Asymptotically hyperbolic 3-metric with Ricci flow foliation

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Constraint Equations with cosmological constant

A triple (M^3, g, k) where g is a Riemannian metric on M and k is a symmetric $(0, 2)$-tensor on M is considered as an initial data set if (M, g, k) satisfies the following equations:

$$\begin{aligned} R_g + (\mathrm{tr}_g k)^2 - ||k||_g^2 &= 2\Lambda + 16\pi\rho, \\ \nabla_j(k_{ij} - (\mathrm{tr}_g k)g_{ij}) &= 8\pi J. \end{aligned} \qquad \text{(CE)}$$

where ρ and J are the energy density and the momentum vector respectively.

Foliation method with prescribed scalar curvature

In 1993, R. Bartnik introduced a quasi-spherical foliation approach.
Consider a metric g on $\mathbb{R}^+ \times \mathbb{S}^2$ of the form

$$g = u^2 dr^2 + (\beta_1 dr + r d\theta)^2 + (\beta_2 dr + r \sin\theta d\phi)^2.$$

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By solving the PDE with prescribed functions β_i and scalar curvature R on $\mathbb{R}^+ \times \mathbb{S}^2$, one can get an AF 3-metric with nonnegative scalar curvature.

In 2002, Y. Shi and L. -F. Tam proved an interesting result about certain type of compact manifolds with boundary by using the foliation technique.

Theorem

Let (Ω^n, g), $3 \leq n \leq 7$ be a compact manifold with smooth boundary and with nonnegative scalar curvature. Suppose $\partial\Omega$ is connected and can be embedded in $\mathbb{R}^n$ as a strictly convex closed hypersurface. Then

$$\int_{\partial\Omega} H \, d\sigma \leq \int_{\partial\Omega} H_0 \, d\sigma.$$

Moreover, if equality holds then Ω is a domain in $\mathbb{R}^n$.

Idea of proof of Shi-Tam's result

1. Consider $\partial\Omega$ embedded in $\mathbb{R}^n$. One can foliate the unbounded region of $\mathbb{R}^n$ by the distance function r from $\partial\Omega$.

2. From Bartnik's idea, with prescribed zero scalar curvature, one can construct an AF metric g on that region, which is of the form

$$g = u^2\, dr^2 + g_r$$

3. Let Σ_r be level sets of r. Prove that

$$m_{BY}(r) = \int_{\Sigma_r} (H_0 - H)\, d\sigma$$

is decreasing.

4. Prove that $m_{BY}(r)$ converges to the ADM mass of g. Hence the result follows by the positive mass theorem.

Foliation by Ricci flow on a closed surface

C. -Y. Lin, 2014 : using foliation by a solution to the **Ricci flow** on a closed surface.

Let (Σ, g_0) be a closed Riemannian surface. For some technical reason, we will consider the modified Ricci flow:

$$\begin{cases} \frac{\partial}{\partial t} g = (r - R_{g(t)})g + 2\text{Hess}_g f =: 2M_{ij}, \\ g(1) = g_0. \end{cases}$$

Here, $r = \frac{1}{|\Sigma_t|} \int_\Sigma R_{g(t)} d\mu_{g(t)}$ and f is the Ricci potential, i.e., $\Delta f = R - r$.

Ricci flow on a closed surface

Theorem (R. Hamilton '86, B. Chow '91)

Let (Σ, g_0) be a closed Riemannian surface. Then there exists a unique global solution $g(t), t \in [0, \infty)$, to the modified Ricci flow. As $t \to \infty$, the metric $g(t)$ converges exponentially fast in any C^k-norm to a smooth metric g_∞ with constant curvature.

Assume that (Σ, g_0) is diffeomorphic to $\mathbb{S}^2$ and has area 4π. Then by the uniformization theorem, g_∞ is isometric to the standard metric on $\mathbb{S}^2$.

Let (Σ, g_0) be a closed surface diffeomorphic to $\mathbb{S}^2$ with area 4π. Consider a metric g on $[1, \infty) \times \Sigma$ of the form

$$\bar{g} = u^2 dr^2 + r^2 g(r)$$

where $\{g(r)\}$ is a family of metrics on Σ which solves the Ricci flow equation. Similarly, one can get a parabolic equation of a function u.

By solving the derived equation and investigating the behavior of u, it has been proved that the metric is AF (assuming some conditions).

AH extension using Ricci flow

Recall one expression for a hyperbolic metric:

$$g_{\mathbb{H}^3} = \frac{1}{1+r^2}dr^2 + r^2 g_{\mathbb{S}^2}.$$

Inspired by this and Lin's result, one might consider constructing AH 3-metrics on $[1,\infty) \times \mathbb{S}^2$ of the form

$$\bar{g} = \frac{u^2}{1+r^2}dr^2 + r^2 g(r)$$

by an analogous procedure.

