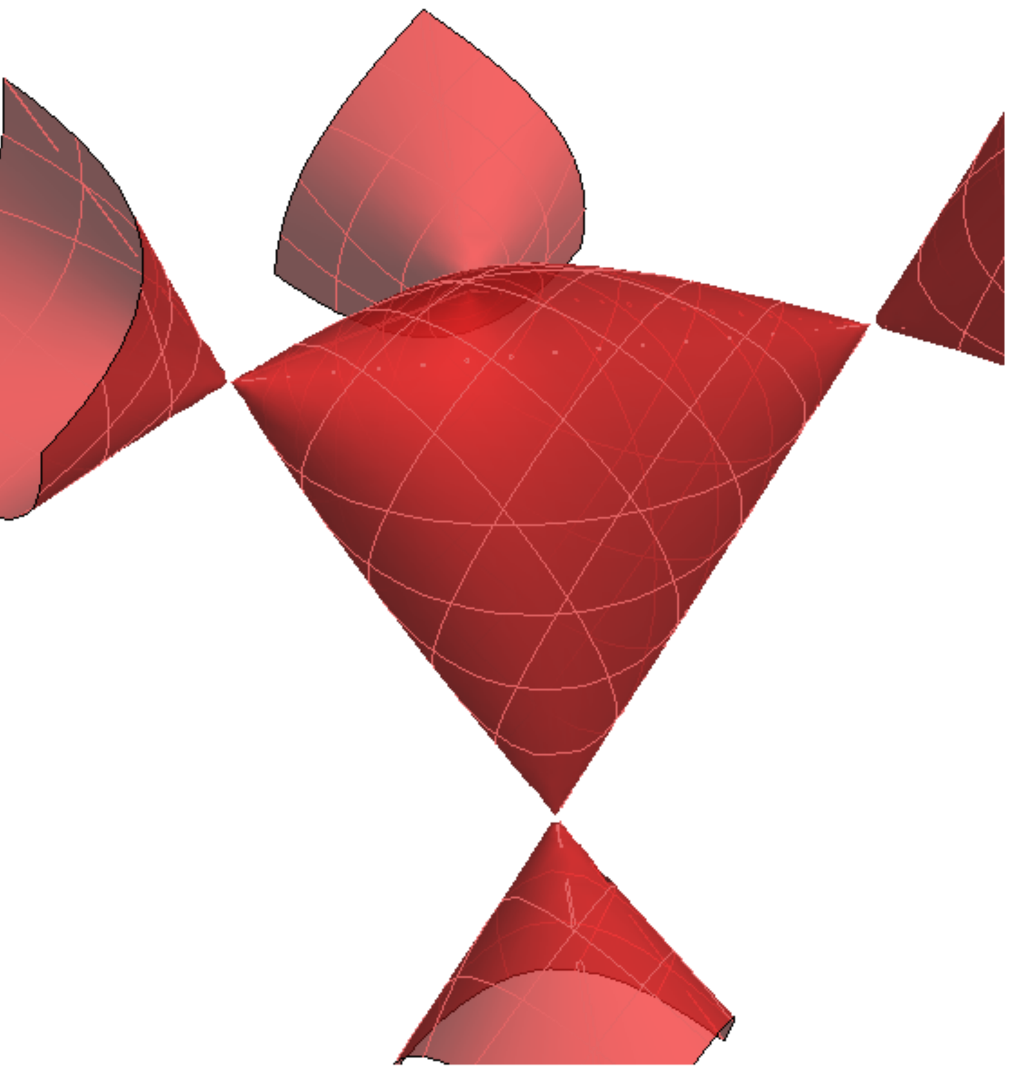
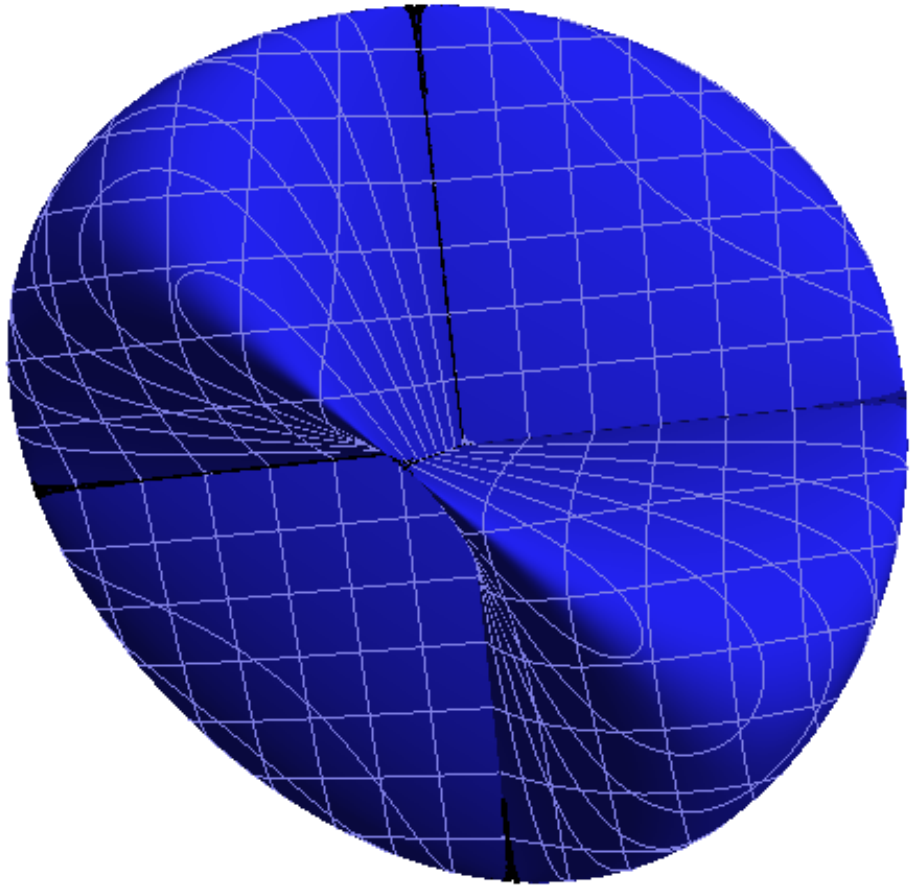
KIPPENHAHN'S THEOREM FOR THE JOINT NUMERICAL RANGE



Daniel Plaumann
(TU Dortmund)

Rainer Sinn
(FU Berlin)

Stephan Weis
(University of Coimbra)



Banff Workshop on
Geometry of Polynomials, Convexity and Optimization
Berkeley, 27 May, 2019

The numerical range

Let A be a complex $d \times d$-matrix.

