M} \mathsf{SNR}_i(\boldsymbol{w}) \\ \text{subject to} & \|\boldsymbol{w}\|_2^2 \le P, \\ & \mathsf{INT}_j(\boldsymbol{w}) \le \beta_j \quad \text{for } j = 1, \ldots, J. \end{array}$$

  Here,

  – $P > 0$: maximum allowable transmit power of the ST;

  – $\beta_j > 0$: interference threshold of the PU j.

Incorporating Robustness

- Right now, our problem formulation assumes that the channels $\{\boldsymbol{h}_i\}_{i=1}^M$ and $\{\boldsymbol{g}_j\}_{j=1}^J$ are perfectly known.

- This may be reasonable for the ST-SU channels $\{\boldsymbol{h}_i\}_{i=1}^M$ (as the ST and the SUs can exchange channel state information), but certainly not for the ST-PU channels $\{\boldsymbol{g}_j\}_{j=1}^J$ (as the PUs will not voluntarily reveal information to the ST).

- Thus, the channels between the ST and the PUs have to be estimated.

- A commonly adopted model for the ST-PU channels:

$$\boldsymbol{g}_j = \bar{\boldsymbol{g}}_j + \boldsymbol{\Delta}_j, \quad \|\boldsymbol{\Delta}_j\|_2 \le \delta_j,$$

  where

  - $\bar{\boldsymbol{g}}_j \in \mathbb{C}^N$: estimated channel between the ST and PU j;
  - $\boldsymbol{\Delta}_j \in \mathbb{C}^N$: channel estimation error associated with PU j;
  - $\delta_j \ge 0$: error threshold associated with PU j.

Robust Multicast Beamforming in CR Networks

- This yields the following robust formulation:

$$\begin{array}{rll} v^\star = & \max\limits_{\boldsymbol{w} \in \mathbb{C}^N} & \min\limits_{i=1,\ldots,M} \mathsf{SNR}_i(\boldsymbol{w}) \\ & \text{subject to} & \|\boldsymbol{w}\|_2^2 \le P, \\ & & \max\limits_{\|\boldsymbol{\Delta}_j\|_2 \le \delta_j} \mathsf{INT}_j(\boldsymbol{w}, \boldsymbol{\Delta}_j) \le \beta_j \quad \text{for } j = 1, \ldots, J. \end{array} \qquad \text{(R-MMF-BF)}$$

  Here,

$$\mathsf{SNR}_i(\boldsymbol{w}) = \frac{\left|\boldsymbol{h}_i^H \boldsymbol{w}\right|^2}{\sigma_i^2}, \quad \mathsf{INT}_j(\boldsymbol{w}, \boldsymbol{\Delta}_j) = \left|(\bar{\boldsymbol{g}}_j + \boldsymbol{\Delta}_j)^H \boldsymbol{w}\right|^2.$$

  By re-defining $\boldsymbol{h}_i$ if necessary, we may assume that $\sigma_i^2 = 1$ for $i = 1, \ldots, M$.

- Even when $\delta_j = 0$ for $j = 1, \ldots, J$, (R-MMF-BF) is still a non-convex QCQO problem and is NP-hard in general **[Luo-Sidiropoulos-Tseng-Zhang'07] [Karipidis-Sidiropoulos-Luo'08]**.

Semidefinite Relaxation of (R-MMF-BF)

- A classic approach to tackling (R-MMF-BF) is to apply semidefinite relaxation. First, observe that

$$\boldsymbol{W} = \boldsymbol{w}\boldsymbol{w}^H \quad \Longleftrightarrow \quad \boldsymbol{W} \succeq \boldsymbol{0}, \quad \mathrm{rank}(\boldsymbol{W}) \leq 1.$$

