A critical equation with Hardy potential

(jointly with N. Ghoussoub, A. Pistoia and G. Vaira)

Pierpaolo Esposito,
Department of Mathematics and Physics,
University of Roma Tre

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$$S_{\gamma}(\mathbb{R}^{N})\left(\int_{\mathbb{R}^{N}}|U|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}}\leq \int_{\mathbb{R}^{N}}|\nabla U|^{2}-\gamma\int_{\mathbb{R}^{N}}\frac{U^{2}}{|x|^{2}}\quad\forall\;U\in\mathcal{D}^{1,2}(\mathbb{R}^{N})$$

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and have the form

$$U_{\mu}(x) = \frac{\alpha_{N} \mu^{\Gamma}}{|x|^{\beta^{-}} (\mu^{\frac{4\Gamma}{N-2}} + |x|^{\frac{4\Gamma}{N-2}})^{\frac{N-2}{2}}}, \quad \mu > 0$$

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- F. Catrina, Z.Q. Wang, CPAM 53 (2000)
- S. Terracini, Adv. Differential Equations 2 (1996)



On Ω bdd domain with $0 \in \Omega$ set

$$S_{\gamma}(\Omega)=\inf\{\int_{\Omega}[|\nabla u|^2-\gamma\frac{u^2}{|x|^2}]:\ u\in H^1_0(\Omega)\ \mathrm{s.t.}\ \int_{\Omega}|u|^{\frac{2N}{N-2}}=1\}$$

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with E-L equation

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Remark: no soln's in general for $\lambda \leq 0$ and no positive soln's for

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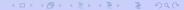
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- N. Ghoussoub, F. Robert, Calc. Var., to appear
- E. Jannelli, JDE 156 (1999)
- D. Ruiz, M. Willem, JDE 190 (2003)



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See also the survey

• N. Ghoussoub, F. Robert, Bull. Math. Sci. 6 (2016)



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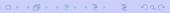
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- D. Cao, P. Han, JDE 205 (2004)
- D. Cao, S. Peng, JDE 193 (2003)
- D. Cao, S. Yan, Calc. Var. 38 (2010)
- Z. Chen, W. Zou, JDE 252 (2012)
- A. Ferrero, F. Gazzola, JDE 177 (2001)



For $\gamma = 0$

- H. Brezis, L. Nirenberg, CPAM 36 (1983) (no ground states in Ω and no positive soln's in B when N=3)
- Adimurthi, S.L. Yadava, Nonlinear Anal. 14 (1990) & F.V. Atkinson, H. Brezis, L. Peletier, JDE 85 (1990) (no radial sign-changing soln's in B when N = 3, 4, 5, 6

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Q1: What about positive soln's for $\gamma < 0$?



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- Q1: What about positive soln's for $\gamma < 0$?
- <u>Q2</u>: What about sign-changing soln's for $\gamma < 0$ or $\gamma \ge \frac{(N-2)^2}{4} 4$?

Attack existence issues by a perturbative approach for λ small:

Theorem 1

i) $\gamma \leq \frac{(N-2)^2}{4} - 1 \Rightarrow$ positive solution u_{λ} developing a bubble at 0 ii) $\gamma < \frac{(N-2)^2}{4} - 4 \Rightarrow$ sign-changing solution u_{λ} shaped as a tower

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By a fine asymptotic analysis:

Theorem 2

 $\gamma > \frac{(N-2)^2}{4} - 4 \Rightarrow$ no radial sign-chang. soln's in B for $\lambda > 0$ small

A more general result

Set $\sigma_j = \frac{\Gamma}{2(\Gamma-1)} (\frac{\Gamma}{\Gamma-2})^{j-1} - \frac{1}{2}$ with $\Gamma = \sqrt{\frac{(N-2)^2}{4} - \gamma}$. Let u_n be solutions in B with $\lambda_n \to 0^+$. Theorem 2 follows by

Theorem 3

- i) $u_n>0$ then $\gamma\leq \frac{(N-2)^2}{4}-1$ and $u_n\sim U_{\mu_n^1}$ on the scale $\mu_1^n\sim \lambda_n^{\sigma_1}$
- ii) u_n sign-changing, then $\gamma < \frac{(N-2)^2}{4} 4$
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Change of variables:
$$v(r) \sim r^{\frac{(N-2)\beta_{-}}{2\Gamma}} u(r^{\frac{N-2}{2\Gamma}})$$
, $\alpha = \frac{2\beta_{-}}{\Gamma}$, satisfies $-\Delta v = |v|^{\frac{4}{N-2}} v + \lambda |x|^{\alpha} v$ in $B \setminus \{0\}$, $v = 0$ on ∂B $(P)_{\lambda}$

