Simplicity of algebras associated to non-Hausdorff groupoids

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The Grigorchuk group

$X = \{0, 1\}$

$X^* = \bigcup_{n \geq 0} X^n$ — finite words in 0, 1

$\mathrm{Aut}(X^*)$ = length-preserving bijections of X^*

$\supset$ graph automorphisms of the rooted binary tree

The **Grigorchuk group** is the subgroup of $\mathrm{Aut}(X^*)$ generated by the following elements.

The Grigorchuk group

For $w \in X^*$, define $a, b, c, d \in \mathrm{Aut}(X^*)$ recursively by

$$\begin{aligned} a \cdot (0w) &= 1w & \qquad c \cdot (0w) &= 0(a \cdot w) \\ a \cdot (1w) &= 0w & c \cdot (1w) &= 1(d \cdot w) \end{aligned}$$

$$\begin{aligned} b \cdot (0w) &= 0(a \cdot w) & \qquad d \cdot (0w) &= 0w \\ b \cdot (1w) &= 1(c \cdot w) & d \cdot (1w) &= 1(b \cdot w) \end{aligned}$$

The group $\Gamma = \langle a, b, c, d \rangle$ is called the **Grigorchuk group** (Grigorchuk, 1980).

The Grigorchuk group

Some relations:

$a^2 = b^2 = c^2 = d^2 = e$

$bc = cb = d, \qquad cd = dc = b \qquad bd = db = c$

$\implies$ reduced words alternate between a and elements from $\{b, c, d\}$

E.g. $abaca$, $bacad$

Note: these are not all the relations, for example $(ad)^4 = e$.

The Grigorchuk group

$\Gamma = \langle a, b, c, d \rangle \subset \mathrm{Aut}(X^*)$

Theorem (Grigorchuk)

Γ is an infinite, finitely generated torsion group.

Theorem (Grigorchuk)

Γ is just infinite (all quotients are finite).

Theorem (Grigorchuk)

Γ is amenable, but not elementary amenable.

Theorem (Grigorchuk)

Γ has intermediate word growth.

Self-similar groups

Suppose we have an finite set X, a countable discrete group G,

1. a faithful action $G \times X^* \to X^*$ which preserves lengths, and
2. a restriction $G \times X \to G$

$$(g, x) \mapsto g|_x .$$

such that the action on X^* can be defined recursively

$$g(x\alpha) = (gx)(g|_x \alpha)$$

The pair (G, X) is called a self-similar action.

Restriction extends to words

$$g|_{\alpha_1\alpha_2\cdots\alpha_n} := g|_{\alpha_1}|_{\alpha_2}\cdots|_{\alpha_n}$$

$$g(\alpha\beta) = (g\alpha)(g|_\alpha \beta)$$

The action extends to infinite words $X^{\mathbb{N}}$

Self-similar group C*-algebras

Nekrashevych defined a C*-algebra $\mathcal{O}_{G,X}$ from (G, X).

$\mathcal{O}_{G,X}$ is the universal C*-algebra for

1. a Cuntz family of isometries $\{s_x\}_{x \in X}$ and
2. a unitary representation $\{u_g\}_{g \in G}$,

subject to

$$u_g s_x = s_{g \cdot x} u_{g|_x} \qquad x \in X, g \in G.$$

$$\mathcal{O}_{G,X} = \overline{\text{span}\{s_\alpha u_g s_\beta^* : \alpha, \beta \in X^*, g \in G\}}$$

$$\supset \overline{\text{span}\{s_\alpha u_g s_\beta^* : \alpha, \beta \in X^*, g \in G, |\alpha| = |\beta|\}}$$

Self-similar graph C*-algebras

Exel and Pardo extended the idea of a self-similar group action to act on a graph rather than an alphabet.

This idea unified Nekrashevych's algebras with Katsura algebras associated to a pair $A, B \in M_N(\mathbb{N})$.

Each $\mathcal{O}_{A,B} \cong \mathcal{O}_{\mathbb{Z},E_A}$.

Katsura: Every UCT Kirchberg algebra is modeled by some $\mathcal{O}_{A,B}$.

Self-similar group C*-algebras

For the Grigorchuk group Γ, it was left open whether $\mathcal{O}_{\Gamma,X}$ is simple or not.

$\mathcal{O}_{G,X} = C^*(\mathcal{G}_{G,X})$ for an étale groupoid $\mathcal{G}_{G,X}$.

Nekrashevych showed $\mathcal{G}_{G,X}$ Hausdorff $\implies$ $\mathcal{O}_{G,X}$ simple.

The issue: $\mathcal{G}_{\Gamma,X}$ is not Hausdorff.