Moreover, by the $\mathcal{S}$-lemma, we have

$$\mathsf{INT}_j(\boldsymbol{w}, \boldsymbol{\Delta}_j) = \max_{\|\boldsymbol{\Delta}_j\|_2 \leq \delta_j} \left| (\bar{\boldsymbol{g}}_j + \boldsymbol{\Delta}_j)^H \boldsymbol{w} \right|^2 \leq \beta_j$$

if and only if there exists a $\kappa_j \geq 0$ such that

$$\begin{bmatrix} \kappa_j \boldsymbol{I} - \boldsymbol{w}\boldsymbol{w}^H & -\boldsymbol{w}^H \bar{\boldsymbol{g}}_j \boldsymbol{w} \\ -\bar{\boldsymbol{g}}_j^H \boldsymbol{w}\boldsymbol{w}^H & \beta_j - \left| \boldsymbol{w}^H \bar{\boldsymbol{g}}_j \right|^2 - \delta_j^2 \kappa_j \end{bmatrix} \succeq \boldsymbol{0}.$$

Semidefinite Relaxation of (R-MMF-BF)

- Putting everything together, we can reformulate (R-MMF-BF) as the following rank-constrained SDP:

$$
\begin{array}{ll}
\displaystyle\max_{\boldsymbol{W},t,\kappa_1,\ldots,\kappa_J} & t \\
\text{subject to} & \boldsymbol{h}_i^H \boldsymbol{W} \boldsymbol{h}_i \geq t \quad \text{for } i = 1,\ldots,M, \\
& \mathrm{Tr}(\boldsymbol{W}) \leq P, \\
& \begin{bmatrix} \kappa_j \boldsymbol{I} - \boldsymbol{W} & -\boldsymbol{W}\bar{\boldsymbol{g}} \\ -\bar{\boldsymbol{g}}_j^H \boldsymbol{W} & \beta_j - \bar{\boldsymbol{g}}_j^H \boldsymbol{W} \bar{\boldsymbol{g}}_j - \delta_j^2 \kappa_j \end{bmatrix} \succeq \boldsymbol{0}, \ \ \kappa_j \geq 0 \\
& \quad \text{for } j = 1,\ldots,J, \\
& \boldsymbol{W} \succeq \boldsymbol{0}, \ \ \mathrm{rank}(\boldsymbol{W}) \leq 1.
\end{array}
\qquad \text{(SDR)}
$$

- By dropping the rank constraint, we obtain a semidefinite relaxation of the original problem.

Quality of the Semidefinite Relaxation

- Let $(\boldsymbol{W}^\star, t^\star, \kappa_1^\star, \ldots, \kappa_J^\star)$ be an optimal solution to (SDR). Clearly, we have $t^\star \geq v^\star$ because (SDR) is a relaxation of (R-MMF-BF).

- If we have $\mathrm{rank}(\boldsymbol{W}^\star) \leq 1$, then we have found an optimal solution to the original problem (R-MMF-BF).

- In general, there is no guarantee that $\boldsymbol{W}^\star$ will satisfy the rank constraint.

- Questions

  Let $\boldsymbol{W}^\star$ be an optimal solution to (SDR). If $\mathrm{rank}(\boldsymbol{W}^\star) > 1$, how do we generate from $\boldsymbol{W}^\star$ a feasible solution $\hat{\boldsymbol{w}}$ to the original problem (R-MMF-BF)? Can we say something about the quality of the approximate solution $\hat{\boldsymbol{w}}$?

  – In particular, can we find $\alpha \in (0, 1)$ (called the approximation ratio) such that

$$v^\star \geq \min_{i=1,\ldots,M} \mathsf{SNR}_i(\hat{\boldsymbol{w}}) \geq \alpha \cdot v^\star?$$

- A classic idea: Perform "randomized rounding"

Handling High-Rank SDP Solutions

- Suppose that $\mathrm{rank}(\boldsymbol{W}^\star) > 1$. Generate $\boldsymbol{\xi} \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{W}^\star)$.

- Why may this be a reasonable idea? Observe that

$$\mathbb{E}\left[\mathsf{SNR}_i(\boldsymbol{\xi})\right] = \mathbb{E}\left[\left|\boldsymbol{h}_i^H \boldsymbol{\xi}\right|^2\right] = \boldsymbol{h}_i^H \boldsymbol{W}^\star \boldsymbol{h}_i \geq t^\star,$$

$$\mathbb{E}\left[\|\boldsymbol{\xi}\|_2^2\right] = \mathrm{Tr}(\boldsymbol{W}^\star) \leq P.$$

  In other words, in expectation, $\boldsymbol{\xi}$ satisfies the power constraint and achieves the optimal value of (SDR).

