When is the conic hull of a curve a hyperbolicity cone?

Mario Kummer

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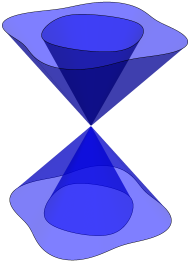

joint work with Rainer Sinn

Hyperbolic Polynomials

A homogeneous polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]_d$ is hyperbolic with respect to a point $e \in \mathbb{R}^n$ if $f(e) \neq 0$ and for every $v \in \mathbb{R}^n$, all roots of $f(te - v) \in \mathbb{R}[t]$ are real.

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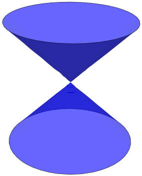

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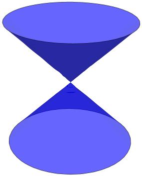

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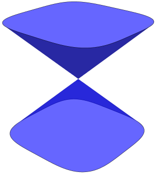

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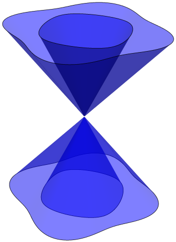
Hyperbolicity Cones

Its hyperbolicity cone, denoted $C(f, e)$, is the set of all $v \in \mathbb{R}^n$ where $f(te - v) \in \mathbb{R}[t]$ has only nonnegative roots.



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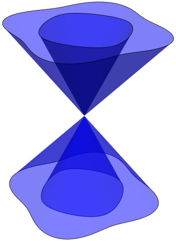

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Gårding showed that

- $C(f, e)$ is convex.
- $C(f, e)$ is the closure of the connected component of e in $\{x \in \mathbb{R}^n : f(x) \neq 0\}$.
- f is hyperbolic with respect to any point $a \in \operatorname{int} C(f, e)$.

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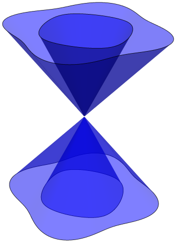

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One can use interior point methods to optimize a linear function over an affine section of a hyperbolicity cone (Güler, Renegar). This solves a *hyperbolic program*.

Example

The determinant $\det : \mathrm{Sym}_n \to \mathbb{R}$ of symmetric matrices is hyperbolic with respect to the identity matrix I_n:

- $\det(tI_n - X)$ has only real zeros for every symmetric matrix $X \in \mathrm{Sym}_n$.
- The hyperbolicity cone is the set Sym_n^+ of positive semidefinite matrices.

Why convex hulls?

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Let $f \in \mathbb{R}[x_1, \ldots, x_n]_d$ be strictly hyperbolic with respect to a point $e \in \mathbb{R}^n$, i.e., for every $v \in \mathbb{R}^n$, all roots of $f(te - v) \in \mathbb{R}[t]$ are real and distinct.

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If $C(f, e)$ is the covex hull of some low-dimensional set, then f is usually far from being strictly hyperbolic.

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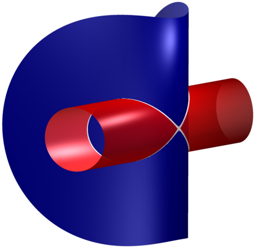
- $\mathbb{R}^n$ is also contained in $\mathbb{R}^{n+1}$ as the closed subet of all points $(1, x_1, \dots, x_n)$: Hyperbolicity cones live in $\mathbb{R}^{n+1}$.

Curves in space

Theorem. (K., Sinn) Let $X \subset \mathbb{R}^3$ be a one-dimensional semi-algebraic set. Assume that the closed convex hull $\overline{\operatorname{conv}(X)}$ of X is (the affine slice $x_0 = 1$ of) the hyperbolicity cone of some $f \in \mathbb{R}[x_0, x_1, x_2, x_3]$. Then for every irreducible factor f_0 of f there exists an invertible linear change of coordinates T such that $f_0(Tx) \in \mathbb{R}[x_0, x_1, x_2]$.



Convex hull of a rational quartic. From: "On the convex hull of a space curve" by Ranestad, Sturmfels.

Secant Varieties

Let $X \subset \mathbb{P}^n$ be a projective variety. The kth secant variety $\sigma_k(X)$ is the Zariski closure of the union of all linear spaces spanned $k+1$ points on X.

