Quantifying the magic resources for quantum computation

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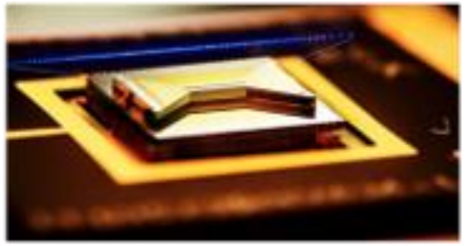
TOWARDS QUANTUM COMPUTERS

- State of the art: well-characterized qubits with well-controlled interactions and long coherence times.
- Leading candidate systems: trapped ions, superconducting qubits, linear optics, silicon-based qubits.

Google



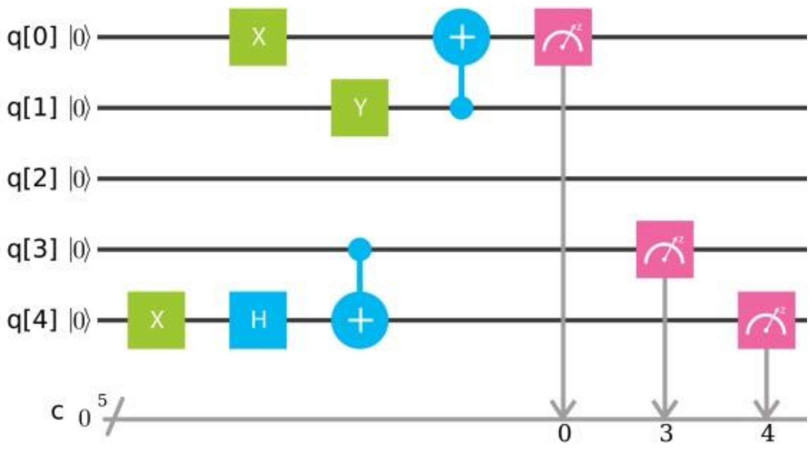
UMD

etc

- There are errors in the physical realization of quantum computation.
- Need error correction to protect our quantum computation.
- Fault-tolerant quantum computation (FTQC) [Shor'96] provides a framework to overcome this difficulty.
- It allows reliable quantum computation when the physical error rate is below a certain threshold value.

FAULT-TOLERANT QUANTUM COMPUTATION

- The model of FTQC allows stabilizer operations
  - Preparation of stabilizer states
  - Implementation of Clifford unitaries
  - Pauli measurements



- Clifford group: easy to implement, cheap

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad C_X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

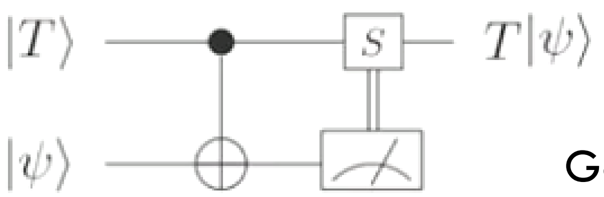
- The Gottesman–Knill theorem: A quantum circuit comprised of only Clifford gates can be simulated efficiently on a classical computer. Thus such circuit confers no quantum computational advantage.

- However, the additional non-stabilizer (or magic) resources **promote Clifford to universal QC**.

MOTIVATIONS

- Magic states are necessary resource to achieve universal quantum computation.
- Magic states can allow non-Clifford gates.

$$|T\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi/4}|1\rangle)$$



$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

Gate injection [Zhou, Leung, Chuang'00]

- QC via magic state distillation is the **leading model** for experimentally realizing FTQC.
- A central problem in quantum information is to determine the minimal physical resources that are required for quantum computational speed-up.
- Better understand the resource cost of killer apps of quantum computing.

Factoring a 1024-bit number [Kutin'06]	3132 qubits	5.7×10^9 T gates
Simulating 50 spins (PF) [Childs et al.'18]	50 qubits	1.8×10^8 T gates
Simulating FeMoco [Reiher et al. 16]	111 qubits	1×10^{14} T gates

MAGIC QUANTUM RESOURCES

Stabilizer states: States of the form $U_{\text{Clif}}|0\rangle^{\otimes n}$

Stabilizer operations: preparation/measurement in the computational basis + Clifford unitaries.

Q: What are magic quantum resources?

A: Non-stabilizer states or non-stabilizer operations.

Examples

- Qubit states that are not from

$$\{|0\rangle, |1\rangle, |+\rangle, |-\rangle, |+i\rangle, |-i\rangle\}$$

Examples: computational basis states, maximally entangled states.

Let us start by defining the magic states.

Definition 1. *Consider pure states $|H\rangle, |T\rangle \in \mathbb{C}^2$ such that*

$$|T\rangle\langle T| = \frac{1}{2}\left[I + \frac{1}{\sqrt{3}}(\sigma^x + \sigma^y + \sigma^z)\right]$$

and

$$|H\rangle\langle H| = \frac{1}{2}\left[I + \frac{1}{\sqrt{2}}(\sigma^x + \sigma^z)\right].$$

The images of $|T\rangle$ and $|H\rangle$ under the action of one-qubit Clifford operators are called magic states of T-type and H-type, respectively.

[Bravyi, Kitaev'05]: the Clifford group operations combined with magic states preparation are sufficient for UQC.

MAIN MESSAGE

- Further development of the resource theory of magic states
- Introduce the resource theory of quantum channels.
- New magic measures to quantify the magic of both quantum states and quantum channels.
- Applications in operational tasks
  - Magic state distillation and magic state generation via channels.
  - Magic cost of quantum operations and gate synthesis.
  - Classical simulation of noisy quantum circuits.
- Ideas from quantum resource theories play important roles.

Resource theory of magic states & its applications

RESOURCE THEORY OF MAGIC STATES

- The resource theory of magic was first proposed by Veitch, Mousavian, Gottesman, and Emerson in 2014.
- As the quantum resource theory developed fast in the past five years, it makes sense to further develop the resource theory of magic states.

	Magic	Entanglement
Free states	Stabilizer states	Separable states
Free operations	Stabilizer operations	LOCC operations
Key task	Distilling magic state (e.g. T state)	Entanglement distillation

- How to quantify the magic of quantum states?
- What is the limit of magic state manipulation (e.g., magic state distillation)?
- What is the ability of quantum channels to generate or consume magic resources?
- Our work uses information-theoretic approaches to answer the above questions.

STABILIZER FORMALISM

- For a prime number d, we define the unitary boost and shift operators X,Z in terms of their action on the computational basis:

$$X|j\rangle = |j \oplus 1\rangle, \quad Z|j\rangle = \omega^j |j\rangle, \omega = e^{2\pi i/d}.$$

- The Heisenberg-Weyl operators $T_{\mathbf{u}} = \tau^{-a_1 a_2} Z^{a_1} X^{a_2}$,

  where $\tau = e^{(d+1)\pi i/d}$, $\mathbf{u} = (a_1, a_2) \in \mathbb{Z}_d \times \mathbb{Z}_d$.

