A new gluing phenomenon for metrics of prescribed Q-curvature in dimension 6

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Conformal changes of metric, Gauss equation

Let (S, g) = surface, K_g=Gauss curvature

Conformal change $g_u := e^{2u} g$

$$\Rightarrow -\Delta_g u + K_g = K_{g_u} e^{2u} \quad \text{Gauss equation.}$$

Similar with Q-curvature in (M, g) of dimension $2m$: $g_u := e^{2u} g$

$$P_g^{2m} u + Q_g = Q_{g_u} e^{2mu}. \tag{1}$$

If $M = \Omega \subset \mathbb{R}^{2m}$,

$$(1) \Leftrightarrow (-\Delta)^m u = Q_{g_u} e^{2mu}, \quad g_u = e^{2u} |dx|^2.$$

Remark Similar to a mean-field equation, but no boundary condition.

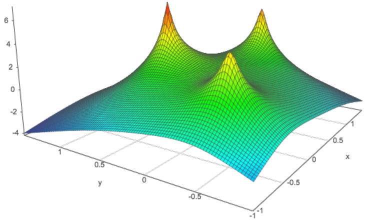
Compactness

Theorem (H. Brézis - F. Merle 1991) $\Omega \subset \mathbb{R}^2$, $(u_k)_{k\in\mathbb{N}}$ solutions to

$$-\Delta u_k = V_k e^{2u_k} \quad \text{in } \Omega, \quad \int_\Omega e^{2u_k} dx \le \bar{A}$$

with $V_k \ge 0$, $\|V_k\|_{L^\infty} \le \bar{K}$. Then up to a subsequence either
(i) (compactness) $u_k \to u_\infty$ in $C^1_{\mathrm{loc}}(\Omega)$, or
(ii) (blow-up) $S_1 := \left\{ x \in \Omega : \lim_{r\to 0} \liminf_{k\to\infty} \int_{B_r(x)} V_k e^{2u_k} dx \ge 2\pi \right\}$,

$$u_k \to -\infty \text{ in } \Omega \setminus S_1, \quad K_k e^{2u_k} \overset{*}{\rightharpoonup} \sum_{j=1}^N \alpha_j \delta_{a_j}, \quad \alpha_j \ge 2\pi.$$



Theorem (YY. Li - I. Shafrir) If $V_k \to V_\infty \ge 0$ in C^0, then $\alpha_j \in 4\pi\mathbb{N}$.

Dimension 4

$$\Delta^2 u_k = V_k e^{4u_k} \quad \text{in } \Omega \subset \mathbb{R}^2, \quad \int_\Omega e^{4u_k} dx \le C \tag{2}$$

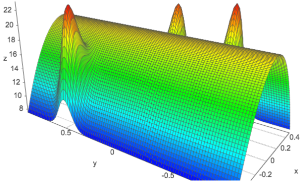
Theorem 1 (Adimurthi, F. Robert, M. Struwe '06)
$(u_k)_{k\in\mathbb{N}}$ solutions to (2) with $V_k > 0$, $\|V_k\|_{L^\infty} \le \bar{K}$. Set

$$S_1 := \left\{ x \in \Omega : \lim_{r\to 0^+} \liminf_{k\to\infty} \int_{B_r(x)} V_k e^{4u_k} dx \ge 3|S^4| \right\},$$

Then up to a subsequence either
(i) (compactness) $u_k \to u_\infty$ in $C^3_{\mathrm{loc}}(\Omega)$ ($\Rightarrow S_1 = \emptyset$), or
(ii) (blow-up) $\exists \varphi \in C^\infty(\Omega \setminus S_1)$ and $\beta_k \to \infty$ s.t.

$$\Delta^2 \varphi \equiv 0, \quad \varphi \le 0, \quad \varphi \not\equiv 0, \quad \frac{u_k}{\beta_k} \to \varphi \text{ in } C^3_{loc}(\Omega \setminus S_1)$$

$(\Rightarrow u_k \to -\infty$ in $\Omega \setminus S_1 \cup S_\varphi$, $S_\varphi := \{\varphi = 0\})$



