On the problem of stability of AdS

Piotr Bizoń

Jagiellonian University
Kraków

Banff, 18 May 2018

Anti-de Sitter spacetime in $d+1$ dimensions

- AdS$_{d+1}$ can be defined as the quadric

$$X_1^2 + \cdots + X_d^2 - U^2 - V^2 = -l^2$$

embedded in a flat $d+2$ dimensional space with metric

$$ds^2 = dX_1^2 + \cdots + dX_d^2 - dU^2 - dV^2$$

- For $X = r\omega, U = \sqrt{r^2+l^2}\sin(t/l), V = \sqrt{r^2+l^2}\cos(t/l)$ the induced metric on the quadric

$$g = -(1+r^2/l^2)dt^2 + \frac{dr^2}{1+r^2/l^2} + r^2 d\omega^2_{S^{d-1}}$$

solves the vacuum Einstein equations $G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 0$ with $\Lambda = -\dfrac{2}{d(d-1)l^2}$. The (maximal) symmetry group is $O(2, d-1)$.

- In the following by AdS we mean the universal covering space with the time coordinate t unrolled to $(-\infty, \infty)$.

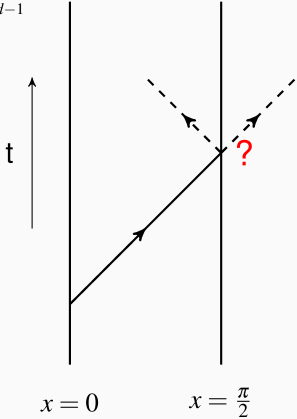
Causal structure

Using $x = \arctan(r/l) \in [0, \pi/2)$ we get

$$g = \frac{l^2}{\cos^2 x}\left(-dt^2 + dx^2 + \sin^2 x\, d\omega^2_{S^{d-1}}\right)$$

Spatial infinity $x = \pi/2$ is the timelike cylinder $\mathscr{I} = \mathbb{R} \times S^{d-1}$ with the conformal boundary metric $ds^2_{\mathscr{I}} = -dt^2 + d\Omega^2_{S^{d-1}}$

- Null geodesics get to infinity in finite time
- AdS is not globally hyperbolic
- Asymptotically AdS spacetimes by definition have the same conformal boundary as AdS



Brief history of AdS

- **AdS metric:** A. Friedmann, *On the possibility of a world with a constant negative curvature of space*, Zeitschrift für Physik 21, 326 (1924)
- *"de Sitter universe with K negative involves ideas of altogether too revolutionary a character for physics as it exists today."*
  J.L. Synge in Relativity: The General Theory (1960)
- **Linear stability** : P. Breitenlohner and D.Z. Freedman, *Stability of gauge extended supergravity*, Annals of Physics 14, 249 (1982)
- **Local well-posedness of the initial-boundary value problem** for 4-dim vacuum Einstein's equations with AdS asymptotics:
  H. Friedrich, *Einstein equations and conformal structure: existence of anti-de Sitter-type space-times,* J. Geom. Phys. 17, 125 (1995)
- **AdS/CFT duality**: J. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. 2, 231 (1998) (cited 13665 times)

Is AdS stable?

- By the positive energy theorem AdS space is the unique ground state among asymptotically AdS spacetimes (much as Minkowski space is the unique ground state among asymptotically flat spacetimes).
- Basic question for any equilibrium solution: **do small perturbations at $t = 0$ remain small for all future times?**
- Minkowski space is asymptotically stable (Christodoulou-Klainerman '93)
- Key difference between Minkowski and AdS: **the main mechanism of stability of Minkowski - dissipation of energy by dispersion - is absent in AdS** (for no-flux boundary conditions $\mathscr{I}$ acts as a mirror).
- Stability of AdS has not been explored until '11; notable exceptions: local well-posedness (Friedrich '95), boundedness of linearized perturbations (Ishibashi-Wald '04), rigidity (Anderson '06):

  > *One expects that g_{AdS} is in fact dynamically stable, with the behavior of the nonlinear exact solutions nearby to g_{AdS} well-modeled on the linearized behavior.*