- Phase-space point operators $A_0 = \frac{1}{d} \sum_{\mathbf{u}} T_{\mathbf{u}}, \qquad A_{\mathbf{u}} = T_{\mathbf{u}} A_0 T_{\mathbf{u}}^{\dagger}.$

- For each point $\mathbf{u} \in \mathbb{Z}_d \times \mathbb{Z}_d$ in the discrete phase space, the value of the discrete Wigner representation of a state at this point is given by

$$W_\rho(\mathbf{u}) := \frac{1}{d} \mathrm{Tr}[A_{\mathbf{u}} \rho].$$

DISCRETE WIGNER FUNCTION $W_\rho(\mathbf{u}) := \frac{1}{d}\mathrm{Tr}[A_\mathbf{u}\rho].$

- Discrete Wigner function for qudits (in odd dimensions)
  - A quasi-probability distribution.
  - Discrete Hudson's theorem: a pure state has positive representation if and only if it is a stabilizer state.
  - Information complete: $\rho = \sum_\mathbf{u} W_\rho(\mathbf{u}) A_\mathbf{u}.$
- Phase space distributions in continuous variable quantum information: characteristic function $\chi(\mathbf{u}) = \mathrm{Tr}[\rho D_\mathbf{u}]$ and Wigner function.
- Here, we focus on a similar formalism in finite dimensions.

Example: Wigner function of the qutrit state $|\mathbb{S}\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$

$$W_\mathbb{S}(\mathbf{u}) := \frac{1}{3}\mathrm{Tr}[A_\mathbf{u}|\mathbb{S}\rangle\langle\mathbb{S}|]$$

	0	1	2
0	-1/3	1/6	1/6
1	1/6	1/6	1/6
2	1/6	1/6	1/6

MAGIC MEASURES

- Recall that states with positive W. function are bound magic states [Veitch et al.'12] (similar to PPT states in entanglement theory).
- We may borrow ideas from the resource theory of NPT entangled states.

- Logarithmic negativity (mana) [Veitch et al.'14]:

$$\mathcal{M}(\rho) = \log \|\rho\|_{W,1}$$

- Wigner trace norm and Wigner spectral norm

$$\|V\|_{W,1} = \sum_{\mathbf{u}} |W_V(\mathbf{u})| = \sum_{\mathbf{u}} |\operatorname{Tr}[A_{\mathbf{u}} V]/d|,$$
$$\|V\|_{W,\infty} = d \max_{\mathbf{u}} |W_V(\mathbf{u})| = \max_{\mathbf{u}} |\operatorname{Tr}[A_{\mathbf{u}} V]|.$$

- Example: for $|\mathbb{S}\rangle = (|1\rangle - |2\rangle)/\sqrt{2}$ with Wigner function

$$\mathcal{M}(\mathbb{S}) = \log \frac{5}{3}.$$

	0	1	2
0	-1/3	1/6	1/6
1	1/6	1/6	1/6
2	1/6	1/6	1/6

- Relative entropy of magic [Veitch et al.'14]

MAGIC MEASURE: THAUMA

- We introduce free sub-states with non-positive mana: $\mathcal{W} := \{\sigma : \mathcal{M}(\sigma) \leq 0, \ \sigma \geq 0\}$.
- We introduce magic measures based on divergence between a state and $\mathcal{W}$.

- Google: How to quantify magic? There's a universally accepted unit of what it takes to pull a rabbit out of a hat, called **thauma** (Greek for "wonder" or "marvel").

- A new family of magic measures called **thauma**

$$\boldsymbol{\theta}(\rho) := \inf_{\sigma \in \mathcal{W}} \mathbf{D}(\rho \,\|\, \sigma)$$

- The generalized divergence $\mathbf{D}(\rho \,\|\, \sigma)$ is any function of a quantum state ρ and a positive semi-definite operator σ that obeys data processing.

INSPIRATION

$$D(\rho\|\sigma) = \mathrm{Tr}\{\rho[\log\rho - \log\sigma]\}$$

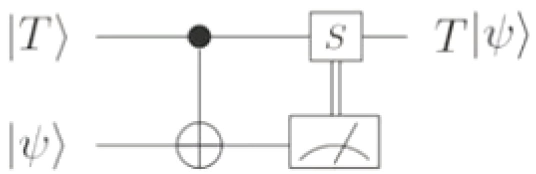
$D_{\max}$ Max-relative entropy [Datta'09]

- Inspired by the idea behind the Rains bound: sub-normalized states with non-positive log negativity. (It leads to better upper bounds for ent. Distillation and quantum communication [Rains'01; Tomamichel, Wilde, Winter'17; Wang and Duan'16], etc).

	Entanglement	Magic
Logarithmic negativity (mana)	$E_N(\rho) = \log \|\rho^{T_B}\|_1$ [VW02, Ple05]	$\mathcal{M}(\rho) = \log \|\rho\|_{W,1}$ [VMGE'14]
Rel. entropy of ent. (magic)	$E_R(\rho) = \inf_{\sigma\in\mathrm{Sep}} D(\rho\|\sigma)$	$R_{\mathcal{M}}(\rho) = \inf_{\sigma\in\mathrm{Stab}} D(\rho\|\sigma)$ [VMGE'14]
Rains bound (thauma)	$R(\rho) = \inf_{\sigma\in\mathrm{PPT'}} D(\rho\|\sigma)$ [Rains'01]	$\theta(\rho) = \inf_{\sigma\in\mathcal{W}} D(\rho\|\sigma)$ [WWS'18]
Max-Rains bound (max-thauma)	$R_{\max}(\rho) = \inf_{\sigma\in\mathrm{PPT'}} D_{\max}(\rho\|\sigma)$ [WD'16]	$\theta_{\max}(\rho) = \inf_{\sigma\in\mathcal{W}} D_{\max}(\rho\|\sigma)$ [WWS'18]

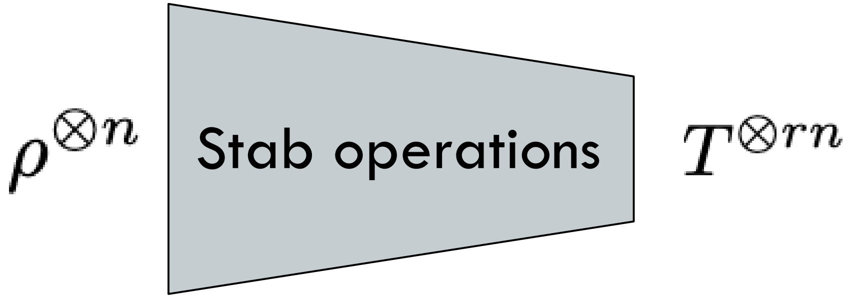
- The family of thauma measures can be seen as the analog of the Rains bound in the resource theory of magic states.