Dimension 2m

$$(-\Delta)^m u_k = V_k e^{2mu_k} \quad \text{in } \Omega \subset \mathbb{R}^{2m}, \quad \int_\Omega e^{2mu_k} dx \le C \tag{3}$$

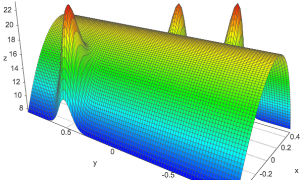
Theorem 2 (Adimurthi, F. Robert, M. Struwe '06, M. '09, A.Hyder'18)
$(u_k)_{k\in\mathbb{N}}$ solutions to (3) with $V_k \ge 0$, $\|V_k\|_{L^\infty} \le \bar{K}$. Set

$$S_1 := \left\{ x \in \Omega : \lim_{r\to 0^+} \liminf_{k\to\infty} \int_{B_r(x)} V_k e^{2mu_k} dx \ge \frac{\Lambda_1}{2} \right\}, \quad \Lambda_1 := (2m-1)|S^{2m}|.$$

Then up to a subsequence either
(i) (compactness) $u_k \to u_\infty$ in $C^{2m-1}_{\mathrm{loc}}(\Omega)$ ($\Rightarrow S_1 = \emptyset$), or
(ii) (blow-up) $\exists \varphi \in C^\infty(\Omega)$ and $\beta_k \to \infty$ s.t.

$$\Delta^m \varphi \equiv 0, \quad \varphi \le 0, \quad \varphi \not\equiv 0, \quad \frac{u_k}{\beta_k} \to \varphi \text{ in } C^{2m-1}_{loc}(\Omega \setminus S_1)$$

($\Rightarrow u_k \to -\infty$ in $\Omega \setminus S_1 \cup S_\varphi$, $S_\varphi := \{\varphi = 0\}$)



Problem: Can $S_1, S_\varphi \subset \Omega \subset \mathbb{R}^{2m}$ be prescribed?

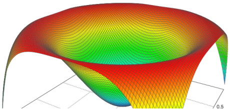
$$\mathcal{K}(\Omega) := \{\varphi \in C^\infty(\Omega) : \Delta^m \varphi \equiv 0, \quad \varphi \le 0, \quad \varphi \not\equiv 0\}, \quad S_\varphi := \{\varphi = 0\}$$

Problem Given $\varphi \in \mathcal{K}(\Omega)$ and $S_1 \subset \Omega$ finite, find (u_k) sol. to

$$(-\Delta)^m u_k = V_k e^{2mu_k} \quad \text{in } \Omega \subset \mathbb{R}^{2m}, \quad \Lambda_k := \int_\Omega V_k e^{2mu_k} dx \le C \qquad (4)$$

blowing up as in Theorem 2 on $S_1 \cup S_\varphi$.

Theorem 3 (A.Hyder, S.Iula, M.'17) $S_1 = \emptyset \Rightarrow \forall \varphi \in \mathcal{K}(\Omega)$, $\Lambda_k \in (0, \frac{\Lambda_1}{2})$ there exists u_k sol. to (4) with $u_k \to \infty$ on S_φ (polyharmonic blow up).



Idea Look for solutions of the form

$$u_k = c_k \varphi + k - \alpha_k \log(1 + |x|^2) - |x|^2 + v_k, \quad \frac{c_k}{k} \to \infty, \alpha_k \to 0, v_k \to 0.$$

Based on a previous work of A. Hyder on prescribed fractional Q-curv., Schauder fixed point and a Brézis-Merle estimate.
Open problems Try with $\Lambda_k \ge \frac{\Lambda_1}{2}$. Can we have $S_1 \ne \emptyset$?

Example of point concentration Let u solve

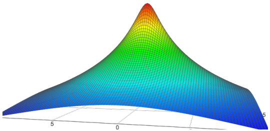
$$(-\Delta)^m u = (2m-1)!e^{2mu}, \quad \int_{\mathbb{R}^{2m}} e^{2mu}dx < +\infty. \tag{5}$$

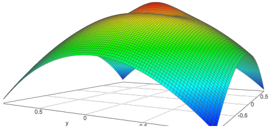
$\Rightarrow u_k(x) := u(kx) + \log k$ also solves (5).
u_k blows up at 0 ($S_1 = \{0\}$, concentration blow up).