The groupoids of some Katsura algebras are also non-Hausdorff (as well as some Li semigroup algebras, foliations, pseudogroups).

Our question: can we characterize simplicity for C*-algebras of non-Hausdorff groupoids?

Groupoids

- Let $\mathcal{G}$ be a second countable topological groupoid such that $\mathcal{G}^{(0)}$ is Hausdorff.
- We say $\mathcal{G}$ is ample if $\mathcal{G}$ has a basis of compact open bisections.
  - Ample groupoids are always locally compact and étale.
- We say $\mathcal{G}$ is minimal if every orbit is dense.
- We say $\mathcal{G}$ is topologically principal if the units with trivial isotropy are dense in $\mathcal{G}^{(0)}$.
- We say $\mathcal{G}$ is effective if the interior of the isotropy bundle is $\mathcal{G}^{(0)}$.
- If $\mathcal{G}$ is Hausdorff and second countable, then effective and topologically principal are equivalent.

Non-Hausdorff groupoids

Suppose $\mathcal{G}$ is an étale groupoid such that $\mathcal{G}^{(0)}$ is Hausdorff.

Then $\mathcal{G}^{(0)}$ is open in $\mathcal{G}$.

$\mathcal{G}^{(0)}$ is closed in $\mathcal{G} \iff \mathcal{G}$ is Hausdorff

$\mathcal{G}$ not Hausdorff $\implies$ there is a sequence of units that converges to something outside the unit space

Elements of $\overline{\mathcal{G}^{(0)}} \setminus \mathcal{G}^{(0)}$ are isotropy.

$\implies$ principal étale groupoids are Hausdorff.

Non-Hausdorff groupoids

"I feel like I just don't speak non-Hausdorff, and when I try, I have a terrible Hausdorff accent." — Aidan Sims

If $\mathcal{G}$ is not Hausdorff, then compact sets might not be closed and the intersection of compact sets might not be compact.

Many authors take "compact" to include Hausdorff.

If $\mathcal{G}$ is not Hausdorff, then effective and topologically principal are different.

Étale groupoid C*-algebras

Renault, Connes: $\mathcal{G} \to C^*(\mathcal{G})$

$C_c(U)$ = compactly supported continuous functions on U

$\mathcal{C}(\mathcal{G})$ = linear span of the $C_c(U)$ as U ranges over all bisections.

WARNING: $f \in C_c(U)$ might not be continuous on $\mathcal{G}$!

$u \in \mathcal{G}^{(0)} \quad \longrightarrow \quad L_u : \mathcal{C}(\mathcal{G}) \to \mathcal{B}(\ell^2(\mathcal{G}u))$ satisfying

$$L_u(f)\delta_\gamma = \sum_{\alpha \in \mathcal{G}r(\gamma)} f(\alpha)\delta_{\alpha\gamma} \text{ for } f \in \mathcal{C}(\mathcal{G}).$$

$C_r^*(\mathcal{G})$ = closure of the image of $\mathcal{C}(\mathcal{G})$ under $\bigoplus_{u \in \mathcal{G}^{(0)}} L_u$

$\exists$ universal $C^*(\mathcal{G}) \to C_r^*(\mathcal{G})$

Étale groupoid C*-algebras

$\mathcal{C}(\mathcal{G})$ dense in $C^*_r(\mathcal{G})$

We view elements of $C^*_r(G)$ as functions from $\mathcal{G} \to \mathbb{C}$:

For $a \in C^*_r(\mathcal{G})$, define $j(a) : \mathcal{G} \to \mathbb{C}$ by

$$j(a)(\gamma) = \big(L_{s(\gamma)}(a)\delta_{s(\gamma)} \mid \delta_\gamma\big).$$

$j : C^*_r(\mathcal{G}) \to \ell^\infty(\mathcal{G})$ linear, contractive, identity on $\mathcal{C}(\mathcal{G})$.

$\mathrm{supp}(f) = \{\gamma \in \mathcal{G} : f(\gamma) \neq 0\}$ “open support” may not be open.

We say f is singular if interior of $\mathrm{supp}(f)$ is empty.

Simplicity of étale groupoid C*-algebras

Theorem (Renault, Brown-Clark-Farthing-Sims)

Let $\mathcal{G}$ be a Hausdorff étale groupoid. Then $C^*(\mathcal{G})$ is simple if and only if

1. *$\mathcal{G}$ is minimal (every orbit is dense)*
2. *$\mathcal{G}$ is effective (the interior of the isotropy group bundle is the unit space), and*
3. *$\mathcal{G}$ satisfies weak containment ($C_r^*(\mathcal{G}) \cong C^*(\mathcal{G})$).*

Proof: ($\Longleftarrow$) uses a Cuntz-Krieger uniqueness theorem saying an ideal intersects $C_0(\mathcal{G}^{(0)})$.