DISTILLING MAGIC FROM STATES



Gate injection

Magic state distillation



- Goal: transform a large number of noisy magic states into a small number of high-quality key magic states.
- MSD + gate injection is a popular approach to perform universal QC.

Key question: what is the maximum number of T states we can distill from given states approximately?

- Rate estimation for finite-copy and asymptotic regime:
  - One-shot distillable T magic $\mathcal{M}_T^{\varepsilon}(\rho) = \sup\{r : \Lambda(\rho) \approx_{\varepsilon} |T\rangle\langle T|^{\otimes r},\ \Lambda \in \mathrm{SO}\}$
  - Distillable T magic $\mathcal{M}_T(\rho) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \mathcal{M}_T^{\varepsilon}(\rho^{\otimes n})$

ESTMITAING MAGIC STATE DISTILLATION

For the qutrit T state $|T\rangle := \frac{1}{\sqrt{3}}(e^{2\pi i/9}|0\rangle + |1\rangle + e^{-2\pi i/9}|2\rangle)$, we have

$$\mathcal{M}_T^{\varepsilon}(\rho) \leq \frac{\min_{\sigma\in\mathcal{W}} D_H^{\varepsilon}(\rho\|\sigma)}{\log_2(1+2\sin(\pi/18))}, \quad \mathcal{M}_T(\rho) \leq \frac{\theta(\rho)}{\log_2(1+2\sin(\pi/18))}.$$

$$D_H^{\varepsilon}(\rho\|\sigma) := -\log_2 \min \operatorname{Tr} G\rho \quad \text{s.t.} \operatorname{Tr} G\sigma \geq 1-\varepsilon,\ 0 \leq G \leq \mathbb{1}.$$

- Motivations for distilling qutrit T states [Dawkins, Howard, PRL'15]
- Establish the fundamental limits of magic state distillation.
- The one-shot estimation can be computed efficiently via SDP.
- The asymptotic estimation can be computed efficiently via convex optimization.
- The approach also applies to distilling some other basic magic states.

MAIN IDEA

$$\max_{\sigma \in \mathcal{W}} \operatorname{tr} |T\rangle\langle T|^{\otimes n} \sigma$$

- Step 1: Estimate the maximum overlap between the n copies of T states and the set of free states. (Note that this quantity is also essential for the resource theories of entanglement and coherence.)

  - Based on the min-thauma $\theta_{\min}(\rho) := \min_{\sigma \in \mathcal{W}} D_0(\rho \,\|\, \sigma) := \min_{\sigma \in \mathcal{W}} [-\log_2 \operatorname{Tr} P_\rho \sigma]$
  - We have $\theta_{\min}(T) = \log_2(1 + 2\sin(\pi/18))$
  - Then, for any state in W, it holds that $\operatorname{Tr} |T\rangle\langle T| \sigma \leq (1 + 2\sin(\pi/18))^{-1}$
  - Furthermore, we prove the additivity of min-thauma, which implies that

$$\forall \sigma \in \mathcal{W}, \qquad \operatorname{Tr} |T\rangle\langle T|^{\otimes n} \sigma \leq (1 + 2\sin(\pi/18))^{-n}$$

- Step 2: Use the data processing inequality to establish the one-shot bound.

- The asymptotic bound utilizes the quantum Stein's lemma.

A FUNDAMENTAL DIFFERENCE BETWEEN RESOURCE THEORIES OF MAGIC AND ENTANGLEMENT

- In entanglement theory, the states with maximum entanglement measure (e.g., log negativity) can be transformed to each other at **a rate equal to one**.
- However, this is **not** the case in the resource theory of magic.
- The Strange state and the Norrell state each has maximum mana (log negativity)

$$|\mathbb{S}\rangle := (|1\rangle - |2\rangle)/\sqrt{2}, \quad |\mathbb{N}\rangle := (-|0\rangle + 2|1\rangle - |2\rangle)/\sqrt{6}$$

$$\mathcal{M}(\mathbb{S}) = \mathcal{M}(\mathbb{N}) = \log_2(5/3)$$

  and is thus the most costly resource to simulate on a classical computer (Pashayan et al.'15).

- Using the thauma measures, we prove that it is **not possible** to transform the Norrell state to the Strange state at a rate equal to one, even asymptotically:

$$R(\mathbb{N} \to \mathbb{S}) \leq \frac{\log_2(3/2)}{\log_2(5/3)} < 1$$

- Also answers an open question in [Veitch, Mousavian, Gottesman, Emerson'14].

Quantifying the magic of quantum channels

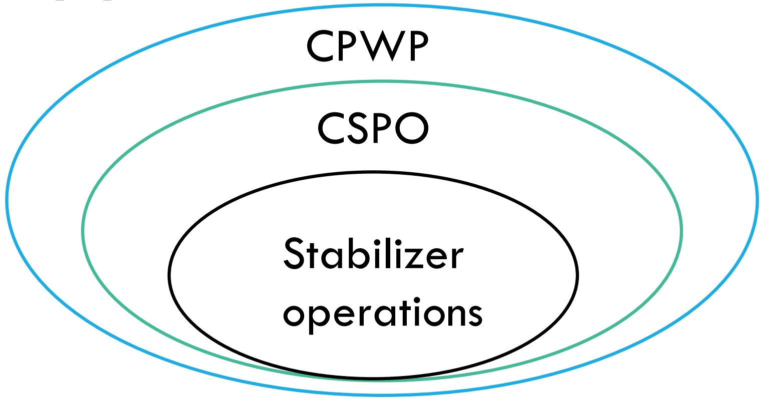
FREE OPERATIONS BEYOND STABILIZER OPERATIONS

	Magic	Entanglement (PPT)
Free states	States with positive W func.	PPT states
Free operations	?	PPT operations (completely preserve PPT)

- Completely Positive-Wigner-Preserving operations (CPWP)
  - **Intuition:** A quantum circuit consisting of an initial quantum state, unitary evolutions, and measurements, each having non-negative Wigner functions, can be classically simulated.
  - A Hermiticity-preserving linear map Π is called **completely positive Wigner preserving (CPWP)** if

$$\forall \rho_{RA} \in \mathcal{W}_+, \quad (\mathrm{id}_R \otimes \Pi_{A\to B})(\rho_{RA}) \in \mathcal{W}_+.$$

- The completely stabilizer-preserving operations (CSPO) were introduced in [Seddon, Campbell'19].



