VAUGHT'S CONJECTURE FOR MONOMORPHIC THEORIES

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THE VAUGHT CONJECTURE

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- L-sentences: $\varphi \in \mathrm{Sent}_L$ and theories $\mathcal{T} \subset \mathrm{Sent}_L$
- The binary language $L_b = \langle R \rangle$, $\mathrm{ar}(R) = 2$

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$\mathrm{Th}(\mathbb{Y}) := \{\varphi \in \mathrm{Sent}_L : \mathbb{Y} \models \varphi\}$ is the **complete theory of** $\mathbb{Y}$

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- Morley (1970): $I(\mathcal{T},\omega) > \omega_1 \Rightarrow I(\mathcal{T},\omega) = \mathfrak{c}$

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- Several generalizations

MONOMORPHIC STRUCTURES

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- the **betweeness relation**, $D_{\varphi_b} \subset X^3$, defined by the L_b-formula

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- the **separation relation**, $D_{\varphi_s} \subset X^4$, defined by the formula

$$\begin{aligned} \varphi_s \quad := \quad & (v_0 < v_1 < v_2 < v_3) \vee (v_0 < v_3 < v_2 < v_1) \vee \\ & (v_1 < v_0 < v_3 < v_2) \vee (v_1 < v_2 < v_3 < v_0) \vee \\ & (v_2 < v_1 < v_0 < v_3) \vee (v_2 < v_3 < v_0 < v_1) \vee \\ & (v_3 < v_0 < v_1 < v_2) \vee (v_3 < v_2 < v_1 < v_0). \end{aligned}$$

  saying: v_0, v_1, v_2 and v_3 are different and the pair $\{v_0, v_2\}$ separates the pair $\{v_1, v_3\}$.

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Such structures are called **constant** by Fraïssé.

MONOMORPHIC THEORIES

Establishing monomorphic theories

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Proposition

If L is a relational language (of any size) and $\mathcal{T}$ a complete L-theory with infinite models, then the following conditions are equivalent:

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Proof. (c) $\Rightarrow$ (b) is trivial.
(b) $\Rightarrow$ (a) If $\mathbb{Y} \models \mathcal{T}$ is a monomorphic structure, there is a Π_1 theory $\mathcal{T}_{\mathrm{Age}(\mathbb{Y})} \subset \mathrm{Th}(\mathbb{Y}) = \mathcal{T}$ such that each model $\mathbb{Z}$ of $\mathcal{T}_{\mathrm{Age}(\mathbb{Y})}$ (and, in particular of $\mathcal{T}$) is monomorphic and $\mathrm{Age}(\mathbb{Z}) = \mathrm{Age}(\mathbb{Y})$.

Proposition: (a) $\Rightarrow$ (c)

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Claim

If $\mathcal{T}$ is a complete monomorphic L-theory with infinite models and $|I| > \omega$, then $\mathcal{T}$ has a countable model and there are
- a countable language $L_J \subset L$ and
- a complete monomorphic L_J-theory $\mathcal{T}_J$ such that

$$\left|\mathrm{Mod}_L^{\mathcal{T}}(\omega)/\cong\right| = \left|\mathrm{Mod}_{L_J}^{\mathcal{T}_J}(\omega)/\cong\right|. \tag{1}$$

Proof. Let $\mathbb{Y} = \langle Y, \langle R_i^{\mathbb{Y}} : i \in I\rangle\rangle \in \mathrm{Mod}_L^{\mathcal{T}}$ and let $\langle Y, <\rangle$ chain $\mathbb{Y}$.
$|\mathrm{Form}_{L_b}| = \omega$ so there is a partition $I = \bigcup_{j\in J} I_j$, where $|J| \le \omega$,
such that, picking $i_j \in I_j$, we have $R_i^{\mathbb{Y}} = R_{i_j}^{\mathbb{Y}}$, for all $i \in I_j$. So

$$\mathcal{T}_\eta := \bigcup_{j\in J} \left\{\forall \bar{v}\ (R_i(\bar{v}) \Leftrightarrow R_{i_j}(\bar{v})) : i \in I_j\right\} \subset \mathrm{Th}_L(\mathbb{Y}) = \mathcal{T}$$