Secant Varieties

Example. Let $C \subset \mathbb{P}^{2n}$ be the rational normal curve of degree $2n$. Then $\sigma_k(C)$ is cut out by the $(k+2) \times (k+2)$-minors of the Hankel matrix

$$H(x) = \begin{pmatrix} x_0 & x_1 & x_2 & \cdots & x_n \\ x_1 & x_2 & & & \\ x_2 & & \ddots & & \vdots \\ \vdots & & & \ddots & \\ x_n & & \cdots & & x_{2n} \end{pmatrix}$$

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Thus $\sigma_{n-1}(C)$ is the hypersurface cut out by the hyperbolic polynomial $\det H(x)$. Its hyperbolicity cone is the convex hull of C.

Hyperbolic Varieties

A projective variety $X \subset \mathbb{P}^n$ is hyperbolic with respect to a linear subspace $E \subset \mathbb{P}^n$ of dimension $n - \dim X - 1$ if $X \cap E = \emptyset$ and for every linear subspace $E \subset H$ with $\dim H = \dim E + 1$, all points in $X \cap H$ are real.

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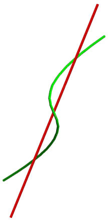

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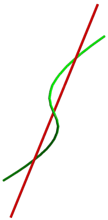

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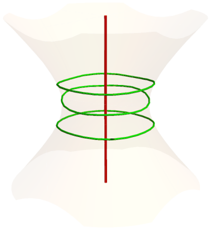
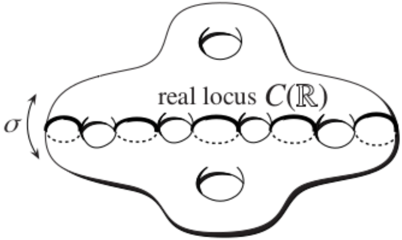


A space sextic

Hyperbolic Curves

Theorem. (K., Shamovich) Let C be a smooth, geometrically irreducible, projective, real curve. Then the following are equivalent:

- C can be embedded to $\mathbb{P}^n$ as a hyperbolic curve for some n.
- There is a morphism $f : C \to \mathbb{P}^1$ with $f^{-1}(\mathbb{P}^1(\mathbb{R})) = C(\mathbb{R})$.
- $C(\mathbb{C}) \setminus C(\mathbb{R})$ is not connected.



Riemann surface of dividing type. From: "Ahlfors circle maps and total reality: from Riemann to Rohlin", Gabard.

Hyperbolic Secant Varieties

Lemma. (K., Sinn) Let $C \subset \mathbb{P}^n$ be an irreducible nondegenerate real curve with $C(\mathbb{R})$ Zariski dense in C. Suppose that $\sigma_k(C) \neq \mathbb{P}^n$ and let $E \subset \mathbb{P}^n$ be a real linear subspace of codimension $2k+2$ with $E \cap \sigma_k(C) = \emptyset$. The following are equivalent:

- $\sigma_k(C)$ is hyperbolic with respect to E.
- Every hyperplane $H \subset \mathbb{P}^n$ with $E \subset H$ intersects C in at most $2k$ non-real points.

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Note that the latter corresponds to a linear subspace $V \subset \Gamma(C, \mathcal{O}_C(1))$ of dimension $2k+2$ such that every section $s \in V$ has at most $2k$ non-real zeros on C.

Example

Let $C \subset \mathbb{P}^{2n}$ be the rational normal curve of degree $2n$. Finding a linear space with respect to which $\sigma_k(C)$ is hyperbolic amounts finding a vector space $V \subset \mathbb{R}[t]_{\leq 2n}$ of dimension $2k+2$ such that every $f \in V$ has at most $2k$ non-real zeros.

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- Thus H is defined by a linear form $L : \mathbb{R}[t]_{\leq 2n} \to \mathbb{R}$ that is nonnegative on the cone of nonnegative polynomials.
- This means the bilinear form on $\mathbb{R}[t]_{\leq d}$ defined by $(f, g) \mapsto L(f \cdot g)$ is positive semidefinite. Its representing matrix is the Hankel matrix $H(x)$ that we have already seen.

A more general construction

Theorem. (K., Sinn) Let $C \subset \mathbb{P}^n$ be a smooth, irreducible, projectively normal, real curve of genus g. Assume that C is an M-curve, i.e., $C(\mathbb{R})$ has $g+1$ connected components. Assume furthermore that at most one component C_0 of $C(\mathbb{R})$ realizes the trivial homology class in $H_1(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z})$. Then every secant variety $\sigma_k(C)$ is hyperbolic.