The **numerical range** of A is the set

$$W(A) = \left\{ \overline{x^T} A x \;\middle|\; x \in \mathbb{C}^d \text{ with } \|x\| = 1 \right\} \subset \mathbb{C}$$

The numerical range

Let A be a complex $d \times d$-matrix.
The **numerical range** of A is the set

$$W(A) = \left\{ \overline{x^T} A x \;\middle|\; x \in \mathbb{C}^d \text{ with } \|x\| = 1 \right\} \subset \mathbb{C}$$

Remark. (1) $W(A)$ contains the eigenvalues of A.
(2) A is Hermitian if and only if $W(A)$ is a real line segment.
(3) If A is normal, $W(A)$ is the convex hull of the eigenvalues.

The numerical range

Let A be a complex $d \times d$-matrix.
The **numerical range** of A is the set

$$W(A) = \left\{ \overline{x^T} A x \;\middle|\; x \in \mathbb{C}^d \text{ with } \|x\| = 1 \right\} \subset \mathbb{C}$$

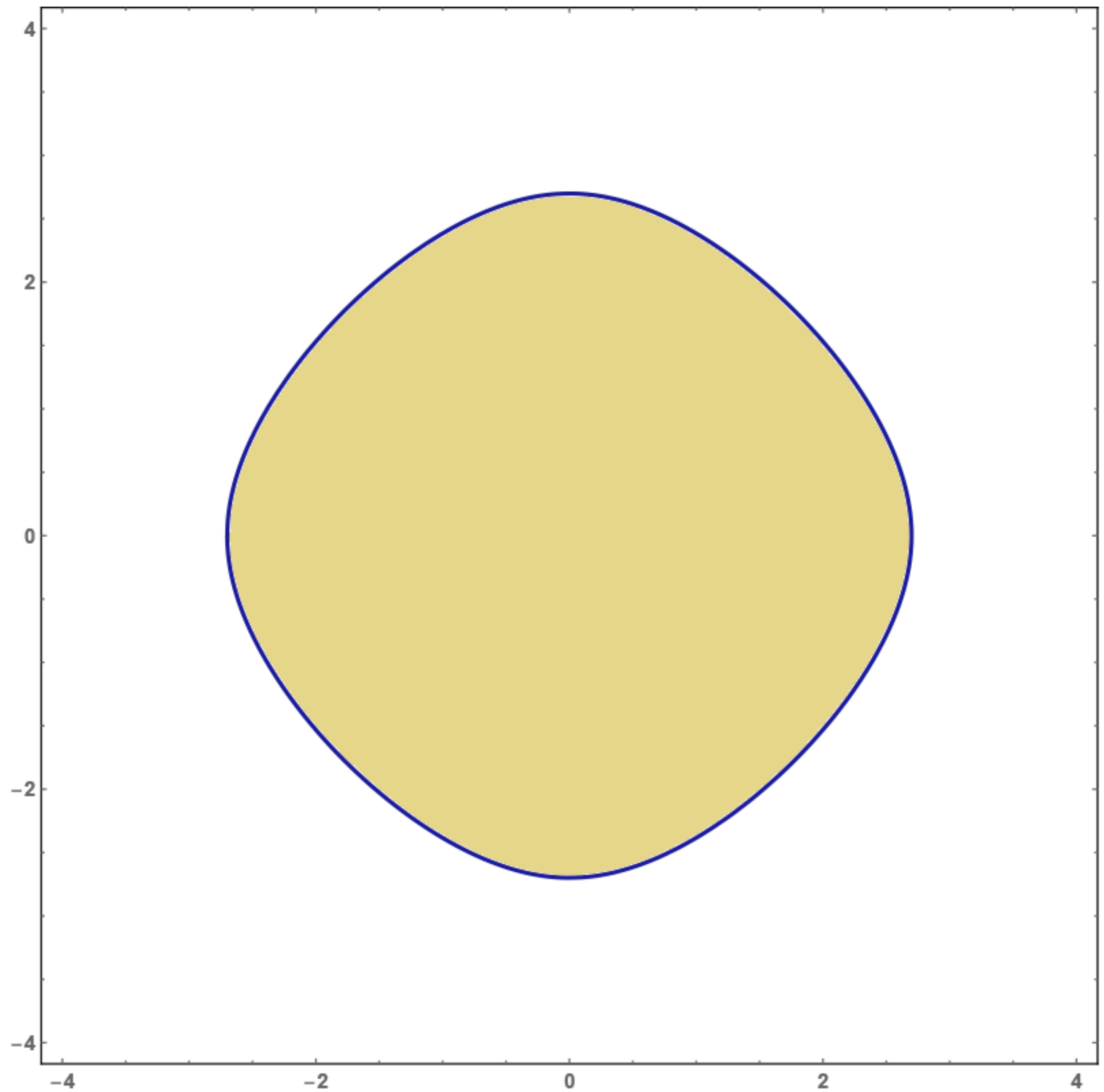
Remark. (1) $W(A)$ contains the eigenvalues of A.
(2) A is Hermitian if and only if $W(A)$ is a real line segment.
(3) If A is normal, $W(A)$ is the convex hull of the eigenvalues.

Toeplitz-Hausdorff Theorem (1919).
The set $W(A)$ is a convex subset of $\mathbb{C} = \mathbb{R}^2$.

