Worst-Case Law Invariant Risk Measures and Distributions: The Case of Nonlinear DRO

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Distributionally Robust Optimization Workshop
BIRS, Banff, 2018

Outline

1. Distributionally Robust Optimization (DRO)
   - Moment-based DRO
   - Worst-Case Distributions
2. DRO Formulation based on Risk Measures
   - Law Invariant Risk Measures
   - Worst-Case Risk Measures
   - Worst-Case Distributions
   - DRO with Worst-Case Risk Measures
3. Conclusion & Future Work

Outline

Moment-based DRO

A moment-based DRO can be generally formulated as

$$\max_{x \in D} \inf_{F_\xi \in \mathscr{M}} \mathbb{E}_{F_\xi}[h(x,\xi)],$$

where

$$\mathscr{M} := \left\{ F_\xi : \Re^n \to \Re_{\geq 0} \;\middle|\; \mathbb{E}_{F_\xi}[G(\xi)] \in \mathscr{K} \subseteq \Re^m \right\}.$$

- $h(x,\xi)$: some form of perceived benefit
- x: a decision variable vector
- ξ: a random parameter vector with distribution F_ξ

Example: Robust Mean-Covariance Solutions

In the case that only mean and covariance are available, Popescu (2007) consider the following DRO problem

$$\max_{x \in D} \inf_{F_\xi \in \mathscr{M}} \mathbb{E}_{F_\xi}[u(\xi^\top x)],$$

where

$$G(\xi) := \begin{pmatrix} \xi \\ \xi\xi^\top \end{pmatrix}, \mathscr{K} := \begin{pmatrix} \mu \\ \Sigma + \mu\mu^\top \end{pmatrix}.$$

- Popescu (2007) shows that for a large family of utility functions $u(\cdot)$, the above problem can be reduced to a parametric quadratic program.
- If $u(\cdot)$ is piecewise linear, several others (Bertsimas et al., Natarajan et al., and Delage et al. (2010)) show that the problem can be solved as a conic program.

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Worst-Case Distributions

Worst-case distributions with finite supports

- Popescu (2007) exploits the structure of worst-case distributions $F_{\xi^\top x}$ supported by at most three points.
- For general moment-based DRO, worst-case distributions have been identified as discrete distributions with at most $m+1$ supports (Rogosinsky (1958)).
- This structure of worst-case distributions follows the fact that moment-based DRO is linear in the distribution (e.g. Smith (1995)).

Drawback

- Such worst-case distributions appear implausible as descriptions of real worst-case scenarios

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DRO based on Law Invariant Risk Measures

W.l.o.g., we now consider $h(x,\xi)$ represents some form of loss and replace the expectation in DRO by a law invariant risk measure ρ

$$\min_{x\in D} \sup_{F_\xi \in \mathscr{M}} \rho_{F_\xi}(h(x,\xi)).$$

- A risk measure ρ is law invariant (or distribution-based) if $\rho(Z_1) = \rho(Z_2)$ holds for any Z_1, Z_2 that satisfy $F_{Z_1} \equiv F_{Z_2}$.
- This comprises all risk measures that one would encounter in DRO.
- The above DRO, in general, is non-linear in F_ξ.

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Example: DRO based on Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR)

- El Ghaoui et al. (2003) studied the case where ρ_{F_ξ} is VaR.
- Many others have addressed the case where ρ_{F_ξ} is CVaR, e.g. Delage et al., Natarajan et al. (2010), Chen et al. (2011).

On the Family of Coherent Risk Measures

Our interest is to study DRO based on a more general family of risk measures, since

- VaR and CVaR do not well represent one's true risk preference,
- there is growing interest to seek an alternative framework other than expected utility (as studied in Popescu (2007)) to address risk

In particular, we only assume ρ_{F_ξ} is coherent (Artzner et al. (1999)), i.e. it satisfies

- monotonicity
- convexity
- translation invariance
- scale invariance

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Examples of Coherent Risk Measures

1. Expectation: $\rho(Z) := \mathbb{E}[Z]$
2. Conditional Value-at-Risk (CVaR): $\rho(Z) := \frac{1}{1-\alpha}\int_\alpha^1 F_Z^{-1}(p)dp$
3. Wang Transform (WT): $\rho(Z) := \int_0^1 F_Z^{-1}(p)dH(p)$, where $H(p) = -\Phi[\Phi^{-1}(1-p)+\lambda]$
4. Gini Measure: $\rho(Z) := \mathbb{E}[Z] + r\mathbb{E}(|Z - Z'|)$ (Z' : ind. copy of Z)
5. Deviation from the Median: $\rho(Z) := \mathbb{E}[Z] + a\mathbb{E}[|Z - F_Z^{-1}(0.5)|]$
6. Higher order risk measures: $\rho(Z) := \inf_t\{t + c \cdot ||(Z-t)^+||_p\}$, $c \geq 1, p \geq 1$
7. Higher order semideviation: $\rho(Z) := \mathbb{E}[Z] + \lambda||(Z - \mathbb{E}[Z])^+||_p$, $p \geq 1, 0 \leq \lambda \leq 1$
8. Many others...