Let $L_J := \langle R_{i_j} : j \in J\rangle$. To each $\varphi \in \mathrm{Form}_L$, replacing R_i by R_{i_j}, we adjoin $\varphi_J \in \mathrm{Form}_{L_J}$ and by induction prove that

$$\forall \mathbb{Z} \in \mathrm{Mod}_L^{\mathcal{T}_\eta}\ \ \forall \varphi(\bar{v}) \in \mathrm{Form}_L\, \forall \bar{z} \in Z\ \left(\mathbb{Z} \models \varphi[\bar{z}] \Leftrightarrow \mathbb{Z}|L_J \models \varphi_J[\bar{z}]\right). \tag{2}$$

VAUGHT'S CONJECTURE FOR MONOMORPHIC THEORIES

The main result

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Theorem

If $\mathcal{T}$ is a complete monomorphic theory having infinite models, then

$$I(\mathcal{T}, \omega) \in \{1, \mathfrak{c}\}.$$

In addition, $I(\mathcal{T}, \omega) = 1$ iff some countable model of $\mathcal{T}$ is simply definable by an ω-categorical linear order on its domain.

PROOF OF VAUGHT'S CONJECTURE

Part I: Preliminaries

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By Claim 1, w.l.o.g. we suppose $|L| \leq \omega$; thus $\mathrm{Mod}_L^{\mathcal{T}}(\omega) \neq \emptyset$.

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For $\mathbb{Y} \in \mathrm{Mod}_L^{\mathcal{T}}(\omega)$ let

$$\mathcal{L}_{\mathbb{Y}} := \{\langle \omega, \lhd \rangle : \lhd \in LO_\omega \text{ and } \langle \omega, \lhd \rangle \text{ chains } \mathbb{Y}\}$$

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and there are quantifier free L_b-formulas $\varphi_i(v_0, \ldots, v_{n_i-1})$, $i \in I$, such that

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Let

$$\Phi : \mathrm{Mod}_{L_b}(\omega) \to \mathrm{Mod}_L(\omega)$$

be the mapping defined by

$$\Phi(\mathbb{X}) = \mathbb{Y}_{\mathbb{X}}, \text{ for each } \mathbb{X} \in \mathrm{Mod}_{L_b}(\omega).$$

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Proof. (a) If $f \in \mathrm{Iso}(\mathbb{X}_1, \mathbb{X}_2)$, then since f preserves all formulas in both directions, for each $i \in I$ and $\bar{x} \in \omega^{n_i}$ we have: $\bar{x} \in R_i^{\mathbb{Y}_{\mathbb{X}_1}}$ iff $\mathbb{X}_1 \models \varphi_i[\bar{x}]$ iff $\mathbb{X}_2 \models \varphi_i[f\bar{x}]$ iff $f\bar{x} \in R_i^{\mathbb{Y}_{\mathbb{X}_2}}$. Thus $f \in \mathrm{Iso}(\mathbb{Y}_{\mathbb{X}_1}, \mathbb{Y}_{\mathbb{X}_2})$.
(b) For $\varphi(\bar{v}) \in \mathrm{Form}_L$ let $\varphi_b(\bar{v}) \in \mathrm{Form}_{L_b}$ be obtained from φ by replacing of R_i by φ_i. An easy induction shows that

$$\forall \mathbb{X} \in \mathrm{Mod}_{L_b}(\omega) \;\; \forall \varphi(\bar{v}) \in \mathrm{Form}_L \;\; \forall \bar{x} \in \omega^n \left(\mathbb{Y}_{\mathbb{X}} \models \varphi[\bar{x}] \Leftrightarrow \mathbb{X} \models \varphi_b[\bar{x}] \right), \qquad (6)$$

which implies: $\mathbb{Y}_{\mathbb{X}} \models \varphi$ iff $\mathbb{X} \models \varphi_b$, for all $\varphi \in \mathrm{Sent}_L$
(c) If $\mathbb{X} \in \mathrm{Mod}_{L_b}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)$, then $\mathbb{X} \equiv \mathbb{X}_0$ and, by (b), $\Phi(\mathbb{X}) = \mathbb{Y}_{\mathbb{X}} \equiv \mathbb{Y}_{\mathbb{X}_0} = \mathbb{Y}_0 \models \mathcal{T}$. $\square$