The numerical range

Example.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 4 & 0 & 0 & 0 \end{pmatrix}$$



Trace trick

$$x^*Ax = \operatorname{tr}(x^*Ax) = \operatorname{tr}(A(xx^*)) = \langle A, xx^* \rangle$$

Trace trick

$$x^* A x = \operatorname{tr}(x^* A x) = \operatorname{tr}(A(xx^*)) = \langle A, xx^* \rangle$$

By the Toeplitz-Hausdorff Theorem:

$$\begin{aligned} W(A) &= \{\langle A, X\rangle : X \text{ Hermitian and psd}, \operatorname{tr}(X) = 1, \operatorname{rk}(X) = 1\} \\ &= \{\langle A, X\rangle : X \text{ Hermitian and psd}, \operatorname{tr}(X) = 1\} \\ &= \pi_A(\mathrm{Her}_d^+ \cap \{\operatorname{tr} = 1\}) \end{aligned}$$

Trace trick

$$x^*Ax = \mathrm{tr}(x^*Ax) = \mathrm{tr}(A(xx^*)) = \langle A, xx^* \rangle$$

By the Toeplitz-Hausdorff Theorem:

$$\begin{aligned} W(A) &= \{\langle A, X\rangle : X \text{ Hermitian and psd}, \mathrm{tr}(X) = 1, \mathrm{rk}(X) = 1\} \\ &= \{\langle A, X\rangle : X \text{ Hermitian and psd}, \mathrm{tr}(X) = 1\} \\ &= \pi_A(\mathrm{Her}_d^+ \cap \{\mathrm{tr} = 1\}) \end{aligned}$$

Define Hermitian matrices

$$\mathrm{Re}(A) = \frac{1}{2}\left(A + \overline{A}^T\right) \qquad \text{and} \qquad \mathrm{Im}(A) = \frac{1}{2i}\left(A - \overline{A}^T\right)$$

Trace trick

$$x^* A x = \operatorname{tr}(x^* A x) = \operatorname{tr}(A(xx^*)) = \langle A, xx^* \rangle$$

By the Toeplitz-Hausdorff Theorem:

$$\begin{aligned} W(A) &= \{\langle A, X\rangle : X \text{ Hermitian and psd}, \operatorname{tr}(X) = 1, \operatorname{rk}(X) = 1\} \\ &= \{\langle A, X\rangle : X \text{ Hermitian and psd}, \operatorname{tr}(X) = 1\} \\ &= \pi_A(\mathrm{Her}_d^+ \cap \{\operatorname{tr} = 1\}) \end{aligned}$$

Define Hermitian matrices

$$\mathrm{Re}(A) = \frac{1}{2}(A + \overline{A}^T) \qquad \text{and} \qquad \mathrm{Im}(A) = \frac{1}{2i}(A - \overline{A}^T)$$

$$\implies A = \mathrm{Re}(A) + i\,\mathrm{Im}(A)$$

Trace trick

$$x^* A x = \mathrm{tr}(x^* A x) = \mathrm{tr}(A(xx^*)) = \langle A, xx^* \rangle$$

By the Toeplitz-Hausdorff Theorem:

$$\begin{aligned} W(A) &= \{\langle A, X\rangle : X \text{ Hermitian and psd}, \mathrm{tr}(X) = 1, \mathrm{rk}(X) = 1\} \\ &= \{\langle A, X\rangle : X \text{ Hermitian and psd}, \mathrm{tr}(X) = 1\} \\ &= \pi_A(\mathrm{Her}_d^+ \cap \{\mathrm{tr} = 1\}) \end{aligned}$$

Define Hermitian matrices

$$\mathrm{Re}(A) = \frac{1}{2}\left(A + \overline{A}^T\right) \qquad \text{and} \qquad \mathrm{Im}(A) = \frac{1}{2i}\left(A - \overline{A}^T\right)$$

$$\implies A = \mathrm{Re}(A) + i\,\mathrm{Im}(A)$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 4 & 0 & 0 & 0 \end{pmatrix} \qquad \mathrm{Re}(A) = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 2 \\ \frac{1}{2} & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{3}{2} \\ 2 & 0 & \frac{3}{2} & 0 \end{pmatrix} \qquad \mathrm{Im}(A) = \begin{pmatrix} 0 & -\frac{i}{2} & 0 & 2i \\ \frac{i}{2} & 0 & -i & 0 \\ 0 & i & 0 & -\frac{3i}{2} \\ -2i & 0 & \frac{3i}{2} & 0 \end{pmatrix}$$

Kippenhahn's Theorem

Let A be a complex $d \times d$ matrix and let

$$p = \det\left(x_0 I_d + x_1 \mathrm{Re}(A) + x_2 \mathrm{Im}(A)\right)$$

with spectrahedron

$$S(A) = \left\{(a_1, a_2) \in \mathbb{R}^2 \;\middle|\; I_d + a_1 \mathrm{Re}(A) + a_2 \mathrm{Im}(A) \succeq 0\right\}$$

Kippenhahn's Theorem

Let A be a complex $d \times d$ matrix and let

$$p = \det\big(x_0 I_d \;+\; x_1 \mathrm{Re}(A) + \; x_2 \mathrm{Im}(A)\big)$$

with spectrahedron

$$S(A) = \big\{(a_1, a_2) \in \mathbb{R}^2 \;\big|\; I_d \;+\; a_1 \mathrm{Re}(A) + \; a_2 \mathrm{Im}(A) \succeq 0\big\}$$

Theorem. *(Kippenhahn 1951)*
The numerical range $W(A)$ is the convex dual

$$S(A)^\circ = \big\{(u_1, u_2) \in \mathbb{R}^2 \;\big|\; \langle u, a\rangle \geqslant -1 \text{ for all } a \in S(A)\big\}$$

of $S(A)$. It is the convex hull of the points (u_1, u_2) for which $[1, u_1, u_2]$ lies on the ***dual curve*** *of $V = \{p = 0\}$.*