Spectral (Distortion) Risk Measures

Examples 1-5 are also known as spectral (distortion) risk measures.

Definition

A risk measure ρ is called a spectral risk measure if it admits the form

$$\rho_\phi(Z) := \int_0^1 \phi(p) F_Z^{-1}(p) dp,$$

where ϕ is a right-continuous, monotonically nondecreasing density function.

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Worst-Case Risk Measures

- How tractable is it to evaluate $\sup_{F_Z \in \mathscr{M}} \rho(F_Z)$ in general?
- Given that
  - $\mathscr{M}(\mu, \sigma)$ is specified only based on the mean μ and standard deviation σ,
  - the Kusuoka representation (Kusuoka (2001)) of a risk measure, i.e. $\rho(F_Z) = \sup_{\phi \in \Phi} \rho_\phi(F_Z)$, is available

Theorem

(Li (2018)) The worst-case counterpart of a coherent risk measure ρ admits the closed-form

$$\sup_{F_Z \in \mathscr{M}(\mu,\sigma)} \rho(F_Z) = \mu + \kappa \cdot \sigma,$$

where $\kappa := \sqrt{\sup_{\phi \in \Phi} ||\phi||_2^2 - 1}$.

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Example 1: Higher-order risk measures

Consider higher-order risk measures, i.e.

$$\rho(Z) := \inf_t \{t + c \cdot ||(Z-t)^+||_p\}, c \geq 1, p \geq 1$$

- We have $\Phi := \{\phi \mid ||\phi||_q \leq c,\ \phi \in \mathscr{A}\}$
- We can derive $||\phi||_2^2 \leq c^p, \forall \phi \in \Phi$ based on Holder's interpolation inequality
- We can show that there exists a ϕ that attains the upper bound

Corollary

$$\sup_{F_Z \in \mathscr{M}(\mu,\sigma)} \rho(F_Z) = \mu + \sigma\sqrt{c^p - 1}$$

when $1 \leq p \leq 2$ and is infinite otherwise.

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Example 2: Higher-order semideviation

Consider higher-order semideviation, i.e.

$$\rho(Z) := \mathbb{E}[Z] + \lambda ||(Z - \mathbb{E}[Z])^+||_p,\ p \geq 1, 0 \leq \lambda \leq 1$$

- We have $\Phi := \{\phi \mid \phi = (1 - \frac{\lambda}{||\eta||_q}) + \frac{\lambda}{||\eta||_q}\eta,\ \eta \in \mathscr{A}\}$
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Corollary

$$\sup_{F_Z \in \mathscr{M}(\mu,\sigma)} \rho(F_Z) = \mu + \sigma\lambda(\frac{p^{1/2}}{2^{1/p}})(\frac{(2-p)^{1/p}}{(2-p)^{1/2}})$$

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Deriving the Closed-Form

- We start from the special case of worst-case spectral risk measures (WCSRM)

$$\sup_{F_Z \in \mathscr{M}(\mu,\sigma)} \rho_\phi(F_Z).$$

- We reformulate ρ_ϕ in terms of a minimization problem and arrive at

$$\sup_{F_Z \in \mathscr{M}(\mu,\sigma)} \min_{\Psi} G(F_Z, \psi),$$

$$G(F_Z, \psi) := \mathbb{E}_{F_Z}[\phi(0)Z + \int_0^1 [(1-\alpha)\psi(\alpha) + (Z - \psi(\alpha))^+] d\phi(\alpha)].$$

- We establish the equivalency between

$$\sup_{F_Z \in \mathscr{M}} \min_{\psi} G(F_Z, \psi) = \min_{\psi \in \Psi^\uparrow} \sup_{F_Z \in \mathscr{M}} G(F_Z, \psi),$$