CPWP OPERATIONS

- Wigner function of a channel N (Mari, Eisert'12): $W_{\mathcal{N}}(\mathbf{v}|\mathbf{u}) = \mathrm{Tr}[\mathcal{N}_{A\to B}(A_A^{\mathbf{u}})A_B^{\mathbf{v}}]/d_B$

Theorem The following about CPWP operations are equivalent:

- N is CPWP;
- The discrete Wigner function of the Choi matrix of N is non-negative;
- $W_{\mathcal{N}}(\mathbf{v}|\mathbf{u})$ is non-negative for all **u** and **v** (a classical channel).

- The equivalence between 2 and 3 was previously proved by (Mari,Eisert'12).
- Why we need Completely Positive-Wigner-Preserving operations?
- The qutrit Werner-Holevo channel [Werner, Holevo'02] $\mathcal{N}_{\mathrm{WH}}(V) = \frac{1}{2}[(trV)I - V^T]$. erasures the magic if we do not consider ancillary system: $\forall\rho,\ \mathcal{N}_{\mathrm{WH}}(\rho) \in \mathcal{W}_+$. This channel is Positive-Wigner-Preserving but it can indeed generate magic.
- Another motivation: this class of channels can be classically simulated **efficiently.**

MANA OF A QUANTUM CHANNEL

- To quantify the magic of a quantum channel, we introduce the **mana** of a quantum channel $\mathcal{N}_{A\to B}$:

$$\mathcal{M}(\mathcal{N}_{A\to B}) := \log \max_{\mathbf{u}} \|\mathcal{N}_{A\to B}(A_A^{\mathbf{u}})\|_{W,1} = \log \max_{\mathbf{u}} \sum_{\mathbf{v}} |W_{\mathcal{N}}(\mathbf{v}|\mathbf{u})|.$$

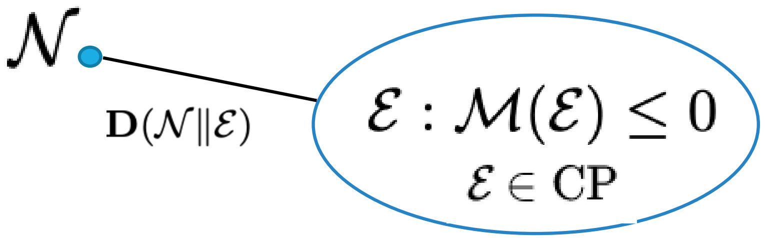
- The mana of a quantum channel has many desirable properties
  1. Reduction to states: for $\mathcal{N}(\rho) = \mathrm{Tr}[\rho]\sigma$, $\mathcal{M}(\mathcal{N}) = \mathcal{M}(\sigma)$.
  2. Faithfulness
  3. Additivity under tensor product: $\mathcal{M}(\mathcal{N}_2 \otimes \mathcal{N}_1) = \mathcal{M}(\mathcal{N}_1) + \mathcal{M}(\mathcal{N}_2)$
  4. Sub-additivity: $\mathcal{M}(\mathcal{N}_2 \circ \mathcal{N}_1) \leq \mathcal{M}(\mathcal{N}_1) + \mathcal{M}(\mathcal{N}_2)$
  5. Monotonicity under CPWP operations.
  6. Quantifies the classical simulation cost of the channel (in the next part).

THAUMA OF A QUANTUM CHANNEL

- Generalized thauma of a quantum channel

$$\boldsymbol{\theta}(\mathcal{N}) = \inf_{\mathcal{E}:\mathcal{M}(\mathcal{E})\leq 0} \mathbf{D}(\mathcal{N}\|\mathcal{E})$$

where the optimization is over CP maps.



- Key idea: **<u>channel divergence</u>**

- Properties
  1. Reduction to states: for $\mathcal{N}(\rho) = \mathrm{Tr}[\rho]\sigma$, we have $\boldsymbol{\theta}(\mathcal{N}) = \boldsymbol{\theta}(\sigma)$
  2. Monotonicity under CPWP operations
  3. Faithfulness

- Provide us the opportunity to explore the magic of quantum channels by choosing specific divergence.

MAX-THAUMA OF A QUANTUM CHANNEL

- A useful measure in this family is the max-thauma

$$\theta_{\max}(\mathcal{N}) = \min_{\mathcal{E}:\mathcal{M}(\mathcal{E})\leq 0} D_{\max}(\mathcal{N}\|\mathcal{E}),$$

where the minimum is taken with respect to all completely positive maps and

$$D_{\max}(\mathcal{N}\|\mathcal{E}) = D_{\max}(\mathcal{N}_{A\to B}(\Phi_{RA})\|\mathcal{E}_{A\to B}(\Phi_{RA})) = \log\min\{t : J^{\mathcal{N}}_{AB} \leq tJ^{\mathcal{E}}_{AB}\}$$

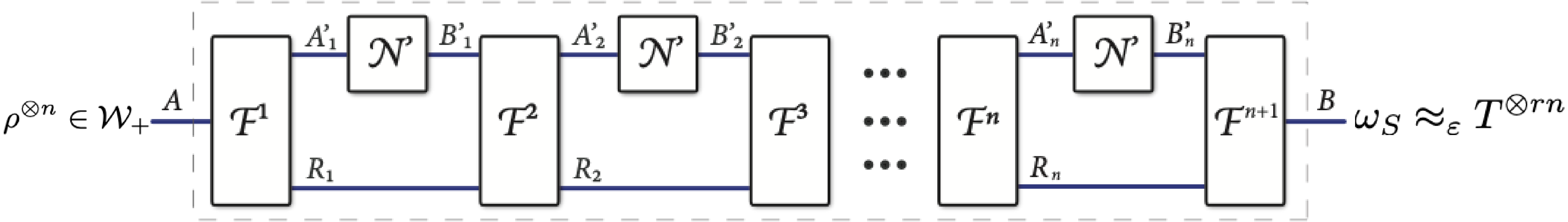
is the max-divergence of channels.