Theorem (J–, 2018)

Let (Σ, g) be a closed surface diffeomorphic to $\mathbb{S}^2$ with area 4π and let N be the product manifold $[1, \infty) \times \Sigma$. Then for any $H \in C^\infty(\Sigma)$ with $H > 0$, there exists an asymptotically hyperbolic 3-metric on N of the form

$$\overline{g} = \frac{u^2}{1+r^2}dr^2 + r^2 g(r), \tag{1}$$

with the scalar curvature $\overline{R} \equiv -6$ where $u \in C^\infty(N)$ is positive everywhere, and $g(r)$ is the solution to Hamilton's modified Ricci flow. Here H is the mean curvature in direction ∂_r on $\{1\} \times \Sigma$.

Idea of Proof

From the Gauss equation for each slice $\{r\} \times \Sigma$, we have

$$\overline{R} = R_r + 2\overline{\mathrm{Ric}}\left(\frac{\sqrt{1+r^2}}{u}\partial_r, \frac{\sqrt{1+r^2}}{u}\partial_r\right) + ||h_r||^2 - H_r^2,$$

where R_r is the scalar curvature on $\{r\} \times \Sigma$ with the induced metric $r^2 g(r)$, h_r is the second fundamental form, and H_r is the mean curvature in direction ∂_r.

Computation

By direct computation, we obtain

$$h_{r,ij} = \frac{\sqrt{1+r^2}}{u}(rg_{ij} + r^2 M_{ij}),$$
$$H_r = \frac{2\sqrt{1+r^2}}{ru}, \quad ||h_r||^2 = \frac{2(1+r^2)}{r^2 u^2} + \frac{1+r^2}{u^2}|M|^2_{g(r)}.$$

$$\overline{\mathrm{Ric}}\left(\frac{\sqrt{1+r^2}}{u}\partial_r, \frac{\sqrt{1+r^2}}{u}\partial_r\right) = -\frac{1}{u}\Delta_{\overline{g}|_{\Sigma_r}} u + \frac{\sqrt{1+r^2}}{u}\frac{\partial H_r}{\partial r} - ||h_r||^2,$$

The semi-linear parabolic equation of u is obtained as the following:

$$r(1+r^2)\frac{\partial u}{\partial r} = \frac{u^2 \Delta_{g(r)} u}{2} - \frac{u^3}{4}(R_{g(r)} - r^2 \overline{R}) + u\left(\frac{1+3r^2}{2} + \frac{r^2(1+r^2)|M|^2_{g(r)}}{4}\right).$$

The local existence is automatically guaranteed from the parabolicity of the equation.

C^0 estimates : Let $w = u^{-2}$ then we get

$$\frac{\partial w}{\partial r} = \frac{1}{r(1+r^2)}\left[\frac{3}{2}u\nabla u\cdot\nabla w + \frac{1}{2w}\Delta w + \frac{1}{2}(R_{g(r)} - r^2\overline{R}) - w\left(1 + 3r^2 + \frac{r^2(1+r^2)|M|^2}{2}\right)\right].$$

Using the maximum principle, one can get the global existence.

To get the estimates for derivatives, let $v = u - 1$ and use a change of coordinate as $s = \log\left(\frac{r}{\sqrt{1+r^2}}\right) + 1$ then

$$
\begin{aligned}
\frac{\partial v}{\partial s} &= \frac{u^2}{2} g^{ij} \frac{\partial^2 v}{\partial x^i \partial x^j} - \frac{u^2}{2} \frac{\partial}{\partial x^i}\left(\sqrt{|g|} g^{ij}\right) \frac{\partial v}{\partial x^j} \\
&\quad - \frac{u^3}{4}(R_{g(r)} - r^2 \overline{R}) + u\left(\frac{1+3r^2}{2} + \frac{r^2(1+r^2)|M|^2}{4}\right) \\
&:= Lv - \frac{1}{4}(R_{g(r)} - r^2 \overline{R}) + \frac{1+3r^2}{2} + \frac{r^2(1+r^2)|M|^2}{4} \\
&= Lv + f.
\end{aligned}
$$

Now by the usual Schauder estimates, we can prove that the obtained metric is asymptotically hyperbolic.

The mass integrals of asymptotically hyperbolic manifolds

(Chruściel, Herzlich 2001)
Let V be a C^1 function on $[R,\infty)\times\mathbb{S}^2$. Set

$$\begin{aligned} H_\phi(V) = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty}\int_{S_r} & \left[V(\mathrm{div}_{\mathbb{H}^3} h - d\mathrm{tr}_{\mathbb{H}^3} h)\right. \\ & \left. - h(\nabla_{\mathbb{H}^3} V, \cdot) + (\mathrm{tr}_{\mathbb{H}^3} h) dV\right](\nu) d\sigma_{\mathbb{H}^3} \end{aligned}$$

where $h = g - g_{\mathbb{H}^3}$, ν is the $g_{\mathbb{H}^3}$-unit outward normal to S_r and $d\sigma_{\mathbb{H}^3}$ is the volume element on S_r of the metric induced from $g_{\mathbb{H}^3}$.