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$$\Psi([\mathbb{X}]) = [\mathbb{Y}_{\mathbb{X}}], \text{ for all } [\mathbb{X}] \in \mathrm{Mod}_{L_b}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)/\cong,$$

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Proof. If $\mathbb{X}_1, \mathbb{X}_2 \in \mathrm{Mod}_{L_b}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)$ and $\mathbb{X}_1 \cong \mathbb{X}_2$, then by the previous Claim $\mathbb{Y}_{\mathbb{X}_1} \cong \mathbb{Y}_{\mathbb{X}_2}$, that is $[\mathbb{Y}_{\mathbb{X}_1}] = [\mathbb{Y}_{\mathbb{X}_2}]$. $\square$

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Proof. Let $f \in \mathrm{Iso}(\mathbb{Z}, \mathbb{Y})$ and $\tau \in \mathrm{otp}[\mathcal{L}_{\mathbb{Y}}]$.
Let $\mathbb{X} = \langle Y, < \rangle \in \mathcal{L}_{\mathbb{Y}}$, where $\mathrm{otp}(\mathbb{X}) = \tau$.
Then $\mathbb{X}_1 := \langle Z, f^{-1}[<] \rangle \cong_f \mathbb{X}$; thus, $\mathrm{otp}(\mathbb{X}_1) = \tau$.
For $i \in I$ and $\bar{z} \in Z^{n_i}$ we have

$$\bar{z} \in R_i^{\mathbb{Z}} \text{ iff } f\bar{z} \in R_i^{\mathbb{Y}} \text{ iff } \mathbb{X} \models \varphi_i[f\bar{z}] \text{ iff } \mathbb{X}_1 \models \varphi_i[\bar{z}],$$

which gives $\mathbb{X}_1 \in \mathcal{L}_{\mathbb{Z}}$. So, $\tau = \mathrm{otp}(\mathbb{X}_1) \in \mathrm{otp}[\mathcal{L}_{\mathbb{Z}}]$. $\square$

Size of the fibers of Ψ

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Claim

For each linear order $\mathbb{X} \in \mathrm{Mod}_{L_b}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)$ we have

$$\left|\Psi^{-1}\left[\{[\mathbb{Y}_{\mathbb{X}}]\}\right]\right| \leq \left|\mathrm{otp}[\mathcal{L}_{\mathbb{Y}_{\mathbb{X}}}] \cap \mathrm{otp}[\mathrm{Mod}_{L_b}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)]\right|. \qquad (*)$$

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Proof. We show that $\Lambda([\mathbb{Z}]) = \mathrm{otp}(\mathbb{Z})$ defines an injection

$$\Lambda \ : \ \Psi^{-1}[\{[\mathbb{Y}_{\mathbb{X}}]\}] \xrightarrow{1-1} \mathrm{otp}[\mathcal{L}_{\mathbb{Y}_{\mathbb{X}}}] \cap \mathrm{otp}[\mathrm{Mod}_{L_b}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)].$$

For $[\mathbb{Z}] \in \Psi^{-1}[\{[\mathbb{Y}_{\mathbb{X}}]\}]$ we have $[\mathbb{Y}_{\mathbb{Z}}] = \Psi([\mathbb{Z}]) = [\mathbb{Y}_{\mathbb{X}}]$,
that is, $\mathbb{Y}_{\mathbb{Z}} \cong \mathbb{Y}_{\mathbb{X}}$
and, by Fact, $\mathrm{otp}(\mathbb{Z}) \in \mathrm{otp}[\mathcal{L}_{\mathbb{Y}_{\mathbb{Z}}}] = \mathrm{otp}[\mathcal{L}_{\mathbb{Y}_{\mathbb{X}}}]$.
Since $\mathbb{Z} \in \mathrm{Mod}_{L_b}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)$ we have $\mathrm{otp}(\mathbb{Z}) \in \mathrm{otp}[\mathrm{Mod}_{L_b}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)]$.
Λ is an injection: if $[\mathbb{Z}] \neq [\mathbb{Z}']$, then $\mathbb{Z} \ncong \mathbb{Z}'$, and, hence, $\mathrm{otp}(\mathbb{Z}) \neq \mathrm{otp}(\mathbb{Z}')$. $\square$