- The max-thauma of a quantum channel has additional desirable properties
  1. Additivity under tensor product
  2. Sub-additivity
  3. Efficiently computable via semidefinite programming

$$\begin{aligned} \theta_{\max}(\mathcal{N}) = \log\min\ & t \\ \text{s.t. } & J^{\mathcal{N}}_{AB} \leq Y_{AB} \\ & \sum_{\mathbf{v}} |\operatorname{Tr}[(A^{\mathbf{u}}_A \otimes A^{\mathbf{v}}_B)Y_{AB}]|/d_B \leq t, \quad \forall \mathbf{u}, \end{aligned}$$

DISTILLING MAGIC VIA CHANNELS

The most general protocol for distilling magic from a quantum channel:



$\mathcal{F}^k \in \mathrm{CPWP}$

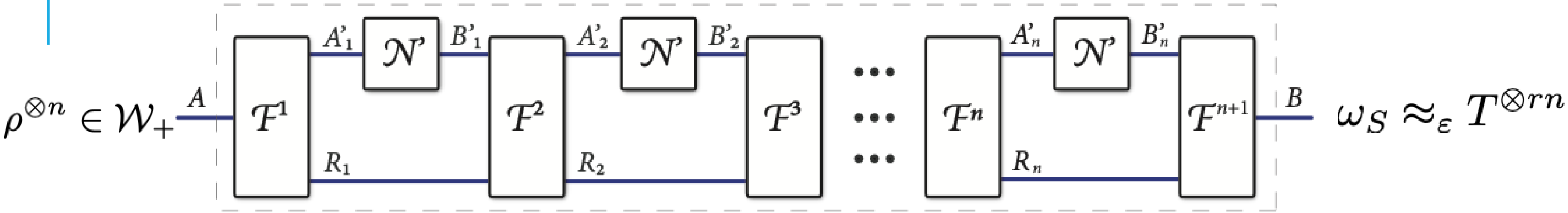
For the above procedure, the rate of magic distillation satisfies that

$$r \leq \frac{1}{\log(1+2\sin(\pi/18))}\left(\theta_{\max}(\mathcal{N}) + \frac{\log(1/[1-\varepsilon])}{n}\right).$$

Moreover, the asymptotic magic generating capacity is bounded by

$$C_T(\mathcal{N}) \leq \frac{\theta_{\max}(\mathcal{N})}{\log(1+2\sin(\pi/18))}.$$

DISTILLING MAGIC VIA CHANNELS (PROOF)



1. Assume that the final state is ω_S and $\omega_S \approx_\varepsilon T^{\otimes k}$.
2. Applying the data processing inequality for the max-relative entropy,

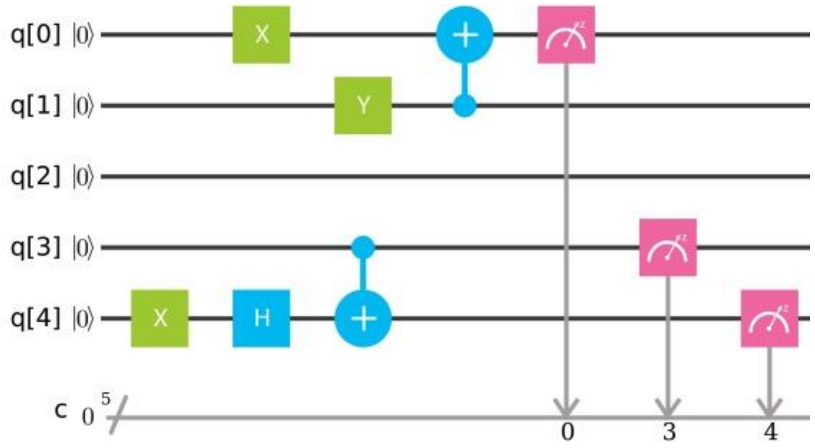
$$\theta_{\max}(\omega_S) \geq \log(1-\varepsilon) + k\log(1+2\sin(\pi/18)).$$

3. The **subadditivity** of max-thauma allows us to estimate the power of sequential protocols:

$$\begin{aligned}\theta_{\max}(\omega_S) &= \theta_{\max}(\mathcal{F}^{(1)} \circ \mathcal{N} \circ \mathcal{F}^{(2)} \circ \mathcal{N} \circ \cdots \circ \mathcal{N} \circ \mathcal{F}^{(n+1)}) \\ &\leq \sum_{i=1}^{n+1} \theta_{\max}(\mathcal{F}^{(i)}) + n\theta_{\max}(\mathcal{N}) \\ &= n\theta_{\max}(\mathcal{N}).\end{aligned}$$

4. Based on the above two inequalities, we have $r = k/n \leq \dfrac{1}{\log(1+2\sin(\pi/18))}\left(\theta_{\max}(\mathcal{N}) + \dfrac{\log(1/[1-\varepsilon])}{n}\right).$

GATE COST IN QC ARCHITECTURES



- Clifford group: easy to implement, cheap

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \quad C_X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

- T gate (expensive)
- Clifford + T $\Rightarrow$ Universal QC.

How many T gates are required to implement certain quantum circuits under FTQC?
What about the case with noise?

- Our focus: How much magic is needed to perform noisy/noiseless quantum circuits when stabilizer operations or their beyond are free?

MAGIC COST OF A QUANTUM CHANNEL

For any qudit quantum channel $\mathcal{N}$, the number of channels $\mathcal{E}$ required to implement it is bounded from below as follows:

$$S_{\mathcal{E}}(\mathcal{N}) \geq \max\left\{\frac{\mathcal{M}(\mathcal{N})}{\mathcal{M}(\mathcal{E})}, \frac{\theta_{\max}(\mathcal{N})}{\theta_{\max}(\mathcal{E})}\right\}.$$

- Applications to the gate synthesis of elementary gates.
- To implement a controlled-controlled-X gate, at least four T gates are required.

$$S_T(CCX) \geq \frac{\mathcal{M}(CCX)}{\mathcal{M}(|T\rangle\langle T|)} \geq \frac{2.1876}{0.6657} \geq 3.2861.$$

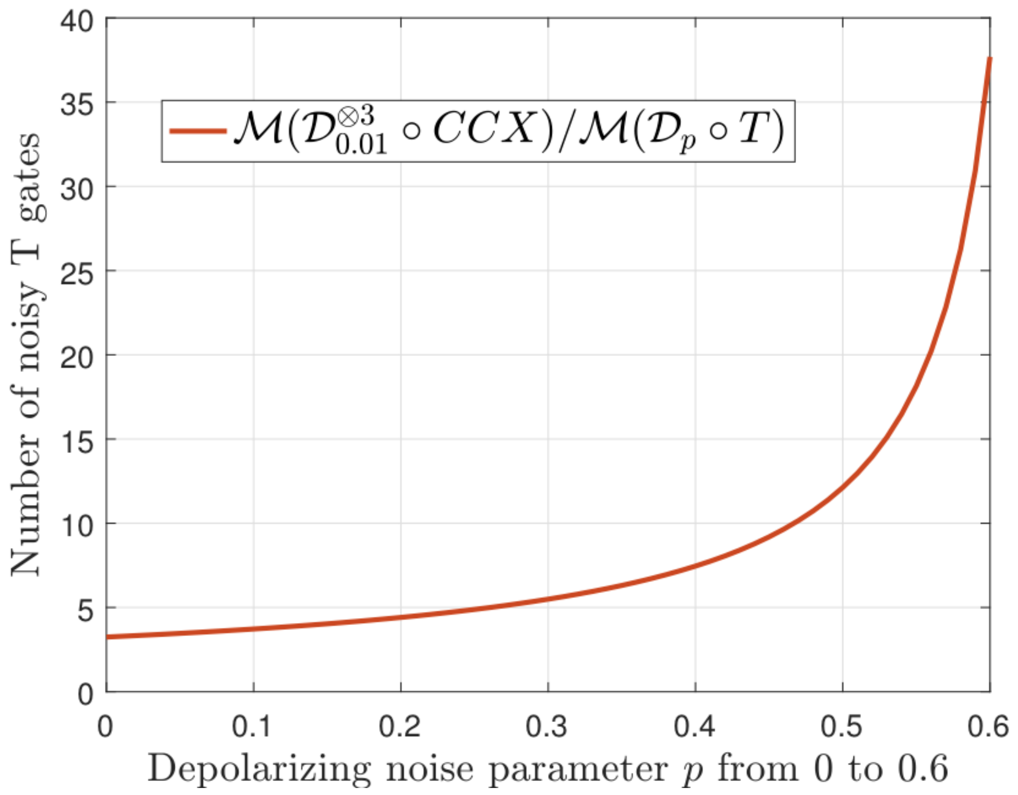
- The approaches in [Howard, Campbell'16] focused on noiseless quantum circuits. However, our lower bound can be applied to NISQ devices.
- We also have preliminary study on the magic cost of approximate gate simulation (gate synthesis with error tolerance).