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- The assumptions are satisfied, e.g., if C is a rational normal curve or if n is even and C is an elliptic normal curve such that $C(\mathbb{R})$ has two connected components.
- Curves satisfying the assumptions exist for any g and n.
- If $\sigma_k(C)$ is a hypersurface, then its hyperbolicity cone is the convex hull of C_0. Moreover, it is a simplicial cone.

Proof Idea for $\sigma_1(C)$ in $\mathbb{P}^4$

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- Thus we want to choose s_0, s_1, s_2, s_3 such that all $\lambda_0 s_0 + \ldots + \lambda_4 s_4$ have at least 2 zeros on the component C_0.

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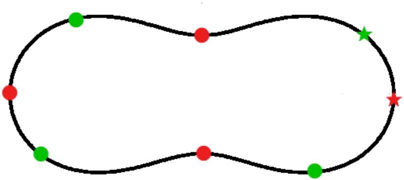

- s_0=red dots, red star
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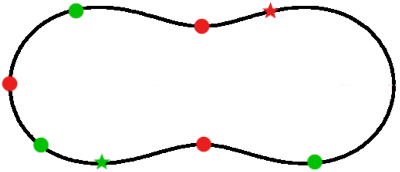

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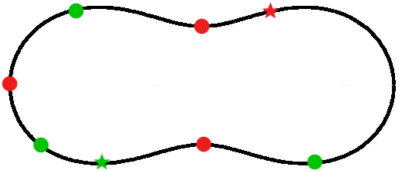

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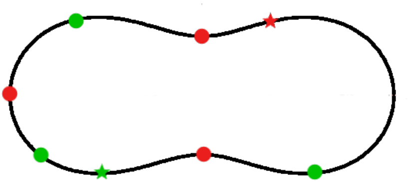

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- Thus $a + b$ has at least two real zeros on C_0

The Polymatroid

Let $C \subset \mathbb{P}^4$ be a curve of genus g that satisfied the assumptions of the theorem.

Then $\sigma_1(C)$ is a hypersurface cut out by a polynomial h that is hyperbolic with respect to some $e \in \operatorname{conv}(C_0)$. Let $S \subset C_0$ be a finite subset:

$$\deg(h(e + t\sum_{x \in S} x)) = \begin{cases} 0, \text{ if } |S| = 0, \\ \frac{1}{2}(g^2 + g + 2), \text{ if } |S| = 1, \\ \frac{1}{2}(g^2 + 3g + 4), \text{ if } |S| = 2, \\ \frac{1}{2}(g^2 + 3g + 6), \text{ if } |S| \geq 3. \end{cases}$$

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Hyperbolic Shadows

Theorem. (K., Sinn) Let $S \subset \mathbb{R}^n$ be a connected component of an M-curve of degree d and genus g. Then $\operatorname{conv}(S)$ is (up to closure) the projection of an affine slice in $\mathbb{R}^m$ with $m \leq d+1$ of a hyperbolicity cone of degree at most $\sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} (j+1)\binom{g}{\lfloor \frac{m}{2} \rfloor - j}$.

Every elliptic curve C has an unramified $2:1$ cover $\tilde{C} \to C$ where $\tilde{C}$ is an elliptic M-curve. Therefore:

Corollary. Let $S \subset \mathbb{R}^n$ be a connected component of an elliptic curve of degree d. Then $\operatorname{conv}(S)$ is (up to closure) the projection of an affine slice in $\mathbb{R}^{2d}$ of a hyperbolicity cone of degree $2d+1$.

Example

Let $C \subset \mathbb{P}^2$ be the smooth cubic curve defined by the equation $x_0^3 - x_0 x_2^2 - x_1^2 x_2$. $C(\mathbb{R})$ has two connected components C_0, C_1. We embed C to $\mathbb{P}^4$ by the map

$$C \to \mathbb{P}^4, (x_0 : x_1 : x_2) \mapsto (x_0^2 : x_0 x_1 : x_0 x_2 : x_1 x_2 : x_2^2).$$

The convex hull of C_0 under this embedding is a hyperbolicity cone. It is the spectrahedron defined by:

$$\begin{pmatrix} -z_2 & z_0 & -z_1 & -z_1 & -z_3 \\ z_0 & -z_2 & 0 & z_3 & 0 \\ -z_1 & 0 & z_0 & 0 & z_2 \\ -z_1 & z_3 & 0 & -z_0 + z_4 & 0 \\ -z_3 & 0 & z_2 & 0 & z_4 \end{pmatrix} \succeq 0$$

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Question. Do these hyperbolic secant varieties all have a determinantal representation?