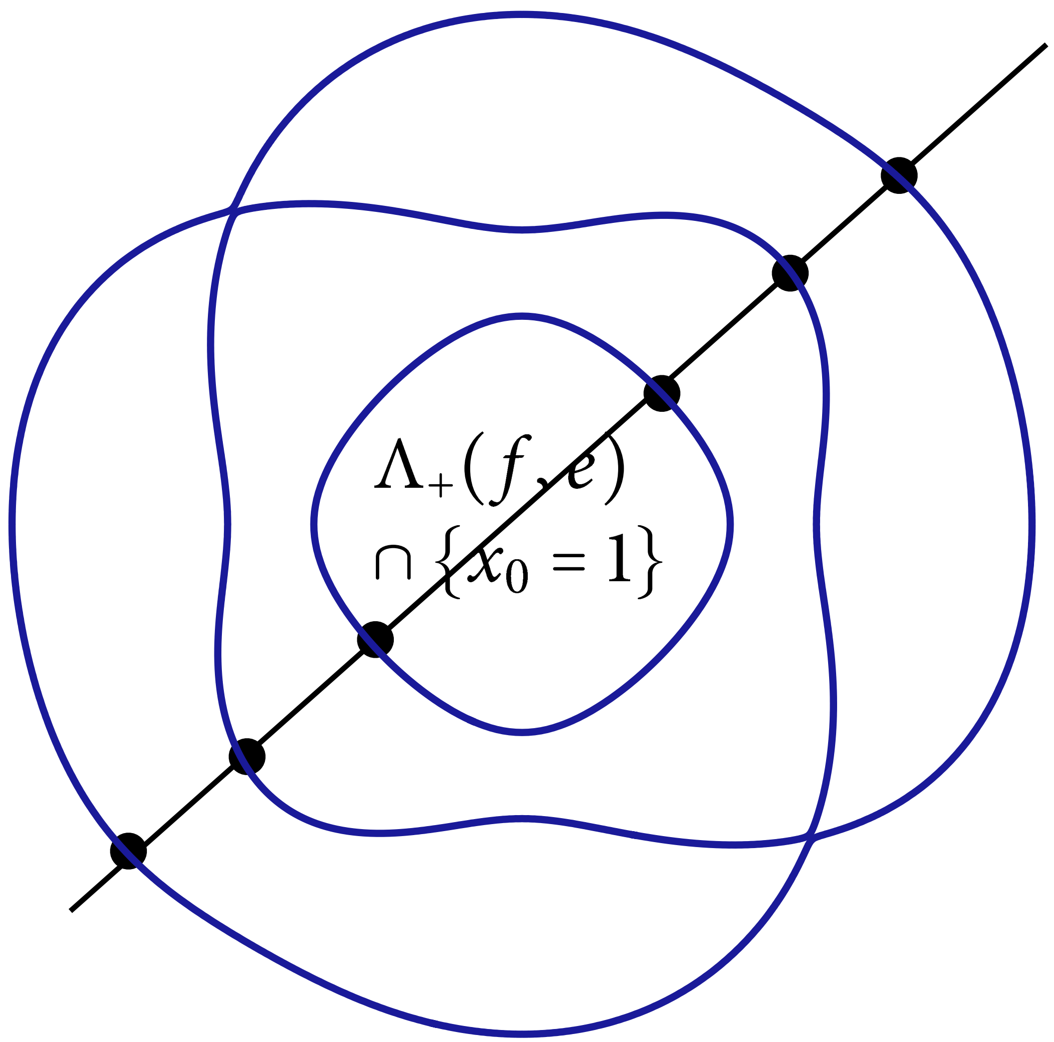
The dual curve V^* is the closure of the set of points $(1, u_1, u_2)$ for which the line

$$x_0 + u_1 x_1 + u_2 x_2 = 0$$

is tangent to V (at some regular point).

Hyperbolic Curves

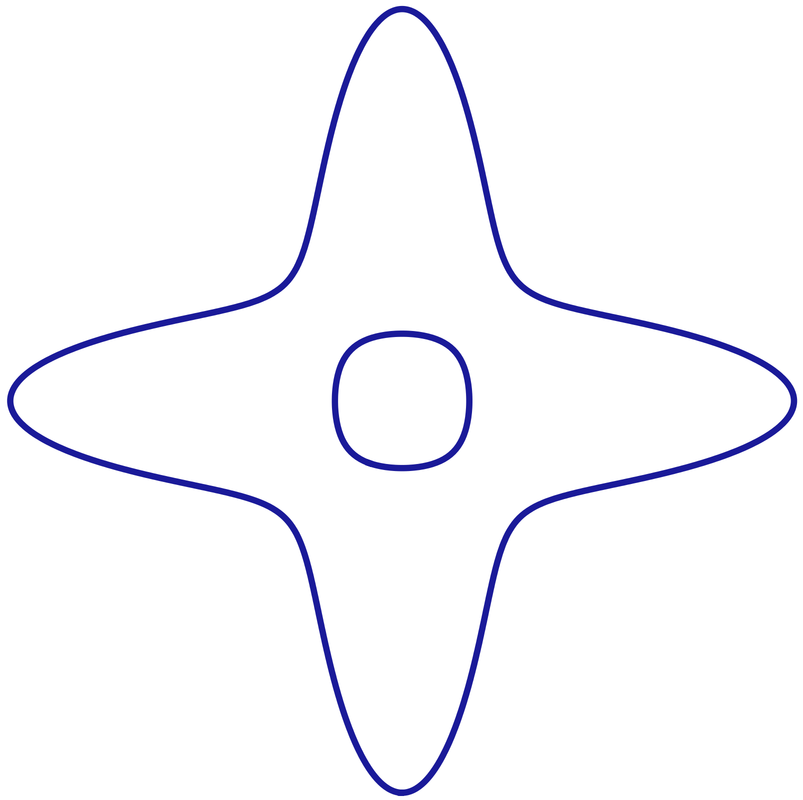
For any hermitian matrices A_1, A_2, the polynomial $f = \det(x_0 I_d + x_1 A_1 + x_2 A_2)$ is hyperbolic with respect to $e = (1, 0, 0)$, i.e. all roots of $f(t, a_1, a_2)$ are real for all $(a_1, a_2) \in \mathbb{R}^2$.