MAGIC COST UNDER NOISE

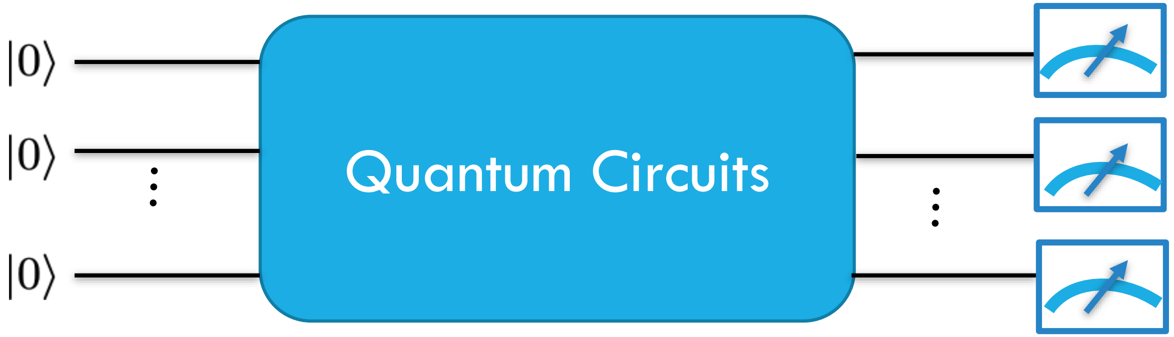
- One common noise model in quantum information processing is the depolarizing channel:

$$\mathcal{D}_p(\rho) = (1-p)\rho + \frac{p}{d^2-1} \sum_{\substack{0\le i,j\le d-1 \\ (i,j)\neq(0,0)}} X^i Z^j \rho (X^i Z^j)^\dagger.$$

- Lower bound on the number of noisy T gates to implement a noisy CCX gate under the depolarizing noise (p=0.01).
- The gate cost will significantly increase if error is beyond some threshold.
- It will be interesting to explore the magic cost under other noise models.



CLASSICAL SIMULATION OF QUANTUM CIRCUITS



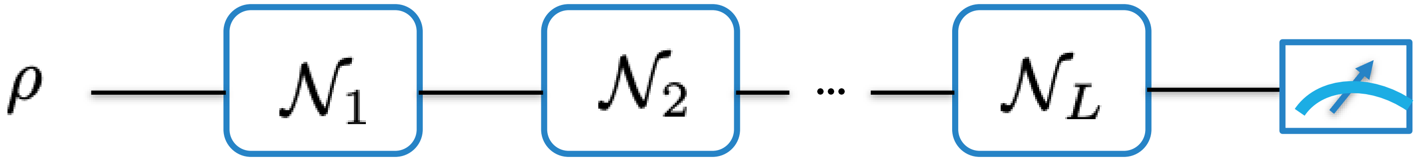
$$z \in \{0,1\}^n$$

$$p(z) = |\langle z|U|0\rangle^{\otimes n}|$$

- Strong simulation: given z, compute p(z).
- Weak simulation: sample a bit string from the distribution p.
- Recent extension of the Gottesman-Knill theorem [Bravyi, Gosset'16].
- Apply Monte Carlo sampling techniques to a quasiprobability representation [Pashayan, Wallman, Bartlett'15].
- Most algorithms have exponential scaling in the number of qubits.
- However, the above approaches cannot be directly applied for noisy quantum circuits.

CLASSICAL SIMULATION OF NOISY CIRCUITS.

- Inspiration from [PWB15]: we could estimate the outcome probabilities of quantum circuits using quasi-probabilities.
- Our contribution: extend the PWB algorithm to noisy quantum circuits.



Our goal is to estimate

$$\mathrm{Tr}\left[E(\mathcal{N}_L \circ \cdots \circ \mathcal{N}_1)(\rho)\right]$$

$$= \sum_{\vec{\mathbf{u}}} W(E|\mathbf{u}_L) \prod_{l=1}^{L} W_{\mathcal{N}_l}(\mathbf{u}_l|\mathbf{u}_{l-1}) W_\rho(\mathbf{u}_0),$$

- We can reformulate both the state preparation and the measurement as quantum channels. W.l.o.g., we could assume $\rho = |0^n\rangle\langle 0^n|, E = |0\rangle\langle 0|.$

1. Sample the initial phase point $\mathbf{u}_0$ according to the distribution $|W_{|0^n\rangle\langle 0^n|}(\mathbf{u}_0)|/\mathcal{M}_{|0^n\rangle\langle 0^n|}$
2. For l=1,…L, we sample a phase point $\mathbf{u}_l$ according to the the conditional distribution $|W_{\mathcal{N}_l}(\mathbf{u}_l|\mathbf{u}_{l-1})|/\mathcal{M}_{\mathcal{N}_l}(\mathbf{u}_{l-1}).$
3. Output the estimate $\mathcal{M}_{|0^n\rangle\langle 0^n|}\mathrm{Sign}\left[W_{|0^n\rangle\langle 0^n|}(\mathbf{u}_0)\right] \prod_{l=1}^{L} \mathcal{M}_{\mathcal{N}_l}(\mathbf{u}_{l-1})\mathrm{Sign}\left[W_{\mathcal{N}_l}(\mathbf{u}_l|\mathbf{u}_{l-1})\right] W(|0\rangle\langle 0||\mathbf{u}_L).$

- This gives an unbiased estimate of the output probability since $\mathbb{E}[\quad] = \mathrm{Tr}\left[|0\rangle\langle 0|(\mathcal{N}_L \circ \cdots \circ \mathcal{N}_1)(|0^n\rangle\langle 0^n|)\right].$

COMPARISON WITH OTHER METHODS

- Cost of samples $\frac{2}{\epsilon^2}\left(\prod_{l=1}^{L}\mathcal{M}_{\mathcal{N}_l}\right)^2 \log\left(\frac{2}{\delta}\right)$ for accuracy ϵ and success probability $1-\delta$.