AdS gravity coupled to a spherically symmetric scalar field

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta}, \quad \Lambda = -\frac{d(d-1)}{2l^2}$$

$$T_{\alpha\beta} = \partial_\alpha \phi \, \partial_\beta \phi - \frac{1}{2} \left((\partial \phi)^2 + m^2 \phi^2 \right) g_{\alpha\beta}$$

$$\Box_g \phi - m^2 \phi = 0$$

All fields are assumed to be spherically symmetric. For $y = \pi/2 - x \to 0$

$$\phi(t,x) \sim c_+(t) \, y^{\frac{d}{2}+\nu} + c_-(t) \, y^{\frac{d}{2}-\nu}, \quad \nu^2 = \frac{d^2}{4} + m^2 l^2$$

"Reflective" boundary conditions: Dirichlet ($c_- = 0$) or Robin ($c_+ + b c_- = 0$).

- For $\nu^2 \geq 1$ the initial-boundary value is locally well-posed only for the Dirichlet boundary conditions (Holzegel-Smulevici '11)
- For $\nu^2 = 1/4$ the system is conformally well-behaved at $\mathscr{I}$ and more general boundary conditions (both reflective and dissipative) are allowed (Holzegel-Warnick '13, Holzegel-Luk-Smulevici-Warnick '15).

- Convenient parametrization of asymptotically AdS spacetimes

$$ds^2 = \frac{l^2}{\cos^2 x}\left(-Ae^{-2\delta}dt^2 + A^{-1}dx^2 + \sin^2 x\, d\omega^2_{S^{d-1}}\right)$$

  where A and δ are functions of (t,x)

- Define mass function $\mu(t,x) = \dfrac{\sin^{d-2} x}{\cos^d x}(1-A)$ and auxiliary variables $\Phi = \phi'$ and $\Pi = A^{-1}e^{\delta}\dot{\phi}$ (where $' = \partial_x, \dot{} = \partial_t$)
- Field equations (using $8\pi G = d-1$) for $m = 0$

$$\dot{\Phi} = \left(Ae^{-\delta}\Pi\right)', \qquad \dot{\Pi} = \frac{1}{\tan^{d-1}x}\left(\tan^{d-1}x Ae^{-\delta}\Phi\right)'$$
$$\mu' = \sin x \cos x A\left(\Phi^2 + \Pi^2\right), \quad \delta' = -\sin x \cos x\left(\Phi^2 + \Pi^2\right)$$

- Dirichlet boundary conditions at $y = \pi/2 - x = 0$

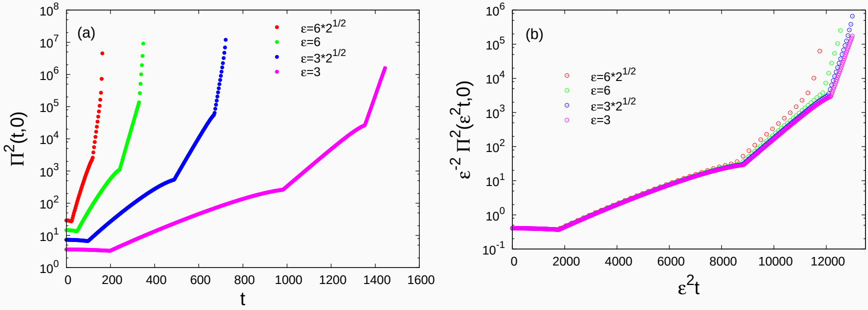
$$\phi \sim y^d, \quad \delta \sim y^{2d}, \quad 1 - A = y^d$$

- The total mass $M = \lim_{x\to\pi/2}\mu(t,x)$ is conserved
- Sample initial data: $\Phi(0,x) = 0, \Pi(0,x) = \varepsilon \exp\left(-\dfrac{\tan^2 x}{\sigma^2}\right)$

Conjecture (B-Rostworowski '11)