Hyperbolic curves

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 4 & 0 & 0 & 0 \end{pmatrix} \qquad \mathrm{Re}(A) = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 2 \\ \frac{1}{2} & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{3}{2} \\ 2 & 0 & \frac{3}{2} & 0 \end{pmatrix} \qquad \mathrm{Im}(A) = \begin{pmatrix} 0 & -\frac{i}{2} & 0 & 2i \\ \frac{i}{2} & 0 & -i & 0 \\ 0 & i & 0 & -\frac{3i}{2} \\ -2i & 0 & \frac{3i}{2} & 0 \end{pmatrix}$$

$$\begin{aligned} p &= \det\left(x_0 I_4 + x_1 \mathrm{Re}(A) + x_2 \mathrm{Im}(A)\right) \\ &= \frac{1}{16}\left(25x_1^4 + 25x_2^4 + 434x_1^2x_2^2 - 120x_0^2x_1^2 - 120x_0^2x_2^2 + 16x_0^4\right) \end{aligned}$$

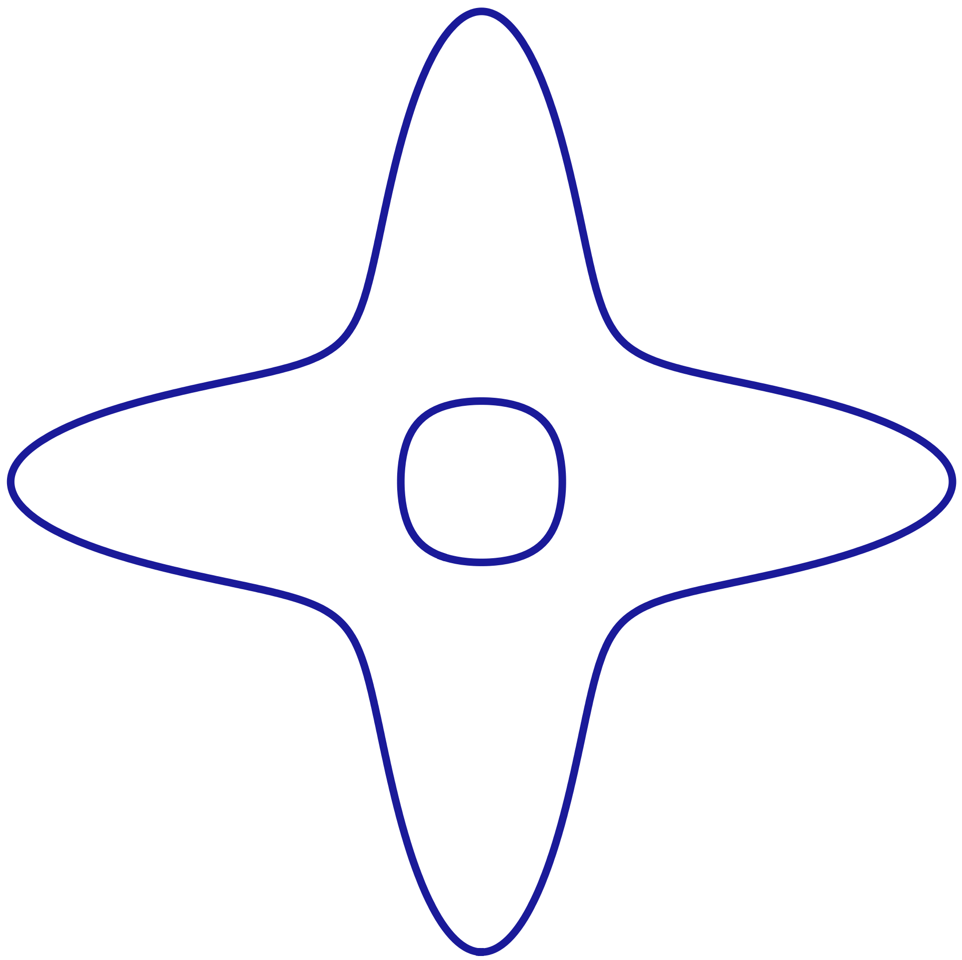


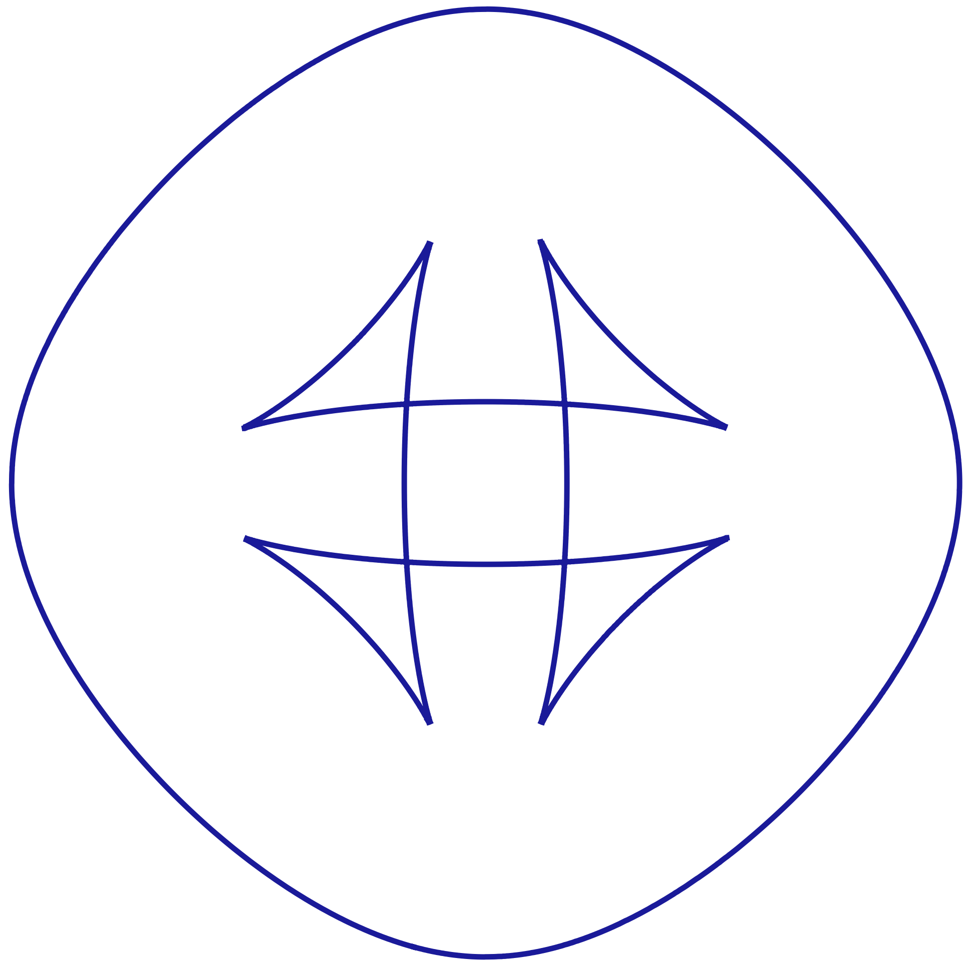
Hyperbolic curves

$$p = \frac{1}{16}\left(25x_1^4 + 25x_2^4 + 434x_1^2x_2^2 - 120x_0^2x_1^2 - 120x_0^2x_2^2 + 16x_0^4\right)$$