Lower bound for the cost via the approaches in [Seddon, Campbell'19]

- [Seddon, Campbell'19] introduced classical algorithms to simulate noisy quantum circuits with cost

$$\frac{2}{\epsilon^2}\mathcal{R}_*(\mathcal{N})^2 \log\left(\frac{2}{\delta}\right), \quad \frac{2}{\epsilon^2}C(\mathcal{N})^2 \log\left(\frac{2}{\delta}\right),$$

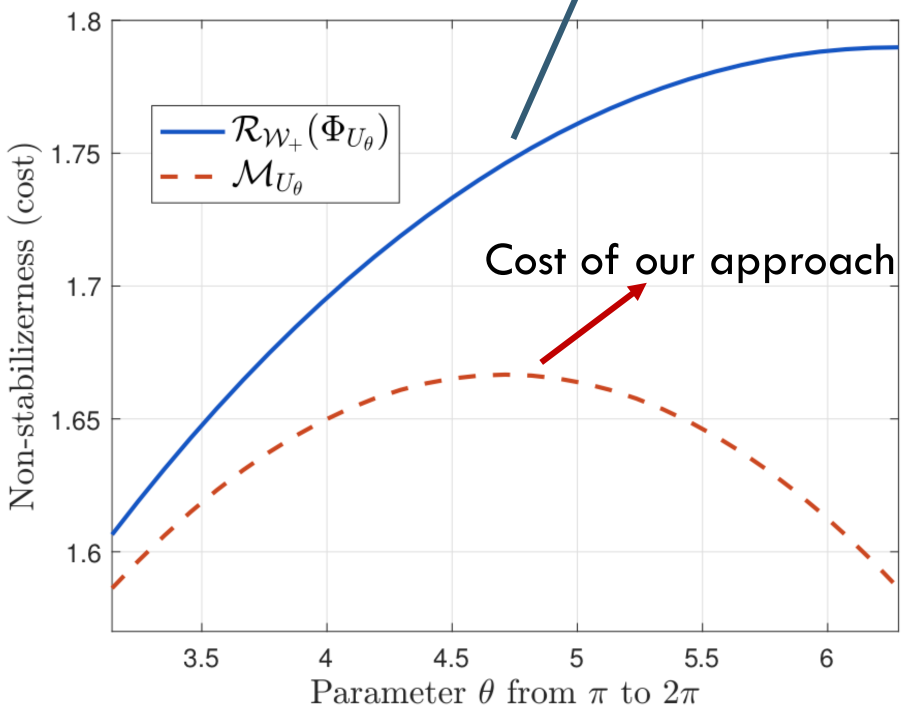
for accuracy ϵ and success probability $1-\delta$.

- Our approach can always outperform their first one since

$$\mathcal{M}_{\mathcal{N}} \leq \mathcal{R}_*(\mathcal{N}), \quad \forall \mathcal{N}.$$

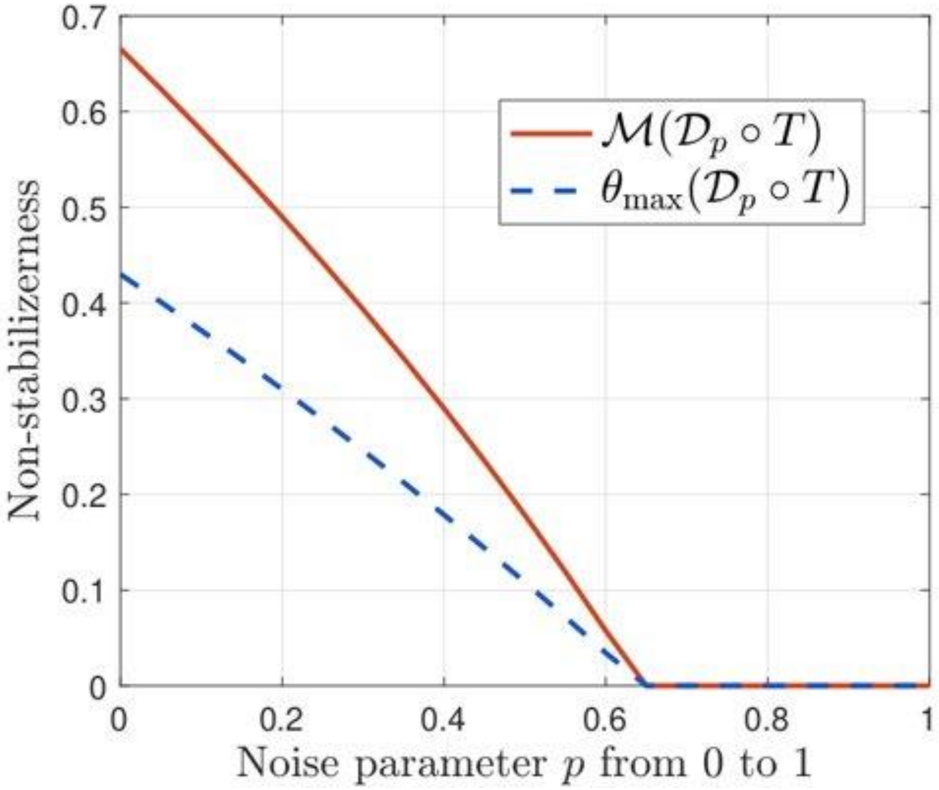
- Our approach can outperform the second one for the unitary

$$U_\theta = \begin{pmatrix} e^{i\theta/9} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\theta/9} \end{pmatrix}.$$