  – However, without knowing the concentration properties of $\mathsf{SNR}_i(\boldsymbol{\xi})$ and $\|\boldsymbol{\xi}\|_2^2$, the above may not mean much.

- Moreover, it is not clear whether the interference constraints are satisfied even in expectation:

$$\mathbb{E}\left[\max_{\|\boldsymbol{\Delta}\|_2 \leq \delta_j} \mathsf{INT}_j(\boldsymbol{\xi}, \boldsymbol{\Delta}_j)\right] = \mathbb{E}\left[\max_{\|\boldsymbol{\Delta}_j\|_2 \leq \delta_j} \left|(\bar{\boldsymbol{g}}_j + \boldsymbol{\Delta}_j)^H \boldsymbol{\xi}\right|^2\right] \overset{?}{\leq} \beta_j.$$

Analysis of the Randomization Procedure: Warm Up

- Consider first the case where $\delta_j = 0$ for $j = 1, \ldots, J$; i.e., there is no channel estimation error. Then, all robust constraints reduce to quadratic constraints.

- This case has been well studied in the literature **[Nemirovski-Roos-Terlaky'99], [Luo-Sidiropoulos-Tseng-Zhang'07], [S-Ye-Zhang'08]**. The key lies in the following deviation inequalities for complex Gaussian quadratic forms:

  Theorem [S-Ye-Zhang'08]: Let $\boldsymbol{Q}, \boldsymbol{W}^\star \succeq \mathbf{0}$ be given. Let $\boldsymbol{\xi} \sim \mathcal{CN}(\mathbf{0}, \boldsymbol{W}^\star)$. If $\boldsymbol{Q}\boldsymbol{W}^\star = \mathbf{0}$, then $\boldsymbol{\xi}^H \boldsymbol{Q} \boldsymbol{\xi} = 0$ a.s. Otherwise, for any $\mu > 1$ and $\nu \in (0,1)$,

$$\Pr\left(\boldsymbol{\xi}^H \boldsymbol{Q} \boldsymbol{\xi} \geq \mu \cdot \mathrm{Tr}(\boldsymbol{Q}\boldsymbol{W}^\star)\right) \leq \exp(1 - \mu + \ln \mu),$$

$$\Pr\left(\boldsymbol{\xi}^H \boldsymbol{Q} \boldsymbol{\xi} \leq \nu \cdot \mathrm{Tr}(\boldsymbol{Q}\boldsymbol{W}^\star)\right) \leq \exp(1 - \nu + \ln \nu).$$

Analysis of the Randomization Procedure: Warm Up

- By the theorem and the union bound, all the following inequalities

$$\mathsf{SNR}_i(\boldsymbol{\xi}) = \left|\boldsymbol{h}_i^H \boldsymbol{\xi}\right|^2 \geq \nu \cdot \mathrm{Tr}\left(\boldsymbol{h}_i \boldsymbol{h}_i^H \boldsymbol{W}^\star\right) \quad \text{for } i = 1, \ldots, M,$$

$$\|\boldsymbol{\xi}\|_2^2 \leq \mu \cdot \mathrm{Tr}(\boldsymbol{W}^\star),$$

$$\mathsf{INT}_j(\boldsymbol{\xi}) = \left|\bar{\boldsymbol{g}}_j^H \boldsymbol{\xi}\right|^2 \leq \mu \cdot \mathrm{Tr}\left(\bar{\boldsymbol{g}}_j \bar{\boldsymbol{g}}_j^H \boldsymbol{W}^\star\right) \quad \text{for } j = 1, \ldots, J$$

  hold with probability at least $1 - (J+1)\exp(1 - \mu + \ln \mu) - M \exp(1 - \nu + \ln \nu)$.