Dual curve is given by

$$\begin{aligned}
&250000u_1^{12} + 4380000u_1^{10}u_2^2 - 5475000u_0^2u_1^{10} + 1446000u_1^8u_2^4 - 68559000u_0^2u_1^8u_2^2 + 47610625u_0^4u_1^8 + 8787776u_1^6u_2^6 \\
&+ 179739600u_0^2u_1^6u_2^4 + 429249700u_0^4u_1^6u_2^2 - 209547000u_0^6u_1^6 + 1446000u_1^4u_2^8 + 179739600u_0^2u_1^4u_2^6 - 1058169786u_0^4u_1^4u_2^4 \\
&- 1493997480u_0^6u_1^4u_2^2 + 476341350u_0^8u_1^4 + 4380000u_1^2u_2^{10} - 68559000u_0^2u_1^2u_2^8 + 429249700u_0^4u_1^2u_2^6 - 1493997480u_0^6u_1^2u_2^4 \\
&+ 2442311100u_0^8u_1^2u_2^2 - 476982000u_0^{10}u_1^2 + 250000u_2^{12} - 5475000u_0^2u_2^{10} + 47610625u_0^4u_2^8 - 209547000u_0^6u_2^6 + 476341350u_0^8u_2^4 \\
&- 476982000u_0^{10}u_2^2 + 82355625u_0^{12} \quad = \quad 0
\end{aligned}$$