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Deriving the Closed-Form

- We apply duality theory for the inner moment problem and after a few additional simplification steps we arrive at

$$\min_{\psi \in \Psi^{\uparrow}, \lambda_0, \lambda_1, \lambda_2} \lambda_0 + \lambda_1 \mu + \lambda_2(\mu^2 + \sigma^2) + \int_0^1 (1-\alpha)\psi(\alpha) d\alpha$$

$$\text{subject to } (\lambda_0 + \int_0^{\beta} \psi(\alpha) d\phi(\alpha)) + (\lambda_1 - \phi(\beta))z + \lambda_2 z^2 \geq 0, \ \forall z, \forall \beta \in (0,1)$$

Deriving the Closed-Form

- We further reduce the problem into

$$
\begin{aligned}
\min_{\psi \in \Psi^{\uparrow}, q, r, s, t} \quad & s + t + \int_0^1 (1-\alpha)\psi(\alpha) d\phi(\alpha) \\
\text{subject to} \quad & s + \int_0^{\beta} \psi(\alpha) d\phi(\alpha) - \phi(\beta)^2 r - \phi(\beta) q \geq 0, \quad \forall \beta \in (0,1) \\
& \begin{pmatrix} q-\mu \\ \sigma \\ r-t \\ r+t \end{pmatrix} \in \mathcal{Q}^4, \;\; r \geq 0
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- We identify a pair of primal-dual optimal solutions in closed-form for the above problem.
- We obtain $\sup_{F_Z \in \mathscr{M}(\mu,\sigma)} \rho_\phi(F_Z) = \mu + \kappa \cdot \sigma$, where $\kappa := \sqrt{||\phi||_2^2 - 1}$, and the general result immediately follows.

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Worst-Case Distributions for WCSRM

Theorem

The worst-case distribution F can be fully characterized by

$$F^{-1}(\beta) = (\mu - \frac{\sigma}{\sqrt{||\phi||_2^2 - 1}}) + \frac{\sigma}{\sqrt{||\phi||_2^2 - 1}}\phi(\beta^-),\ \beta \in (0,1),$$

where $\phi(\beta^-) := \lim_{\alpha \to \beta^-} \phi(\alpha)$ *and* $F^{-1}(\beta) = \mu$ *if* $\kappa = 0$ *(i.e.* $||\phi||_2^2 = 1$*).*
In the case of CVaR, we have

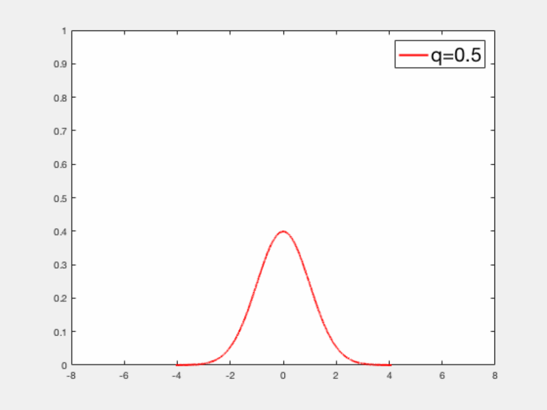
$$F^{-1}(\beta) = \begin{cases} \mu - \sigma\sqrt{\frac{\varepsilon}{1-\varepsilon}} & ,0 < \beta \leq 1-\varepsilon \\ \mu + \sigma\sqrt{\frac{1-\varepsilon}{\varepsilon}} & ,1-\varepsilon < \beta < 1 \end{cases}.$$

As ϕ^{-1} Location-Scale Distributions

- For any $||\phi||_2^2 > 1$, the worst-case distribution can be any distribution bounded from below by the threshold $\mu - \frac{\sigma}{\sqrt{||\phi||_2^2 - 1}}$.
- The worst-case distribution is unimodal if ϕ has at most one inflection point, which is the case for most existing spectral risk measures.
- The limiting distribution, $\lim_{n:||\phi_n||_2^2 \to 1} F_{\phi_n}$, depends on the sequence ϕ_n.

Worst-Case Distributions for Wang Transform (WT)

- WT is a popular spectral (distortion) risk measure, particularly in insurance (Wang (2000)).
- We can derive $\phi(p) = \exp(-\lambda\Phi^{-1}(1-p) - \lambda^2/2)$ and identify that $\lim_{\lambda\to 0} F_{\phi_\lambda} \sim N(\mu, \sigma^2)$.