- Since

$$\mathrm{Tr}\left(\boldsymbol{h}_i \boldsymbol{h}_i^H \boldsymbol{W}^\star\right) \geq t^\star, \quad \mathrm{Tr}(\boldsymbol{W}^\star) \leq P, \quad \mathrm{Tr}\left(\bar{\boldsymbol{g}}_j \bar{\boldsymbol{g}}_j^H \boldsymbol{W}^\star\right) \leq \beta_j,$$

  by choosing $\mu = O(\ln J)$ and $\nu = \Omega(1/M)$, we conclude that with constant probability,

  – $\hat{\boldsymbol{w}} = \boldsymbol{\xi}/\sqrt{\mu}$ is feasible for the original problem (R-MMF-BF),

  – $\mathsf{SNR}_i(\hat{\boldsymbol{w}}) = \left|\boldsymbol{h}_i^H \hat{\boldsymbol{w}}\right|^2 = \left|\boldsymbol{h}_i^H \boldsymbol{\xi}\right|^2 / \mu \geq (\nu/\mu) t^\star \geq (\nu/\mu) v^\star$.

- Hence, $\hat{\boldsymbol{w}}$ is an (ν/μ)-approximate solution to the original problem (R-MMF-BF).

Summary of the Warm-Up Case

- Recall the problem we are trying to solve:

$$\begin{array}{lll} \max\limits_{\boldsymbol{w}\in\mathbb{C}^N} & \min\limits_{i=1,\ldots,M} \mathsf{SNR}_i(\boldsymbol{w}) & \\ \text{subject to} & \|\boldsymbol{w}\|_2^2 \le P, & \\ & \mathsf{INT}_j(\boldsymbol{w}) \le \beta_j \quad \text{for } j=1,\ldots,J. & \end{array} \qquad \text{(MMF-BF)}$$

- The proposed algorithm
  - Solve the semidefinite relaxation and get the optimal solution $\boldsymbol{W}^\star$.
  - Generate $\boldsymbol{\xi} \sim \mathcal{CN}(\mathbf{0}, \boldsymbol{W}^\star)$ and scale it down by

$$\mu = \max\left\{ \frac{\|\boldsymbol{\xi}\|_2^2}{P}, \max_{j=1,\ldots,J} \frac{\mathsf{INT}_j(\boldsymbol{\xi})}{\beta_j} \right\}$$

    to get a feasible solution $\hat{\boldsymbol{w}}$ to (MMF-BF).

- Our analysis showed that with high probability, $\mu = O(\ln J)$ and $\hat{\boldsymbol{w}}$ is an $\Omega(1/M \ln J)$-approximate solution.
- Note that the ratio degrades linearly in the number of SUs (M) but only logarithmically in the number of PUs (J).

Analysis of the Randomization Procedure: General Case

- Now, let us tackle the general case. Again, we generate $\boldsymbol{\xi} \sim \mathcal{CN}(\mathbf{0}, \boldsymbol{W}^\star)$.

- Same as before, with probability at least $1 - \exp(1-\mu+\ln\mu) - M\exp(1-\nu+\ln\nu)$, all the following inequalities hold:

$$\begin{aligned} \mathsf{SNR}_i(\boldsymbol{\xi}) = \left|\boldsymbol{h}_i^H \boldsymbol{\xi}\right|^2 &\geq \nu \cdot \mathrm{Tr}\left(\boldsymbol{h}_i \boldsymbol{h}_i^H \boldsymbol{W}^\star\right) \quad \text{for } i = 1, \ldots, M, \\ \|\boldsymbol{\xi}\|_2^2 &\leq \mu \cdot \mathrm{Tr}(\boldsymbol{W}^\star). \end{aligned}$$