Well-known solutions to (5) are the spherical solutions:

$$u_{\lambda,x_0}(x) = \log \frac{2\lambda}{1+\lambda^2|x-x_0|^2}, \quad \lambda > 0, x_0 \in \mathbb{R}^{2m}.$$

Liouville, Chen-Li '91: $m = 1 \Rightarrow$ all solutions are spherical.
A.Chang-W.Chen '01: $m > 1 \Rightarrow$ there exists other solutions.
(see also A.Chang-P.Yang '97, J-C.Wei-X.Xu '99, X. Xu '96)
C-S.Lin '98, M. '09, A.Hyder '17 u non-spherical
$\Rightarrow u(x) \sim P(x) + \alpha \log|x|$, P polynomial, $2 \leq \deg P \leq 2m-2$, $P \leq C$.



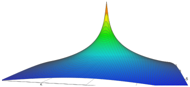


Can we glue polynomial and concentration blow up?

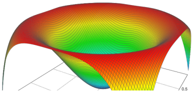
Model case: $B_R \subset \mathbb{R}^6$, $V_k \to V_0 \in C^0_{rad}(B_R)$, $V_0 > 0$, u_k radial sol. to

$$(-\Delta)^3 u_k = V_k e^{6u_k}, \quad \int_{B_R} V_k e^{6u_k} dx \le C.$$

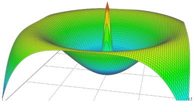
Theorem 4 (A.Hyder-M.'18) In case of blow up, $S = S_1 \cup S_\varphi$ is (up to scaling) either:



(a) $S = \{0\}$



(b) $S = \{|x| = 1\}$



(c) $S = \{0\} \cup \{|x| = 1\}$

In case (c), for $\beta_k \to \infty$, $r_k := 2e^{-u_k(0)} \to 0$

$$\frac{u_k(x)}{\beta_k} \to \varphi(x) = -(1-|x|^2)^2 \quad \text{in } C^5_{loc}(B_R \setminus \{0\}) \quad (\beta_k \to \infty)$$

$$\eta_k(x) := u_k(r_k x) + \log r_k \to \log \frac{2}{1+|x|^2} =: \eta(x) \quad \text{in } C^5_{loc}(\mathbb{R}^6)$$

$$\int_{B_\delta} V_k e^{6u_k} dx \to \Lambda_1 \quad \text{for } 0 < \delta < 1.$$

The proof uses ideas from previous works of Robert and Robert-Struwe.

Proof 0 is local max of $u_k \Rightarrow \Delta u_k(0) \leq 0$.
Green repr. on $B_\tau \Rightarrow$

$$\Delta\eta_k(x) + r_k^2 \underbrace{\left(\frac{\Delta^2 u_k(\tau)\tau^2}{12} - \Delta u_k(\tau)\right)}_{=\beta_k(C_\tau + o(1))}$$

$$= r_k^4 \frac{\Delta^2 u_k(\tau)|x|^2}{12} - r_k^2 \int_{B_\tau} V_k(z) e^{6u_k(z)} \int_{B_\tau} G(r_k x, z) G(y, z) dydz$$

$$\Rightarrow \int_{B_R} |\Delta\eta_k + r_k^2 \beta_k(\underbrace{C_\tau + o(1)}_{<0})| \leq C r_k^4 \beta_k R^8 + C R^4$$

One proves $r_k^2 \beta_k = O(1)$

$$\Rightarrow \int_{B_R} |\Delta\eta_k| \leq C(R) \Rightarrow \eta_k \to \eta_\infty.$$

Classification result $\Rightarrow \Delta\eta_\infty \leq 0$ at $\infty \Rightarrow r_k^2\beta_k = o(1)$ and $\Delta\eta_\infty = o(1)$ at ∞.
$\Rightarrow \eta_\infty$ spherical, by classification result. $\square$

Problem Does case (c) actually occur?