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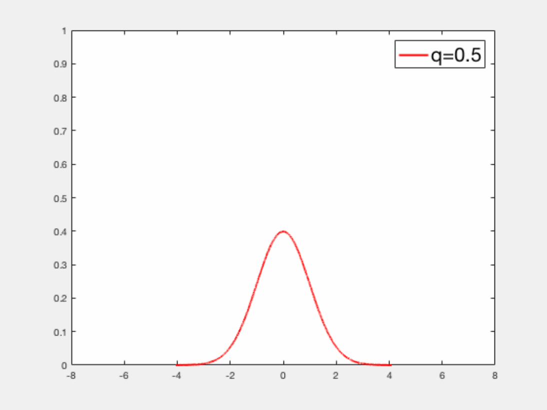

Tightness of WCSRM

- Consider an investor using the power risk spectrum

$$\phi(p) = kp^{k-1},\ k \geq 1,$$

- who is uncertain about which of the following processes will be realized in the coming two years (Lo (1987)):

$$S_t = S_0 e^{r_i(t)}\ (S_t : \text{the price at time } t),$$

$$\text{(Geometric Brownian Motion) } r_1(t) = \mu_1 t + \sigma_1 \sqrt{t} Z,$$

$$\text{(Merton's jump diffusion) } r_2(t) = \mu_2 t + \sigma_2 \sqrt{t} Z_b + \sum_{i=0}^{N(t)} Z_i,$$

where $Z\ (Z_b) \sim \mathcal{N}(0,1)$, $Z_i \sim \mathcal{N}(\beta, \delta^2)$, $N(t) \sim Poisson(\lambda t)$.

- We perform a grid search over all the above processes with $\mathbb{E}[r_1(t)^{(j)}] = \mathbb{E}[r_2(t)^{(j)}]$, $j = 1, 2$.

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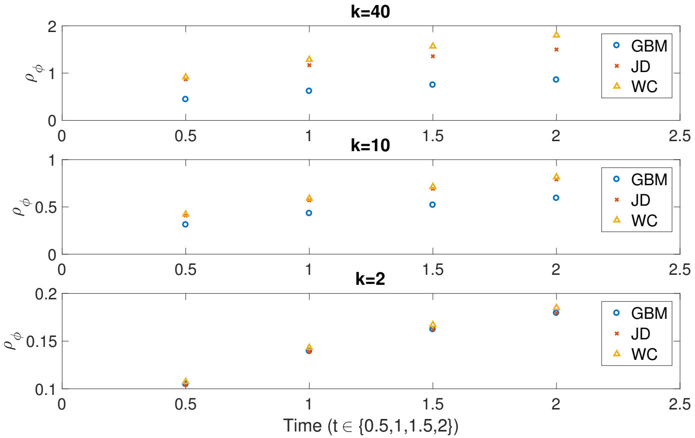

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DRO with Worst-Case Risk Measures

- Consider now again the mean-covariance DRO formulation which now employees certain coherent risk measure

$$\min_{x \in D} \sup_{F_\xi \in \mathscr{M}} \rho_{F_\xi}(-\xi^\top x),$$

$$G(\xi) := \begin{pmatrix} \xi \\ \xi\xi^\top \end{pmatrix}, \mathscr{K} := \begin{pmatrix} \mu \\ \Sigma + \mu\mu^\top \end{pmatrix}.$$

- Applying the same projection property in Popescu (2007), we have the following equivalent formulation

$$\min_{x \in D} \sup_{F_Z \sim (-\mu^\top x, x^\top \Sigma x)} \rho_{F_\xi}(Z)$$

$$\Rightarrow \min_{x \in D} -\mu^\top x + \sqrt{\sup_{\phi \in \Phi} ||\phi||_2^2 - 1} \sqrt{x^\top \Sigma x},$$

which is a second order-conic program if D is a polyhedron.

- Like Popescu (2007), the solutions here are mean-variance efficient but require no further computation to identify the tradeoff coefficient.

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Conclusion & Future Work

We demonstrate in the mean-covariance setting that

- DRO based on coherent risk measures can be simpler to solve than DRO based on utility functions while providing more plausible worst-case distributions;
- moment-based DRO has not to be overly conservative and worst-case distributions can be richly interpreted
- nonlinear DRO is not necessarily harder to solve than linear DRO

Extensions

- We can also identify the worst-case distributions in the case of higher-order moments but not in completed closed-form

Future Work

- Can similar insights be found in other DRO settings? e.g. Pflug et al. (2012) on Wasserstein metric.

Conclusion & Future Work

We demonstrate in the mean-covariance setting that

- DRO based on coherent risk measures can be simpler to solve than DRO based on utility functions while providing more plausible worst-case distributions;
- moment-based DRO has not to be overly conservative and worst-case distributions can be richly interpreted
- nonlinear DRO is not necessarily harder to solve than linear DRO

Extensions

- We can also identify the worst-case distributions in the case of higher-order moments but not in completed closed-form

Future Work

- Can similar insights be found in other DRO settings? e.g. Pflug et al. (2012) on Wasserstein metric.

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