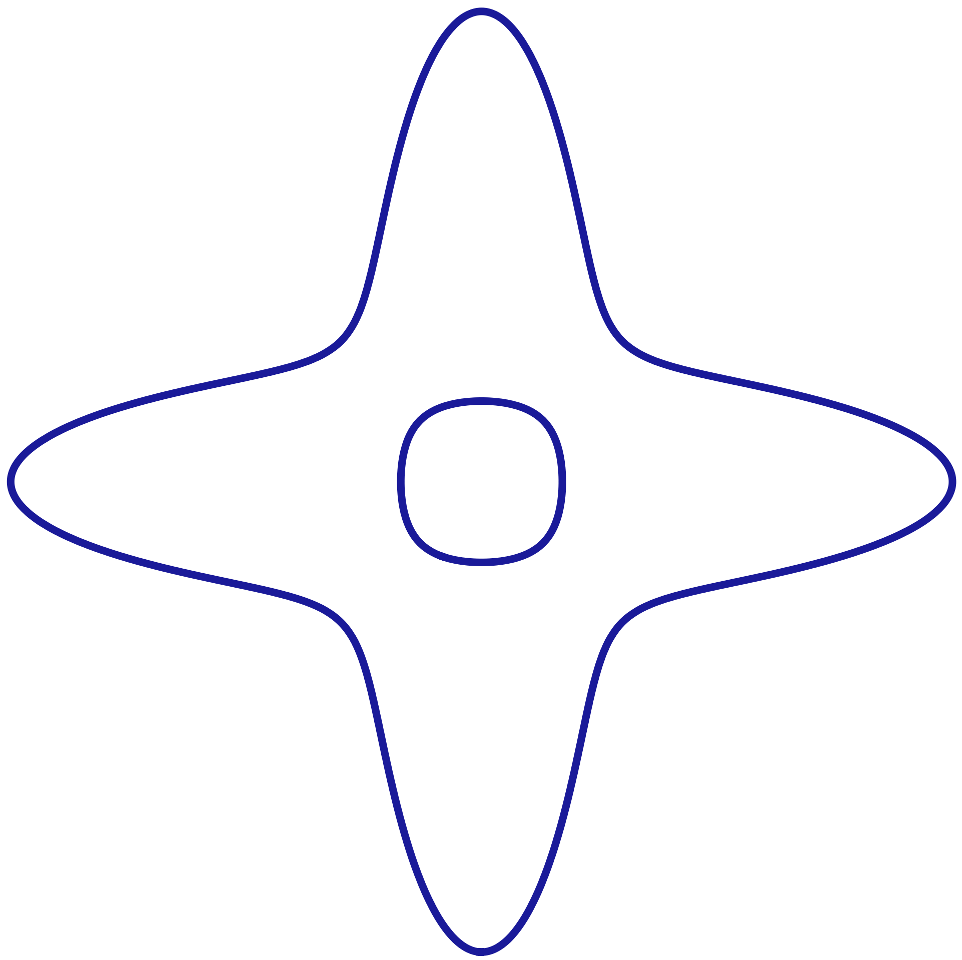
Duality for plane curves

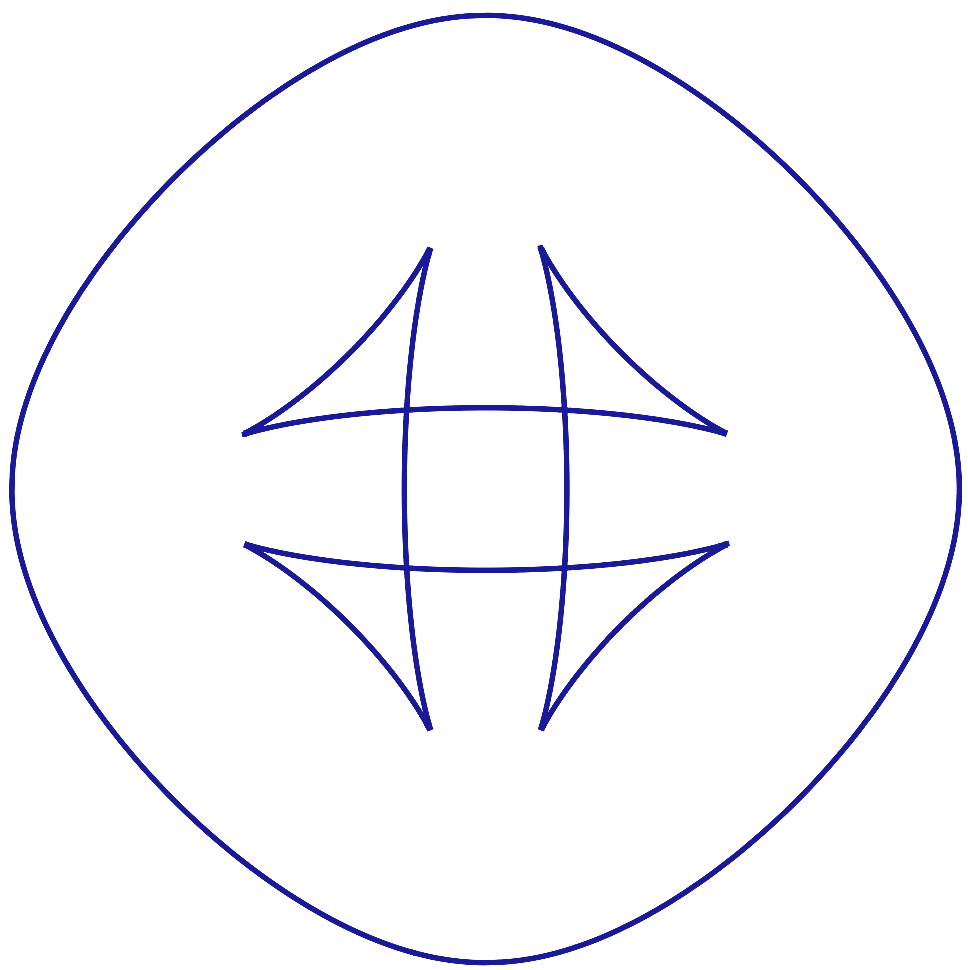
Let $V = \{p = 0\}$ be a plane curve of degree d.

If V is smooth, the dual curve V^* is irreducible of degree $d(d-1)$.

If V is generic (and smooth), then V^* has two types of singularities:

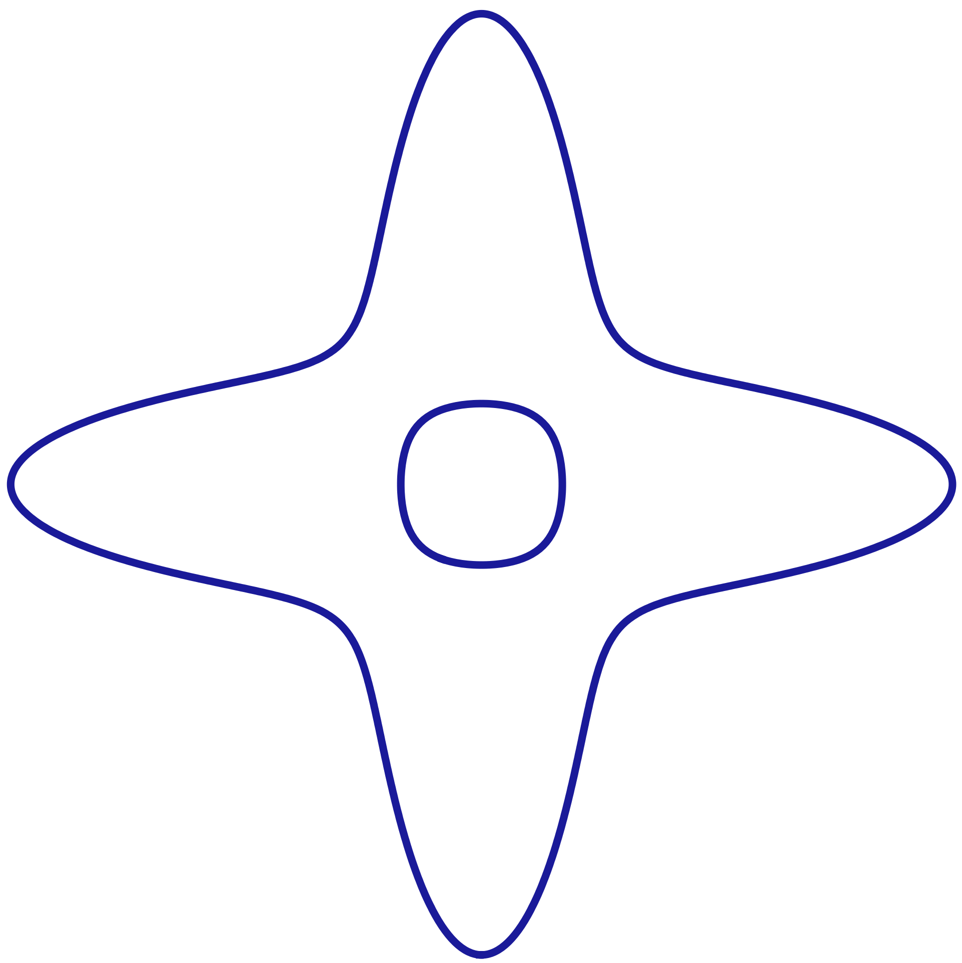
- The **bitangent lines** of V correspond to **nodes** of V^*.
- The **inflection lines** of V correspond to **cusps** of V^*.