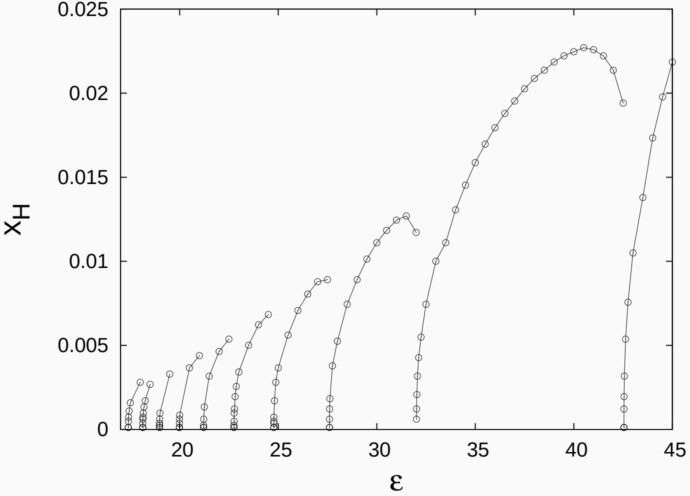
AdS_{d+1}, as the solution of the Einstein-massless-scalar field equations with negative cosmological constant in $d+1$ dimensions (for $d \geq 3$), is unstable under arbitrarily small generic perturbations.

Key numerical evidence:



Gaussian perturbations of size ε collapse in time $\mathscr{O}(\varepsilon^{-2})$.

Multi-critical behavior



Spectral decomposition

- Linear perturbations satisfy $\ddot{\phi} = -L\phi$, where $L = -\frac{1}{\tan^{d-1} x}\partial_x \left(\tan^{d-1} x\, \partial_x\right)$ is essentially self-adjoint on $L^2([0,\pi/2], \tan^d x\, dx)$
- Eigenvalues and eigenmodes of L

$$\omega_n^2 = (d+2n)^2, \quad e_n(x) = N_n \cos^d x\, P_n^{\left(\frac{d-2}{2},\frac{d}{2}\right)}(\cos 2x)$$

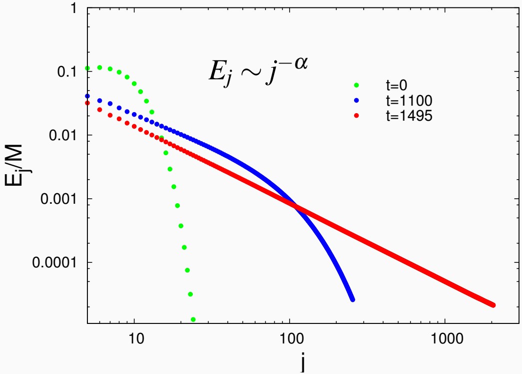
- The linearized perturbations are nondispersive
- Let us define projections $\Phi_n := (\sqrt{A}\,\Phi, e_n')$, $\Pi_n := (\sqrt{A}\,\Pi, e_n)$. Then

$$M = \int\limits_0^{\pi/2} \left(A\Phi^2 + A\Pi^2\right) \tan^2 x\, dx = \sum_{n=0}^{\infty} E_n(t)$$

where $E_n := \Pi_n^2 + \omega_n^{-2}\Phi_n^2$ is the energy of the n-th mode

Heuristic picture

- The linear spectrum is **fully resonant**. Nonlinear interactions between harmonics give rise to **transfer of energy from low to high frequencies**.
- The turbulent cascade leads to concentration of energy on finer and finer spatial scales so eventually a black hole is expected to form.