Analysis of the Randomization Procedure: General Case

- Recalling that $\mathsf{INT}_j(\boldsymbol{\xi}, \boldsymbol{\Delta}_j) = \left| \boldsymbol{g}_j^H \boldsymbol{\xi} \right|^2$ with $\boldsymbol{g}_j = \bar{\boldsymbol{g}}_j + \boldsymbol{\Delta}_j$, consider

$$\max_{\|\boldsymbol{\Delta}_j\|_2 \le \delta_j} \mathsf{INT}_j(\boldsymbol{\xi}, \boldsymbol{\Delta}_j) = \max_{\|\boldsymbol{\Delta}_j\|_2 = \delta_j} \left| \left(\bar{\boldsymbol{g}}_j + \boldsymbol{\Delta}_j\right)^H \boldsymbol{\xi} \right|^2 .$$

For simplicity, let us take $\delta_j = 1$ and define

$$\boldsymbol{\Delta}_j(\boldsymbol{\xi}) = \arg \max_{\|\boldsymbol{\Delta}_j\|_2 = 1} \left| \left(\bar{\boldsymbol{g}}_j + \boldsymbol{\Delta}_j\right)^H \boldsymbol{\xi} \right|^2 .$$

Analysis of the Randomization Procedure: General Case

- Two tempting ideas
  - Apply the previous theorem to claim

$$\left| \left(\bar{\boldsymbol{g}}_j + \boldsymbol{\Delta}_j(\boldsymbol{\xi}) \right)^H \boldsymbol{\xi} \right|^2 \leq \mu \cdot \mathrm{Tr}\left(\left(\bar{\boldsymbol{g}}_j + \boldsymbol{\Delta}_j(\boldsymbol{\xi}) \right) \left(\bar{\boldsymbol{g}}_j + \boldsymbol{\Delta}_j(\boldsymbol{\xi}) \right)^H \boldsymbol{W}^\star \right)$$

    with high probability. However, $\boldsymbol{\Delta}_j(\boldsymbol{\xi})$ depends on $\boldsymbol{\xi}$ and hence we cannot apply the theorem directly.
  - Let $\mathbb{S}^{N-1}$ denote the unit $(N-1)$-sphere. Then, for a fixed $\boldsymbol{\Delta}_j \in \mathbb{S}^{N-1}$, the previous theorem gives

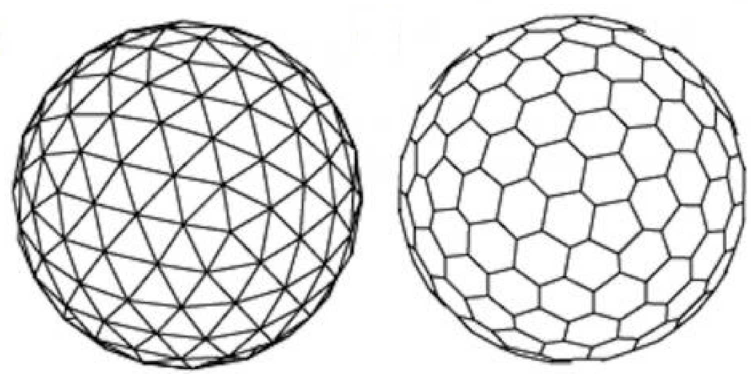
$$\left| \left(\bar{\boldsymbol{g}}_j + \boldsymbol{\Delta}_j \right)^H \boldsymbol{\xi} \right|^2 \leq \mu \cdot \mathrm{Tr}\left(\left(\bar{\boldsymbol{g}}_j + \boldsymbol{\Delta}_j \right) \left(\bar{\boldsymbol{g}}_j + \boldsymbol{\Delta}_j \right)^H \boldsymbol{W}^\star \right)$$

    with high probability. However, we cannot take the union bound over $\mathbb{S}^{N-1}$!

- The notion of an ϵ-net comes to the rescue.

Analysis of the Randomization Procedure: General Case

- **Definition:** Let $\epsilon > 0$ be given. We say that $\mathcal{N} \subset \mathbb{S}^{N-1}$ is an ϵ-net of $\mathbb{S}^{N-1}$ if for any $\boldsymbol{z} \in \mathbb{S}^{N-1}$, there exists a $\boldsymbol{u} \in \mathcal{N}$ such that $\|\boldsymbol{u} - \boldsymbol{z}\|_2 \leq \epsilon$.