Existence and the total curvature problem

Theorem 5 (A.Hyder-M. '18) $V_k \to V_\infty$ in $C^1_{rad}(\mathbb{R}^6)$, $V_\infty' \le 0$.
For every $\Lambda > \Lambda_1$ there are (u_k) rad. symm. sol. to

$$(-\Delta)^3 u_k = V_k e^{6u_k}, \quad \int_{\mathbb{R}^6} V_k e^{6u_k} dx = \Lambda$$

as in case (c) of Theorem 4.

Remark 1 Case $\mathbb{R}^4$ studied by F. Robert '06: only $S = \{0\}$ possible.

Remark 2 Take $V_k = 120 = 5!$ and compare with

$$(-\Delta)^m u = (2m-1)! e^{2mu}, \quad \Lambda := (2m-1)! \int_{\mathbb{R}^{2m}} e^{2mu} dx < +\infty. \quad (6)$$

Theorem (C-S.Lin'98) In 4-d $(m = 2)$ $\Lambda \le \Lambda_1 = 6|S^4|$, with "=" iff u is spherical. Based on **Pohozaev** identity.

Theorem (A.Chang-W.Chen'01) $m \ge 2$, $\Lambda \in (0, \Lambda_1) \Rightarrow \exists$ sol. to (6).

Theorem (J-C.Wei-D.Ye '08, A.Hyder-M.'14) $m \ge 2$, $\Lambda \in (0, \Lambda_1) \Rightarrow \exists$ *non-radial* sol. to (6).

Theorem (M. '12) $m = 3 \Rightarrow \exists$ sol. to (6) with Λ arbitrarily large.

Proof Look for radially symmetric solutions.

$$\begin{cases} (-\Delta)^3 u = 120 e^{6u} \quad \text{in } \mathbb{R}^6 \\ u(0) = u'(0) = u'''(0) = u'''''(0) = 0 \\ u''(0) = a \ll -1, \quad u''''(0) = 1. \end{cases}$$

Claim $\lim_{a \to -\infty} \Lambda = +\infty$. Compare with $\phi(r) := \frac{a}{2} r^2 + \frac{1}{24} r^4$.

$$\Delta^3 u \leq 0 = \Delta^3 \phi, \quad u^{(j)}(0) = \phi^{(j)}(0),\ 0 \leq j \leq 5 \Rightarrow u \leq \phi.$$

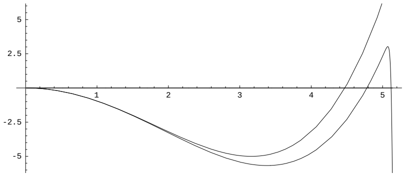


Figure : The functions $u(r) \leq \phi(r)$.

Strategy: Show that in a neighborhood of $\phi(r) = 0$, $\int r^5 e^{6u(r)} dr \to \infty$.

Theorem (X.Huang, D.Ye '15) $m \geq 3$ **odd** (in dim. $6, 10, 14, \dots$) $\Rightarrow \exists$ sol. to (6) with Λ arbitrarily large.

Problem What about dimensions $8, 12, 16, \dots$, or even odd dimension?

Theorem (A.Hyder '17) For $n \geq 5$, $V \in C^0_{rad} \cap L^\infty(\mathbb{R}^n)$, $V > 0$, $\Lambda > 0$, $\exists u$ sol. to

$$(-\Delta)^{\frac{n}{2}} u = Ve^{nu}, \quad \Lambda := \int_{\mathbb{R}^{2m}} Ve^{nu} dx. \tag{7}$$

Idea of proof For $\lambda \in (0, \frac{1}{4n}]$ write $u(x) = v(x) - |x|^4 + c$ where v must solve

$$v(x) = \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{1}{|x-y|}\right) V(y) e^{n(c-|y|^4)} e^{nv(y)} dy + \lambda \Delta v(0)(|x|^4 - 2|x|^2) + c.$$

$\lambda \in (0, \frac{1}{4n}] \Rightarrow \Delta v(0) < 0 \Rightarrow$ compactness $\Rightarrow$ Schauder fixed point. □

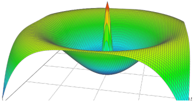
Proof of Theorem 5 Take $\lambda_k \to 0^+$, $u_k(x) = v_k(x) - |x|^4 + c_k$. Prove: 1) $\sup_{B_\varepsilon} u_k \to \infty$, 2) $\lambda_k \Delta v_k(0) \to -\infty$, 3) $u_k(1) \to \infty$.