Evolution of the energy spectrum

Weakly turbulent instability of AdS_3

- Dimensionless measure of gravity's strength is GM/L^{d-2} so in $d=2$ the *total* mass matters (not its concentration)
- AdS-Schwarzschild family in $d=2$

$$g = -A\,dt^2 + A^{-1}dr^2 + r^2\,d\varphi^2\,, \qquad A = 1 - M + r^2/l^2$$

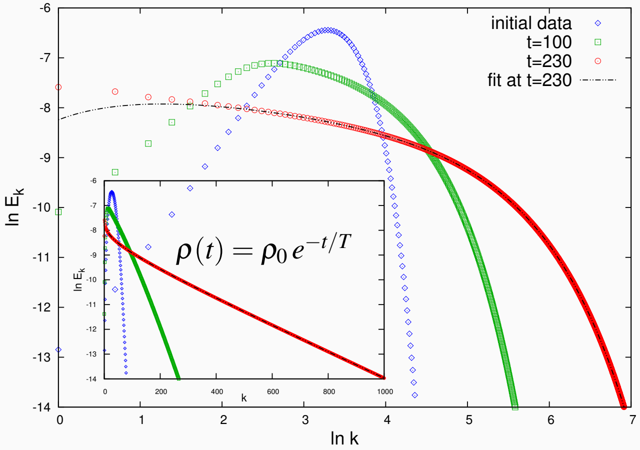
- Mass gap between AdS_3 ($M=0$) and the lightest black hole ($M=1$) Thus, **small perturbations of AdS_3 cannot form black holes**

Conjecture (B-Jałmużna 2013)

Small smooth perturbations of AdS_3 remain smooth for all times but their radius of analyticity shrinks to zero as $t \to \infty$.

- Evidence: analyticity strip method (Sulem-Sulem-Frisch 1984). Idea:
  - Extend the solution $\phi(t,x)$ to complex values of x and determine the imaginary part ρ of a complex singularity nearest to the real axis.
  - Tracing the time evolution of $\rho(t)$ one can predict or exclude blowup
  - The value of ρ is encoded in the asymptotic behavior of Fourier coefficients of $\phi(t,x)$ which decay as $\exp(-\rho k)$ for large k.

Energy spectrum in $2+1$ dimensions



$$E_k(t) = C(t)\, k^{-\alpha(t)} e^{-2\rho(t)k}$$

Similar weakly turbulent loss of regularity has been well known in fluid dynamics (example: incompressible Euler equation in 2d, Yudovich 1974)

Some follow-up studies and open questions

- Turbulent instability is absent for some initial data (**stability islands**). In particular, there exist **stable time-periodic solutions** bifurcating from the eigenmodes (Maliborski-Rostworowski '13)
- Similar phenomenology found for the vacuum Einstein equations in $4+1$ dimensions within the biaxial Bianchi IX ansatz (B-Rostworowski '14)
- Proof of instability of AdS for Einstein-null dust system (Moschidis '17)
- What happens outside spherical symmetry? It is not clear at all if the putative endstate of instability - Kerr-AdS black hole - is stable itself. Key issue: **stable trapping** of waves with large angular momentum ℓ:
  - quasinormal modes decay as $e^{-\Gamma_\ell t}$ where $\Gamma_\ell \sim e^{-c\ell}$ (Gannot 2011)
  - linear perturbations decay as $1/\log(t)$ for $t \to \infty$ (Holzegel-Smulevici 2013)
- Is extrapolation of numerical results to *arbitrarily* small perturbations justified?
- **Resonant approximation**: new approach proposed by Balasubramanian et al. and Craps-Evnin-Vanhoof '14.

Broader perspective: spatially confined nonlinear waves

Unbounded domain	Bounded domain
	
System settles down to equilibrium via dissipation of energy by dispersion	Waves keep interacting for all times, generating out-of-equilibrium dynamics

Understanding of long-time behavior of nonlinear waves in spatially confined systems is challenging. Key questions:

- How the energy injected into the system gets distributed over the degrees of freedom during the evolution?
- Can the energy flow to arbitrarily high frequencies?