- It is well known (and can be easily established via a volume argument) that given $\epsilon > 0$, there exists an ϵ-net $\mathcal{N}_\epsilon$ of $\mathbb{S}^{N-1}$ of size at most $(1 + 2/\epsilon)^{2N}$.
  - More generally, there exists an ϵ-net $\mathcal{N}_\epsilon(\delta)$ of $\delta\mathbb{S}^{N-1}$ of size at most $(\delta(1 + 2/\epsilon))^{2N}$.

Analysis of the Randomization Procedure: General Case

- Now, take an ϵ-net $\mathcal{N}_\epsilon$ of $\mathbb{S}^{N-1}$ with $\epsilon \in (0,1)$. It can be shown that there exist sequences $\{\epsilon_k\}_{k\geq 0}$ and $\{\boldsymbol{u}^k\}_{k\geq 0}$ with $\epsilon_k \in [0,\epsilon^k]$ and $\boldsymbol{u}^k \in \mathcal{N}_\epsilon$ such that

$$\boldsymbol{\Delta}_j(\boldsymbol{\xi}) = \sum_{k\geq 0} \epsilon_k \boldsymbol{u}^k.$$

- Let $S = \left(\sum_{k\geq 0} \epsilon_k\right)^{-1} \geq 1-\epsilon$. Using the above, we can then show that

$$\begin{aligned} \max_{\|\boldsymbol{\Delta}_j\|_2 \leq 1} \mathsf{INT}_j(\boldsymbol{\xi}, \boldsymbol{\Delta}_j) &= \left| \left(\bar{\boldsymbol{g}}_j + \boldsymbol{\Delta}_j(\boldsymbol{\xi})\right)^H \boldsymbol{\xi} \right|^2 \\ &\leq \frac{2}{S^2} \max_{\boldsymbol{u}\in\mathcal{N}_\epsilon} \left| (\bar{\boldsymbol{g}}_j + \boldsymbol{u})^H \boldsymbol{\xi} \right|^2 + 2\left(\frac{1}{S} - 1\right)^2 \left| \bar{\boldsymbol{g}}_j^H \boldsymbol{\xi} \right|^2. \end{aligned}$$

The upshot is that both terms can be tackled by our previous theorem (the first entails taking a union bound over all points in $\mathcal{N}_\epsilon$).

Analysis of the Randomization Procedure: General Case

- Carrying out the necessary calculations, we get the following:

  Theorem: Fix $j \in \{1, \ldots, J\}$. For any $\mu > 1$ and $\epsilon \in (0,1)$, the inequality

$$\max_{\|\boldsymbol{\Delta}_j\|_2 \leq 1} \mathsf{INT}_j(\boldsymbol{\xi}, \boldsymbol{\Delta}_j) \leq \frac{2\mu(1+\epsilon^2)}{(1-\epsilon)^2} \max_{\|\boldsymbol{\Delta}_j\|_2 \leq 1} \mathrm{Tr}\left(\left(\bar{\boldsymbol{g}}_j + \boldsymbol{\Delta}_j\right) \left(\bar{\boldsymbol{g}}_j + \boldsymbol{\Delta}_j\right)^H \boldsymbol{W}^\star \right)$$

  holds with probability at least $1 - (|\mathcal{N}_\epsilon| + 1)\exp(1 - \mu + \ln \mu)$.

- Summing up, with probability at least $1 - (J(|\mathcal{N}_\epsilon| + 1) + 1)\exp(1 - \mu + \ln \mu) - M\exp(1 - \nu + \ln \nu)$, all the following inequalities hold:

$$\mathsf{SNR}_i(\boldsymbol{\xi}) = \left| \boldsymbol{h}_i^H \boldsymbol{\xi} \right|^2 \geq \nu \cdot \mathrm{Tr}\left(\boldsymbol{h}_i \boldsymbol{h}_i^H \boldsymbol{W}^\star \right) \quad \text{for } i = 1, \ldots, M,$$