Proof of 1) $\sup_{B_\varepsilon} u_k \leq C \Rightarrow |\Delta v_k(0)| \leq C \Rightarrow \lambda_k \Delta v_k(0) \to 0$
$\Rightarrow u_k + |x|^4 \to u_\infty$, $u_\infty(x) = \frac{1}{\gamma_6} \int_{\mathbb{R}^6} \log\left(\frac{1}{|x-y|}\right) V_\infty(y) e^{6u_\infty(y)} dy + c_\infty$.
Pohozaev $\Rightarrow \Lambda = \int_{\mathbb{R}^6} V_\infty e^{6u_\infty} \leq \Lambda_1$. Contradiction.

Proof of 2) $|\lambda_k \Delta v_k(0)| \leq C \Rightarrow \|\nabla u_k\|_{L^1(B_R)} \leq C(R)$
$\Rightarrow \int_{B_R} V_k e^{6u_k} \to \Lambda_1$ (quantization by F. Robert '06, M. '11). Contrad.
Proof of 3) $\int_{B_\delta} V_k e^{6u_k} dx \to \Lambda_1$ for $0 < \delta < 1$, $\Lambda > \Lambda_1 \Rightarrow u_k(1) \to +\infty$

A finer analysis

Remember Case (c) of Theorem 5:



$$\eta_k(x) = u_k(r_k x) + \log r_k \to \log \frac{2}{1+|x|^2} =: \eta(x) \quad \text{in } C^5_{loc}(\mathbb{R}^6), \quad r_k := 2e^{-u_k(0)} \to 0$$

$$\frac{u_k(x)}{\beta_k} \to \varphi(x) = -(1-|x|^2)^2 \quad \text{in } C^5_{loc}(B_R \setminus \{0\}) \quad (\beta_k \to \infty)$$

$$\int_{B_\delta} V_k e^{6u_k} dx \to \Lambda_1 \quad \text{for } 0 < \delta < 1.$$

Theorem (A.Hyder-M.) In case (c) of Theorem 5, $\beta_k = u_k(0)(1+o(1))$

$$u_k(x) = \eta(\tfrac{x}{r_k}) + u_k(0)(\varphi(x) + 1 + o(1)), \quad o(1) \to 0 \text{ loc. on } B_1 \qquad (8)$$

$$\eta_k(x) = \eta(x) + \varepsilon_k \psi_0(x) + o(\varepsilon_k)(1+|x|^4), \quad |x| \le \varepsilon_k^{-1/4} \qquad (9)$$

$$\psi_0(x) = 8|x|^2 + O(\log|x|) \quad \text{as } |x| \to \infty, \quad \varepsilon_k := \frac{u_k(0)}{e^{2u_k(0)}}$$

$$\int_{B_\delta} V_k e^{6u_k} dx = \Lambda_1 + 24\Lambda_1 \varepsilon_k + o(\varepsilon_k) > \Lambda_1, \quad \delta \le \frac{1}{2} \qquad (10)$$

Remark (8)-(9) might be used to glue metrics via Lyapunov-Schmidt.