Examples of spatially confined systems

- Nonlinear string

$$\phi_{tt} - \phi_{xx} + \phi^3 = 0, \qquad \phi(t,0) = \phi(t,\pi) = 0$$

- Cubic Klein-Gordon equation on $\mathbb{R} \times S^3$

$$\Box_g \phi - m^2 \phi - \phi^3 = 0, \qquad g = -dt^2 + d\omega^2_{S^3}$$

- Einstein-massless-scalar system with negative cosmological constant

$$R_{\mu\nu} + \frac{d}{l^2} g_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi$$

- Gross-Pitaevskii equation with isotropic harmonic potential

$$i\partial_t \psi = -\Delta \psi + |x|^2 \psi + g|\psi|^2 \psi$$

General strategy

- For a spatially confined system, the associated linearized system has a purely discrete spectrum of frequencies
- Expanding solutions in the basis of linear eigenstates one transforms the original PDE into an infinite-dimensional dynamical system with discrete degrees of freedom ('modes').
- The nonlinearity generates new frequencies that may lead to resonances between the modes. The resonances dominate the transfer of energy.
- Dropping all nonresonant terms from the Hamiltonian one obtains a simplified infinite-dimensional dynamical system, called the resonant system, which accurately approximates the dynamics of small amplitude solutions of the original PDE on long time scales
- Strategy: try to understand the dynamics of the resonant system and then export this knowledge to the original PDE.

Example

- Background geometry: the Einstein cylinder $\mathscr{M} = \mathbb{R} \times \mathbb{S}^3$ with metric

$$g = -dt^2 + dx^2 + \sin^2 x \, d\omega^2, \qquad (t, x, \omega) \in \mathbb{R} \times [0, \pi] \times \mathbb{S}^2$$

  This spacetime has constant scalar curvature $R(g) = 6$.

- On $\mathscr{M}$ we consider a real scalar field ϕ satisfying

$$\left(\Box_g - \frac{1}{6} R(g) \right) \phi - \phi^3 = \Box_g \phi - \phi - \phi^3 = 0 \,.$$

- We assume that $\phi = \phi(t, x)$. Then, $v(t, x) = \sin(x) \phi(t, x)$ satisfies

$$v_{tt} - v_{xx} + \frac{v^3}{\sin^2 x} = 0$$

  with Dirichlet boundary conditions $v(t, 0) = v(t, \pi) = 0$.

- Linear eigenstates: $e_n(x) = \sqrt{\frac{2}{\pi}} \sin(\omega_n x)$ with $\omega_n = n + 1$ $(n = 0, 1, 2, \dots)$

Time averaging

- Expanding $v(t,x) = \sum_{n=0}^{\infty} c_n(t) e_n(x)$ we get

$$\frac{d^2 c_n}{dt^2} + \omega_n^2 c_n = -\sum_{jkl} S_{njkl}\, c_j c_k c_l, \quad S_{jkln} = \int_0^{\pi} \frac{dx}{\sin^2 x} e_n(x) e_j(x) e_k(x) e_l(x)$$

- Using variation of constants

$$c_n = \beta_n e^{i\omega_n t} + \bar{\beta}_n e^{-i\omega_n t}, \qquad \frac{dc_n}{dt} = i\omega_n \left(\beta_n e^{i\omega_n t} - \bar{\beta}_n e^{-i\omega_n t}\right)$$

  we factor out fast oscillations

$$2i\omega_n \frac{d\beta_n}{dt} = -\sum_{jkl} S_{njkl}\, c_j c_k c_l\, e^{-i\omega_n t},$$

- Each term in the sum has a factor $e^{-i\Omega t}$, where $\Omega = \omega_n \pm \omega_j \pm \omega_k \pm \omega_l$. The terms with $\Omega = 0$ correspond to resonant interactions.
- Let $\tau = \varepsilon^2 t$ and $\beta_n(t) = \varepsilon \alpha_n(\tau)$. For $\varepsilon \to 0$ the non-resonant terms $\propto e^{-i\Omega\tau/\varepsilon^2}$ are rapidly oscillating and therefore negligible.

Resonant system

- Keeping only the resonant terms one obtains (B-Craps-Evnin-Hunik-Luyten-Maliborski '16)

$$i\,\omega_n \frac{d\alpha_n}{d\tau} = \sum_{j=0}^{\infty}\sum_{k=0}^{n+j} S_{njk,n+j-k}\,\bar{\alpha}_j \alpha_k \alpha_{n+j-k}\,,$$

  where $S_{njk,n+j-k} = \min\{n,j,k,n+j-k\}+1$

- This system (called the conformal cubic flow) provides an approximation to the conformal cubic wave equation on the timescale $\sim \varepsilon^{-2}$
- Since AdS_4 is conformal to $\mathbb{R}\times S^3_+$, this resonant system also approximates the conformal wave equation on AdS_4
- The resonant system for radial scalar perturbations of AdS_{d+1} has the same form (Balasubramanian et al. '14, Craps-Evnin-Vanhoof '14) but the interaction coefficients S_{njkl} are very complicated.