$$\|\boldsymbol{\xi}\|_2^2 \leq \mu \cdot \mathrm{Tr}(\boldsymbol{W}^\star),$$

$$\max_{\|\boldsymbol{\Delta}_j\|_2 \leq 1} \mathsf{INT}_j(\boldsymbol{\xi}, \boldsymbol{\Delta}_j) \leq \frac{2\mu(1+\epsilon^2)}{(1-\epsilon)^2} \max_{\|\boldsymbol{\Delta}_j\|_2 \leq 1} \mathrm{Tr}\left(\left(\bar{\boldsymbol{g}}_j + \boldsymbol{\Delta}_j\right) \left(\bar{\boldsymbol{g}}_j + \boldsymbol{\Delta}_j\right)^H \boldsymbol{W}^\star \right)$$

$$\text{for } j = 1, \ldots, J.$$

Analysis of the Randomization Procedure: General Case

- Hence, by taking

$$\epsilon = 1/2, \;\; \mu = O(N \ln J), \;\; \nu = \Omega(1/M)$$

and arguing as before, we conclude that $\hat{\boldsymbol{w}} = \boldsymbol{\xi}/\sqrt{\mu}$ is an (ν/μ)-approximate solution to (R-MMF-BF) with constant probability, assuming $\delta_1 = \cdots = \delta_J = 1$.

  – Note that $\nu/\mu = \Omega(1/MN \ln J)$. Compared with the non-robust case, the ratio also degrades linearly in the number of transmit antennae at the ST (N).

Summary of the General Case

- Recall the problem we are trying to solve:

$$\begin{array}{ll} \max\limits_{\boldsymbol{w}\in\mathbb{C}^N} & \min\limits_{i=1,\ldots,M} \mathsf{SNR}_i(\boldsymbol{w}) \\ \text{subject to} & \|\boldsymbol{w}\|_2^2 \le P, \\ & \max\limits_{\|\boldsymbol{\Delta}_j\|_2\le\delta_j} \mathsf{INT}_j(\boldsymbol{w},\boldsymbol{\Delta}_j) \le \beta_j \quad \text{for } j=1,\ldots,J. \end{array} \qquad \text{(R-MMF-BF)}$$

- The proposed algorithm
  - Solve the semidefinite relaxation and get the optimal solution $\boldsymbol{W}^\star$.
  - Generate $\boldsymbol{\xi} \sim \mathcal{CN}(\mathbf{0}, \boldsymbol{W}^\star)$ and scale it down by

$$\mu = \max\left\{ \frac{\|\boldsymbol{\xi}\|_2^2}{P}, \max_{j=1,\ldots,J} \max_{\|\boldsymbol{\Delta}_j\|_2\le\delta_j} \frac{\mathsf{INT}_j(\boldsymbol{\xi},\boldsymbol{\Delta}_j)}{\beta_j} \right\}$$

    to get a feasible solution $\hat{\boldsymbol{w}}$ to (R-MMF-BF).

- Question: How to compute μ?

Summary of the General Case

- The computation of μ involves computing

$$\max_{\|\boldsymbol{\Delta}_j\|_2 \le \delta_j} \mathsf{INT}_j(\boldsymbol{\xi}, \boldsymbol{\Delta}_j) = \max_{\|\boldsymbol{\Delta}_j\|_2 \le \delta_j} \left| \left(\bar{\boldsymbol{g}}_j + \boldsymbol{\Delta}_j \right)^H \boldsymbol{\xi} \right|^2,$$

  which is a non-convex QCQO problem.

- Nevertheless, it is a trust region-type problem, for which there are polynomial-time algorithms **[Ye'92,Ye'94,Adachi-Iwata-Nakatsukasa-Takeda'17]**.

Final Remarks

- We analyzed the approximation quality of a semidefinite relaxation-based algorithm for solving the robust multicast beamforming problem in cognitive radio networks.

- Our techniques can be extended to develop approximation analysis of semidefinite relaxation-based algorithms for a class of rank-constrained robust fractional SDPs.

- There are not many results concerning the approximability of NP-hard robust optimization problems in the literature. This would be an interesting direction for future study.

Thank You!