Comparison with the Moser-Trudinger equation

$u_k \in C^{\infty}_{0,rad}(B_1 \subset \mathbb{R}^2)$ positive sol. to

$$-\Delta u_k = \lambda_k u_k e^{u_k^2} \tag{11}$$

$$\eta_k := \mu_k(u_k(r_k x) - \mu_k), \quad \mu_k := u_k(0), \quad r_k = \frac{1}{\sqrt{\lambda_k \mu_k^2 e^{\mu_k^2}}}.$$

Extending a work with A.Malchiodi:
Theorem (G.Mancini, M.'16) For $u_k(0) \to \infty$,

$$\eta_k = \eta + \frac{w_0}{\mu_k^2} + \frac{z_0}{\mu_k^4} + \frac{O(\log(|x|))}{\mu_k^6}, \quad \text{for } |x| \le e^{\mu_k}, \tag{12}$$

where $w_0(x), z_0(x) \sim -2\log|x|$ at ∞. This implies

$$4\pi + \frac{4\pi}{\mu_k^4} + o(\mu_k^{-1}) \le \int_{B_1} |\nabla u|^2 dx \le 4\pi + \frac{6\pi}{\mu_k^4} + o(\mu_k^{-1}). \tag{13}$$

Remark (12)-(13) used to compute the Leray-Schauder degree of (11):
Work in progress with O.Druet, A.Malchiodi, P-D.Thizy.

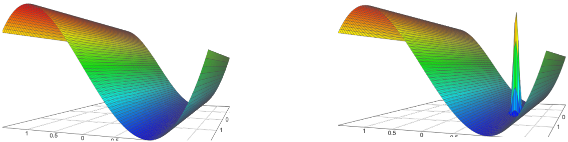
Open problems in the non-radial case

How arbitrarily can we prescribe S_φ, S_1?

Problem 1 Should $S_1 \subset \{\nabla\varphi = 0\}$?

Problem 2 If $S_1, S_\varphi \neq \emptyset$, should blow-ups at S_1 be spherical?

Problem/Example $\varphi(x) = x_1 - \frac{x_1^3}{3} - \frac{2}{3}$ in $\Omega := (-2,2) \times \mathbb{R}^3 \subset \mathbb{R}^4$.
$S_\varphi = \{1\} \times \mathbb{R}^3$ and $\nabla\varphi = 0$ on $\{\pm 1\} \times \mathbb{R}^3$



Remark $\Omega = \mathbb{R}^4 \Rightarrow \varphi = \sum_{i=1}^4 -a_i x_i^2 \Rightarrow \nabla\varphi(x) = 0 \Leftrightarrow x \in S_\varphi$.
Consistent with $\Lambda \leq \Lambda_1$ of C-S. Lin'98.

Remark In the case of mean field (with boundary conditions), there are results. $S_\varphi = \emptyset$, $S_1 \in \{\nabla\Phi = 0\}$, $\Phi =$"reduced functional" (F.Robert, JC. Wei, M.-Petrache, Baracket-Packard, Del Pino, Musso, Clapp, ...)

What about odd dimension?

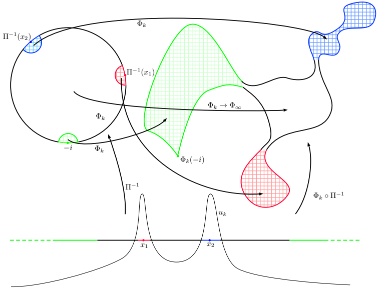
Theorem (Da Lio-M.-Riviere '15-'16) Let $u_k : \mathbb{R} \to \mathbb{R}$ solve

$$(-\Delta)^{\frac{1}{2}} u_k = V_k e^{u_k}, \quad \int_{\mathbb{R}} e^{u_k} dx \leq \bar{L}$$

with $\|V_k\|_{L^\infty} \leq \bar{\kappa}$, $V_k \overset{*}{\rightharpoonup} V_\infty$. Then $\exists S_1 = \{x_1, \ldots, x_N\} \subset \mathbb{R}$ s.t. either

(i) $u_k \to u_\infty$ in $\mathbb{R} \setminus S_1$ and $V_k e^{u_k} \overset{*}{\rightharpoonup} V_\infty e^{u_\infty} + \sum_{a_i \in S_1} \pi \delta_{a_i}$,

(ii) $u_k \to -\infty$ in $\mathbb{R} \setminus S_1$, $V_k e^{u_k} \overset{*}{\rightharpoonup} \sum_{j=1}^N \alpha_j \delta_{x_j}$, $\alpha_j \geq \pi$.



3-d: work in progress with A. de la Torre, A. Hyder, M. Gonzalez.