PROOF OF VAUGHT'S CONJECTURE

Part II: Proof by discussion

(Cases A,B and Subcases B1,B2)

Case A: Some $\mathbb{Y} \in \mathrm{Mod}_L^{\mathcal{T}}(\omega)$ is chained by an ω-categorical linear order

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Claim

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So, $|\,\mathrm{Mod}_L^{\mathcal{T}}(\omega)/\cong\,| = 1$ and we are done.

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Proof. By the theorem of Engeler, Ryll-Nardzewski and Svenonius,
the group $\mathrm{Aut}(\mathbb{X})$ is oligomorphic;
that is, for each $n \in \mathbb{N}$ we have $|\omega^n/\sim_{\mathbb{X},n}| < \omega$,
where $\bar{x} \sim_{\mathbb{X},n} \bar{y}$ iff $f\bar{x} = \bar{y}$, for some $f \in \mathrm{Aut}(\mathbb{X})$.
Since $\mathbb{Y}$ is definable in $\mathbb{X}$ we have $\mathrm{Aut}(\mathbb{X}) \subset \mathrm{Aut}(\mathbb{Y})$,
which implies that for $n \in \mathbb{N}$ and each $\bar{x}, \bar{y} \in \omega^n$ we have
$\bar{x} \sim_{\mathbb{X},n} \bar{y} \Rightarrow \bar{x} \sim_{\mathbb{Y},n} \bar{y}$.
Thus $|\omega^n/\sim_{\mathbb{Y},n}| \leq |\omega^n/\sim_{\mathbb{X},n}| < \omega$, for all $n \in \mathbb{N}$,
and, since $|L| \leq \omega$, by the same theorem, $\mathbb{Y}$ is ω-categorical. □

Case B: The set $\bigcup_{\mathbb{Y}\in \mathrm{Mod}_L^{\mathcal{T}}(\omega)} \mathcal{L}_{\mathbb{Y}}$ does not contain ω-categorical linear orders

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Then, by Rubin's theorem

$$\forall \mathbb{Y} \in \mathrm{Mod}_L^{\mathcal{T}}(\omega) \;\; \forall \mathbb{X} \in \mathcal{L}_{\mathbb{Y}} \;\; \left| \mathrm{Mod}_{L_b}^{\mathcal{T}_{\mathbb{X}}}(\omega)/\cong \right| = \mathfrak{c}.$$

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Clearly, there is no constant $\mathbb{Y} \in \mathrm{Mod}_L^{\mathcal{T}}(\omega)$, that is

$$\forall \mathbb{Y} \in \mathrm{Mod}_L^{\mathcal{T}}(\omega) \;\; \mathcal{L}_{\mathbb{Y}} \neq LO_\omega.$$

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We prove that

$$|\,\mathrm{Mod}^{\mathcal{T}}_{L}(\omega)/\cong\,| = \mathfrak{c},$$

distinguishing subcases B1 and B2.

Subcase B1: For some $\mathbb{Y}_0 \in \mathrm{Mod}_L^{\mathcal{T}}(\omega)$ there is a l.o. $\mathbb{X}_0 \in \mathcal{L}_{\mathbb{Y}_0}$ with at least one end-point

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and recall the general discussion from Part I of the proof.