Solutions of cubic resonant systems

$$i\,\omega_n \frac{d\alpha_n}{d\tau} = \sum_{j=0}^{\infty}\sum_{k=0}^{n+j} S_{njk,n+j-k}\,\bar{\alpha}_j \alpha_k \alpha_{n+j-k}\,,$$

- Such systems are invariant under scaling

  $$\alpha_n(\tau) \to \varepsilon \alpha_n(\varepsilon^2 \tau)$$

  thus they provide access to the regime of arbitrarily small perturbations
- Thanks to enhanced symmetries, the resonant systems are often amenable to rigorous analysis (sometimes explicit solutions can be found)
- Key question: can energy be transferred to arbitrarily high modes?
- Can Sobolev norms $\|\alpha(\tau)\|_{h^s} := \sum_{n=0}^{\infty}(n+1)^{2s}|\alpha_n(\tau)|^2$ with $s > 1$ become unbounded in finite time (strong turbulence) or infinite time (weak turbulence)?

Oscillatory blowup in the AdS resonant system

- Asymptotics for large n

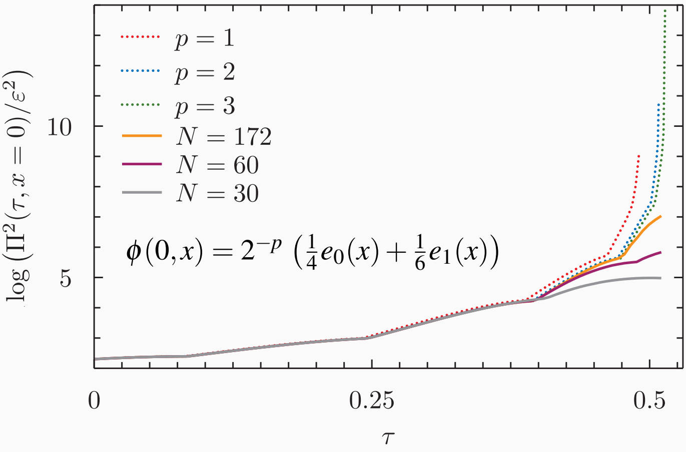
$$|\alpha_n(\tau)| \sim n^{-\beta(\tau)} e^{-\rho(\tau) n}$$

where ρ is the "analyticity radius". If $\lim_{\tau \to \tau_*} \rho(\tau) = 0$ for some $\tau_* < \infty$ then the solution becomes singular

- There is analytic-numerical evidence (B-Maliborski-Rostworowski '15) that for typical initial data $\rho(\tau)$ hits zero in finite time τ_* and

$$\frac{d}{d\tau} \arg(\alpha_n) \sim \ln(\tau_* - \tau) \quad \text{for } \tau \nearrow \tau_*$$

- This indicates that the corresponding solutions of the full Einstein-scalar system collapse on the timescale $\mathcal{O}(\varepsilon^{-2})$ and hints at a possible route to proving the AdS instability conjecture.



Instability on timescale $1/\varepsilon^2$ is captured by the resonant approximation!

Conclusions

- Dynamics of asymptotically AdS spacetimes is an interesting meeting point of fundamental problems in general relativity, PDE theory, and theory of turbulence. Understanding of these connections is at its infancy.
- There is good evidence that AdS spacetime is unstable against arbitrarily small perturbations (for no-flux boundary conditions at $\mathscr{I}$).
- Understanding of the out-of-equilibrium dynamics of small solutions is mathematically challenging even for the simplest nonlinear wave equations on compact manifolds, let alone Einstein's equations.