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$$v(r) \sim r^{\frac{(N-2)\beta_{-}}{2\Gamma}} u(r^{\frac{N-2}{2\Gamma}})$$
, $\alpha = \frac{2\beta_{-}}{\Gamma}$, satisfies $-\Delta v = |v|^{\frac{4}{N-2}} v + \lambda |x|^{\alpha} v$ in $B \setminus \{0\}$, $v = 0$ on ∂B $(P)_{\lambda}$

We recover the non-existence results for $\gamma = 0$, 3 < N < 6 and $\gamma > \frac{(N-2)^2}{4} - 1$ & the asymptotics for $\gamma = 0, \ N \ge 7$ due to

A. lacopetti, Ann. Mat. Pura Appl. 194 (2015)



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Pb: When $\gamma \neq 0$ or $k \geq 3$ several new difficulties arise

• Bubbles of same sign don't superimpose by Pohozaev identity:

$$\int_{\partial A} \left[|x|(v')^{2} + (N-2)vv' + \frac{N-2}{N}|x||v|^{\frac{2N}{N-2}} + \lambda |x|^{\alpha+1}v^{2} \right]$$

$$= (\alpha+2)\lambda \int_{A} |x|^{\alpha}v^{2} > 0$$



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 in $(\hat{M}\delta_j,R_j)$, $\hat{M}>>1$

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A contradiction with $M^{\frac{N-2}{2}}|v|(M) < (\hat{M}\delta_i)^{\frac{N-2}{2}}|v|(\hat{M}\delta_i) << 1$

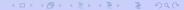
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<u>Crucial estimate</u>: as a by-product $|v| \leq CV_{\delta_i}$ in $[R_{j-1}, R_j]$



• The limiting problem has positive radial solutions on annuli

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$$|v|(r) \sim \frac{(\delta_j)^{\frac{N-2}{2}}}{r^{N-2}} \text{ for } r >> R_{j-1} \ \Rightarrow \ |v|'(R_j) \sim \frac{(\delta_j)^{\frac{N-2}{2}}}{(R_j)^{N-1}} \text{ (left)}$$

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• $\delta_i = \delta_i(\lambda)$ follows by Pohozaev identity if $R_i \sim \sqrt{\delta_i \delta_{i+1}}$



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$$\Rightarrow \frac{1}{\delta_{j+1}^{\frac{N-2}{2}}} = \int_{R_j}^{r_{j+1}} |v|' \sim \frac{\delta_j^{\frac{N-2}{2}}}{R_j^{N-2}} + o\left(\frac{1}{\delta_{j+1}^{\frac{N-2}{2}}}\right)$$

A perturbative approach

Ansatz:
$$u = \sum_{j=1}^k (-1)^j P U_{\mu_j} + \phi$$
, where $P: H^1(\Omega) \to H^1_0(\Omega)$ projection, $\phi \in H^1_0(\Omega)$ small, $\mu_k << \cdots << \mu_1$

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$$\underline{\text{Reduced energy: } E = J_{\lambda} \left(\sum_{i=1}^{k} (-1)^{j} P U_{\mu_{j}} + \phi \right) - k A_{N}, \text{ where } A_{N}}$$

$$J_{\lambda}(u) = \frac{1}{2} \int_{\Omega} [|\nabla u|^2 - \gamma \frac{u^2}{|x|^2} - \lambda u^2] - \frac{N-2}{2N} \int_{\Omega} |u|^{\frac{2N}{N-2}}$$

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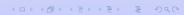
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Energy expansion: As $\lambda \to 0^+$

$$E = (B_N | m_{\gamma,0}(\Omega) | \mu_1^{2\Gamma} - \lambda \mu_1^2 f(\mu_1)) + \sum_{j=2}^k [C_N (\frac{\mu_j}{\mu_{j-1}})^{\Gamma} - D_N \lambda \mu_j^2] + \text{h.o.t}$$

where $f(\mu_1) = \log \frac{1}{\mu_1}/1$ if $\Gamma = 1/>1$



Need to require:

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Problem: the error $o(\lambda^{\theta_1})$ is not sufficiently small

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Key point: split the error as $R_1 + \cdots + R_k$ where

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- A. Iacopetti, G. Vaira, CCM 18 (2016)
- F. Morabito, A. Pistoia, G. Vaira, Potential Anal. (2016)

Thanks for your attention