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That follows from the bound $(*)$ for the size of the fibers of Ψ and the following claim

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$$\left|\mathrm{otp}[\mathcal{L}_{\mathbb{Y}_{\mathbb{X}}}] \cap \mathrm{otp}[\mathrm{Mod}_{L_b}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)]\right| \leq \omega, \text{ for all } \mathbb{X} \in \mathrm{Mod}_{L_b}^{\mathcal{T}_{\mathbb{X}_0}}(\omega).$$

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In the proof of the Claim we will use the following

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(I) $\mathcal{L}_{\mathbb{Y}} = LO_Y$, that is, each linear order $\lhd$ on Y chains $\mathbb{Y}$,

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(III) There are finite subsets K and H of Y such that $\mathbb{X} = \mathbb{K} + \mathbb{M} + \mathbb{H}$ and
$$\mathcal{L}_{\mathbb{Y}} = \bigcup_{\substack{\lhd_K \in LO_K \\ \lhd_H \in LO_H}} \left\{ \langle K, \lhd_K \rangle + \mathbb{M} + \langle H, \lhd_H \rangle, \langle H, \lhd_H \rangle^* + \mathbb{M}^* + \langle K, \lhd_K \rangle^* \right\}.$$ [a]

---

[a] The statement follows from Theorem 9 of [3], which is a modification of similar results obtained independently by Frasnay in [2] and by Hodges, Lachlan and Shelah in [4].

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Since we are in Case B, (I) is impossible.

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If $\mathcal{L}_{\mathbb{Y}_{\mathbb{X}}}$ satisfies (III), then $\mathrm{otp}[\mathcal{L}_{\mathbb{Y}_{\mathbb{X}}}] = \{\tau, \tau^*\}$ and we are done.

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If $\mathcal{L}_{\mathbb{Y}_{\mathbb{X}}}$ satisfies (III), then $\mathrm{otp}[\mathcal{L}_{\mathbb{Y}_{\mathbb{X}}}] = \{\tau, \tau^*\}$ and we are done.
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and, hence, $\mathrm{otp}(\mathbb{F}+\mathbb{I}), \mathrm{otp}(\mathbb{I}^*+\mathbb{F}^*) \notin \mathrm{otp}[\mathrm{Mod}_{L_b}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)]$.
Thus, $\mathrm{otp}[\mathcal{L}_{\mathbb{Y}_\mathbb{X}}] \cap \mathrm{otp}[\mathrm{Mod}_{L_b}^{\mathcal{T}_{\mathbb{X}_0}}(\omega)] \subset \Theta$, where

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Since $|\Theta| = \omega$, the claim is proved. □

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Now, as above, we obtain $|\,\mathrm{Mod}_L^{\mathcal{T}}(\omega)/\cong| = \mathfrak{c}$. □

R. Fraïssé, Theory of relations, Revised edition, With an appendix by Norbert Sauer, Studies in Logic and the Foundations of Mathematics, 145. North-Holland, Amsterdam, (2000)

C. Frasnay, Quelques problémes combinatoires concernant les ordres totaux et les relations monomorphes, Ann. Inst. Fourier (Grenoble) 15,2 (1965) 415–524.

P. C. Gibson, M. Pouzet, R. E. Woodrow, Relational structures having finitely many full-cardinality restrictions, Discrete Math. 291,1-3 (2005) 115134.

W. Hodges, A. H. Lachlan, S. Shelah, Possible orderings of an indiscernible sequence, Bull. London Math. Soc. 9,2 (1977) 212–215.

L. L. Mayer, Vaught's conjecture for o-minimal theories, J. Symbolic Logic 53, 1 (1988) 146–159.

M. D. Morley, The number of countable models, J. Symbolic Logic 35 (1970) 14–18.

M. Pouzet, Application d'une propriété combinatoire des parties d'un ensemble aux groupes et aux relations, Math. Z. 150,2 (1976) 117–134.

M. Pouzet, Application de la notion de relation presque-enchaînable au dénombrement des restrictions finies d'une relation, Z. Math. Logik Grundlag. Math., 27,4 (1981) 289–332.

M. Rubin, Theories of linear order, Israel J. Math. 17 (1974) 392–443.

S. Shelah, End extensions and numbers of countable models, J. Symbolic Logic 43, 3 (1978) 550–562.

S. Shelah, L. Harrington, M. Makkai, A proof of Vaught's conjecture for ω-stable theories, Israel J. Math. 49, 1-3 (1984) 259–280.

R. L. Vaught, Denumerable models of complete theories, 1961 Infinitistic Methods (Proc. Sympos. Foundations of Math., Warsaw, 1959) pp. 303321 Pergamon, Oxford; Panstwowe Wydawnictwo Naukowe, Warsaw.