($\Longrightarrow$) gives effective, not topologically principal.

Simplicity of étale groupoid C*-algebras

Theorem (Clark, Exel, Pardo, Sims, S)

Let $\mathcal{G}$ be an étale groupoid. Then $C^*(\mathcal{G})$ is simple if and only if

1. *$\mathcal{G}$ is minimal (every orbit is dense)*
2. *$\mathcal{G}$ is effective (the interior of the isotropy group bundle is the unit space),*
3. *$\mathcal{G}$ satisfies weak containment ($C_r^*(\mathcal{G}) \cong C^*(\mathcal{G})$), and*
4. *for all $a \in C^*(\mathcal{G})$, $\mathrm{supp}(j(a))$ has nonempty interior*

$j : C^*(\mathcal{G}) \to \ell^\infty(\mathcal{G})$, norm-decreasing, identity on $\mathcal{C}(\mathcal{G})$.

To get ($\Longleftarrow$) we prove a new uniqueness theorem.

$(1) + (2) + (4) \implies C_r^*(\mathcal{G})$ is simple.

Simplicity of étale groupoid C*-algebras

If $\mathcal{G}$ is ample ($=$ has a basis of compact open bisections), then

"every compact open subset of $\mathcal{G}$ is regular open" $\implies$ (4)

Corollary

Let $\mathcal{G}$ be an ample étale groupoid. If $\mathcal{G}$ is minimal, effective, and every compact open subset of $\mathcal{G}$ is regular open, then $C_r^*(\mathcal{G})$ is simple.

This is much easier to verify than $\mathrm{supp}(j(a))^\circ \neq \emptyset \forall a \in C^*(\mathcal{G})$

Even in the case $C_r^*(\mathcal{G}) \cong C^*(\mathcal{G})$, the converse does not hold. Counterexample?

The Grigorchuk Group!

Steinberg algebra

K a field, $\mathcal{G}$ an ample groupoid

$A_K(\mathcal{G}) =$ linear span of characteristic functions of compact bisections.

$A_{\mathbb{C}}(\mathcal{G})$ is a dense subalgebra of $C^*(\mathcal{G})$.

Theorem

Let $\mathcal{G}$ be an ample groupoid. Then $A_K(\mathcal{G})$ is simple if and only if $\mathcal{G}$ is minimal, effective, and $\text{supp}(f)$ has nonempty interior for all nonzero $f \in A_K(\mathcal{G})$

Singular elements: $S(\mathcal{G}) = \{f \in A_K(\mathcal{G}) : \text{supp}(f)^\circ = \emptyset\}$ is always an ideal.

Our results on the Grigorchuk group

Theorem

Let Γ *be the Grigorchuk group and* $\mathcal{G}_{\Gamma,X}$ *the associated groupoid. Then*

1. $\mathcal{G}_{\Gamma,X}$ *contains a compact open set which is not regular open, but...*
2. $C^*(\mathcal{G}_{\Gamma,X}) = \mathcal{O}_{\Gamma,X}$ *is simple.*
3. $A_K(\mathcal{G}_{\Gamma,X})$ *is simple for any characteristic zero field* K*, but...*
4. $A_{\mathbb{Z}_2}(\mathcal{G}_{\Gamma,X})$ *is not simple. (Nekrashevych '16)*

Theorem (Clark, Exel, Pardo, Sims, S)

Let $\mathcal{G}$ be an étale groupoid. Then $C^*(\mathcal{G})$ is simple if and only if

1. *$\mathcal{G}$ is minimal (every orbit is dense)*
2. *$\mathcal{G}$ is effective (the interior of the isotropy group bundle is the unit space),*
3. *$\mathcal{G}$ satisfies weak containment ($C^*_r(\mathcal{G}) \cong C^*(\mathcal{G})$), and*
4. *for all $a \in C^*(\mathcal{G})$, $supp(j(a))$ has nonempty interior*

Question 1: is (4) needed?~~Question 1: is (4) needed?~~ By modifying the Grigorchuk group, get $\mathcal{G}_{G,X}$ minimal, effective, amenable, but with singular functions.

Question 2: Can (4) be replaced by a condition on $\mathcal{G}$?

Question 3: $\mathcal{O}_{\Gamma,X}$ is simple, purely infinite, and nuclear. Does $\mathcal{O}_{\Gamma,X}$ satisfy the UCT?