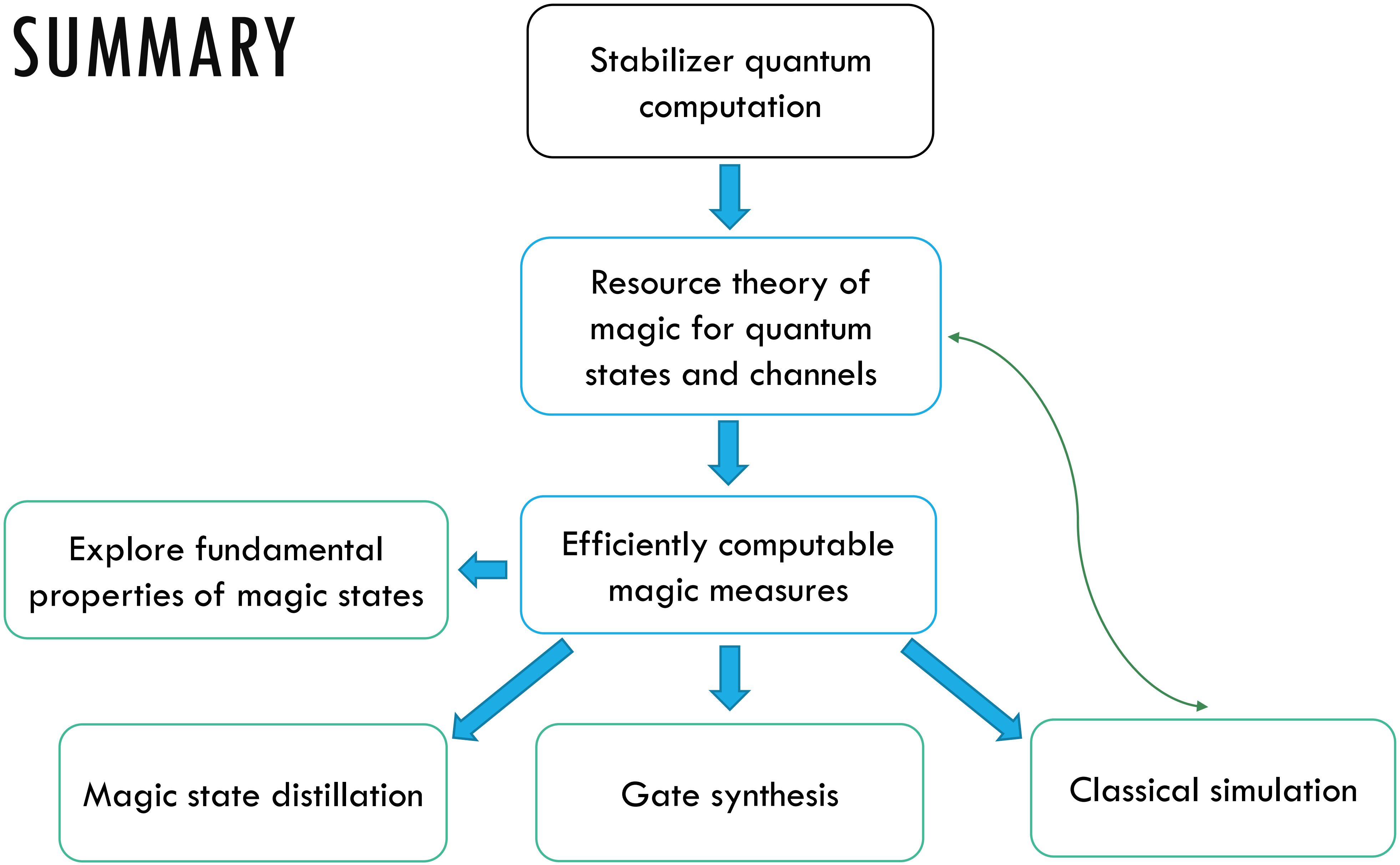
AN EXAMPLE

- Estimate the magic of fundamental noisy quantum circuits.
- Red solid line: Lower bound on the classical simulation cost.
- Blue dashed line: Upper bound on the magic generating capacity.



Summary and outlook

SUMMARY


Stabilizer quantum computation
Resource theory of magic for quantum states and channels
Explore fundamental properties of magic states
Efficiently computable magic measures
Magic state distillation
Gate synthesis
Classical simulation

OUTLOOK

- Generalize our approach to study the resource theory of non-Gaussianity.
- Further study on the magic of multi-qubit quantum operations, in particular, the ability of magic state distillation and classical simulation.
  See [Seddon, Campbell'19] for recent progresses.
- Magic state conversion (like the majorization of pure states in ent. theory)?
  For example, can we find necessary and/or sufficient condition for pure magic state conversion under (stabilizer operations/PWP operations)?
- Better understand the catalysis-assisted magic state manipulation [Campbell'11].
- Exact and approximate unitary synthesis with a focus on important and fundamental circuits (see [Beverland et al.' 19] for recent progresses).

QUESTION ON STABILIZER RANK

Stabilizer rank [Garcia, Markov, Cross'12; Bravyi, Smith, Smolin'16]

Recent progress can be found in [Bravyi et al.'2018].

That stab rank of $\chi(\psi)$ is the smallest integer k such that

Stabilizer states.

$$|\psi\rangle = \sum_{\alpha=1}^{k} c_\alpha |v_k\rangle$$

- This can be understood as the magic version of Schmidt rank in entanglement theory.
- The Gottesman-Knill theorem can be extended to simulation of an arbitrary state ψ undergoing a Clifford circuit with runtime proportional to $\chi(\psi)$.
- If quantum computers are more powerful than classical ones, we then should expect $\chi(\psi^{\otimes n})$ to scale exponentially with n.
- Open question [Bravyi, Smith, Smolin'15]: can we find an exponential lower bound for the stabilizer rank? $\forall \psi \notin \mathrm{Stab}, \exists g > 1, \text{ s.t. } \chi(\psi^{\otimes n}) \geq g^n.$

Thank you for your attention!

See arXiv:1903.04483 & 1812.10145 for further details.

BOUNDS FOR MAGIC STATE DISTILLATION

[WWS'18] For the qutrit T state $|T\rangle := \frac{1}{\sqrt{3}}(e^{2\pi i/9}|0\rangle + |1\rangle + e^{-2\pi i/9}|2\rangle)$, we have

$$\mathcal{M}_T^\varepsilon(\rho) \leq \frac{\min_{\sigma\in\mathcal{W}} D_H^\varepsilon(\rho\|\sigma)}{\log_2(1+2\sin(\pi/18))}, \quad \mathcal{M}_T(\rho) \leq \frac{\theta(\rho)}{\log_2(1+2\sin(\pi/18))}.$$

- Establish the fundamental limits of magic state distillation.
- The estimation can be computed efficiently via convex optimization. (use the min-thauma)
- Proof strategy
  1. Prove the maximum overlap between $\rho^{\otimes n} \in \mathcal{W}$ and $T^{\otimes n}$ is $\frac{1}{(1+2\sin(\pi/18))^n}$
  2. Apply the hypothesis testing and data processing inequality
  3. Minimize over $\sigma \in \mathcal{W}$
  4. Apply the Quantum Stein's Lemma to obtain the asymptotic bound

$$\begin{aligned} k\log_2(1+2\sin(\pi/18)) &\leq D_0(|T\rangle\langle T|^{\otimes k}\|\Lambda(\sigma)) \\ &\leq D_H^\varepsilon(\Lambda(\rho)\|\Lambda(\sigma)) \\ &\leq D_H^\varepsilon(\rho\|\sigma). \end{aligned}$$

Also indicates that the stabilizer fidelity of T state is multiplicative.