The mass integrals of asymptotically hyperbolic manifolds

The mass integrals are

$$p_0 = H_\phi(V_0) \quad \text{and} \quad p_i = H_\phi(V_i) \text{ for } i = 1, \dots, n,$$

where

$$V_0 = \sqrt{1 + |x|^2} \quad \text{and} \quad V_i = x^i \text{ for } i = 1, \dots, n.$$

Hawking mass

Definition

Let (M^3, g) be a 3-dimensional asymptotically hyperbolic manifold, and let $\Sigma \subset M^3$ be a closed 2-surface. Then the Hawking mass $m_H(\Sigma)$ of Σ is defined as

$$m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{16\pi} \int_\Sigma H^2 \, d\sigma + \frac{|\Sigma|}{4\pi}\right) \tag{2}$$

where $d\sigma$ is the induced volume form with respect to g.

Rigidity of the Hawking mass

Theorem (J-, 2018)

Let (Σ, g_1) be a surface diffeomorphic to $\mathbb{S}^2$ with positive mean curvature (not necessarily constant) and let $N = [1, \infty) \times \Sigma$ be an asymptotically hyperbolic extension obtained by Ricci flow foliation. Then $m_H(\Sigma_r)$ is increasing, where $\Sigma_r = \{r\} \times \Sigma$. Furthermore, if

$$p_0 = m_H(\Sigma) \tag{3}$$

then Σ is isometric to the standard unit sphere, and N is rotationally symmetric. If $m_H(\Sigma) = 0$ then N is isometric to a rotationally symmetric region in a hyperbolic space. If $m_H(\Sigma) = m > 0$ then N is isometric to a rotationally symmetric region in anti-de Sitter Schwarzschild space of mass m.

On the manifold we constructed, for each level surface Σ_r we have

$$
\begin{aligned}
m_H(\Sigma_r) &= \sqrt{\frac{|\Sigma_r|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_{\Sigma_r} H^2\, d\sigma_r + \frac{|\Sigma_r|}{4\pi}\right)\\
&= \sqrt{\frac{4\pi r^2}{16\pi}}\left(1 - \frac{1}{16\pi}\int_{\Sigma} \frac{4(1+r^2)}{r^2 u^2} r^2\, d\sigma + \frac{4\pi r^2}{4\pi}\right)\\
&= \frac{r(1+r^2)}{2}\left(1 - \frac{1}{4\pi}\int_{\Sigma} u^{-2} d\sigma\right)\\
&= \frac{1}{4\pi}\int_{\Sigma} \frac{r(1+r^2)}{2}(1-u^{-2}) d\sigma.
\end{aligned}
$$

Monotonicity of the Hawking mass

$$\begin{aligned}\frac{d}{dr}m_H(\Sigma_r) &= \frac{1}{4\pi}\int_\Sigma \frac{3r^2+1}{2}(1-u^{-2}) + \frac{r(1+r^2)}{2}2u^{-3}\frac{\partial u}{\partial r}d\sigma \\ &= \frac{1}{4\pi}\int_\Sigma \frac{3r^2+1}{2} + \frac{u^{-1}\Delta u}{2} - \frac{R}{4} + \frac{r^2\overline{R}}{4} + \frac{r^2(1+r^2)}{4u^2}|M|^2 d\sigma \\ &= \frac{1}{4\pi}\int_\Sigma \frac{(\overline{R}+6)r^2}{4} + \frac{u^{-1}\Delta u}{2} + \frac{r^2(1+r^2)}{4u^2}|M|^2 d\sigma \\ &= \frac{1}{8\pi}\int_\Sigma \frac{(\overline{R}+6)r^2}{2} + \frac{|\nabla u|^2}{u^2} + \frac{r^2(1+r^2)}{2u^2}|M|^2 d\sigma \geq 0\end{aligned}$$

Theorem (P. Miao, L.-F. Tam, N. Xie, 2016)

Let (M^3, g) be an asymptotically hyperbolic manifold. Let Σ_r be the hypersurface in M corresponding to the geodesic sphere of radius r in $\mathbb{H}^3$. Then the following estimate holds:

$$p_0 = \lim_{r\to\infty} m_H(\Sigma_r).$$

If we assume the Hawking mass of the initial surface $m_H(\Sigma)$ is equal to p_0, $m_H(\Sigma_r)$ must be constant by monotonicity. This means that $\overline{R} \equiv -6$, u is constant on each Σ_r, and $|M| \equiv 0$.

Thank you for your attention.