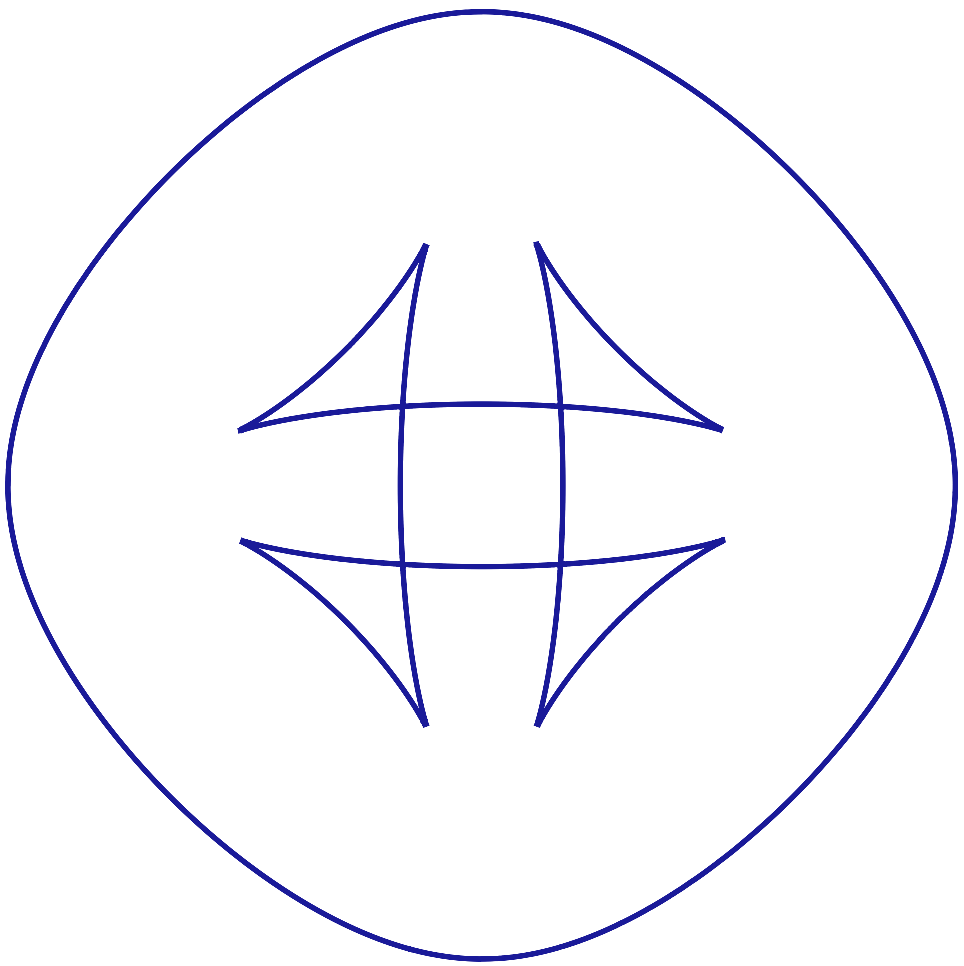




Duality for hyperbolic curves

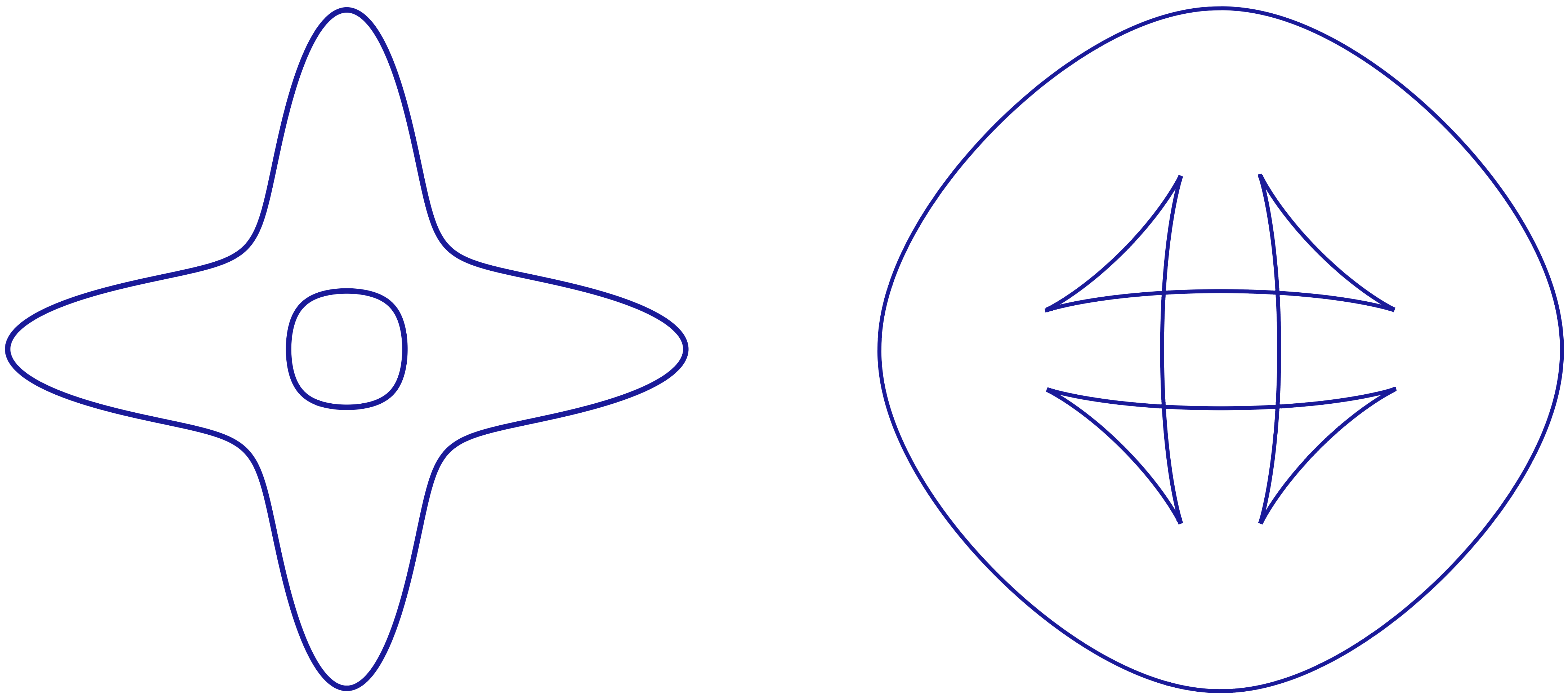
Theorem. *(Kippenhahn for hyperbolic curves)*
Let $p \in \mathbb{R}[x_0, x_1, x_2]$ *be hyperbolic with respect to* $e = (1, 0, 0)$.
The convex dual of the hyperbolicity region $\Lambda_+(f, e) \cap \{x_0 = 1\}$ *is the convex hull of the dual curve of* $\{f = 0\}$ *in the dual plane* $\{u_0 = 1\}$.





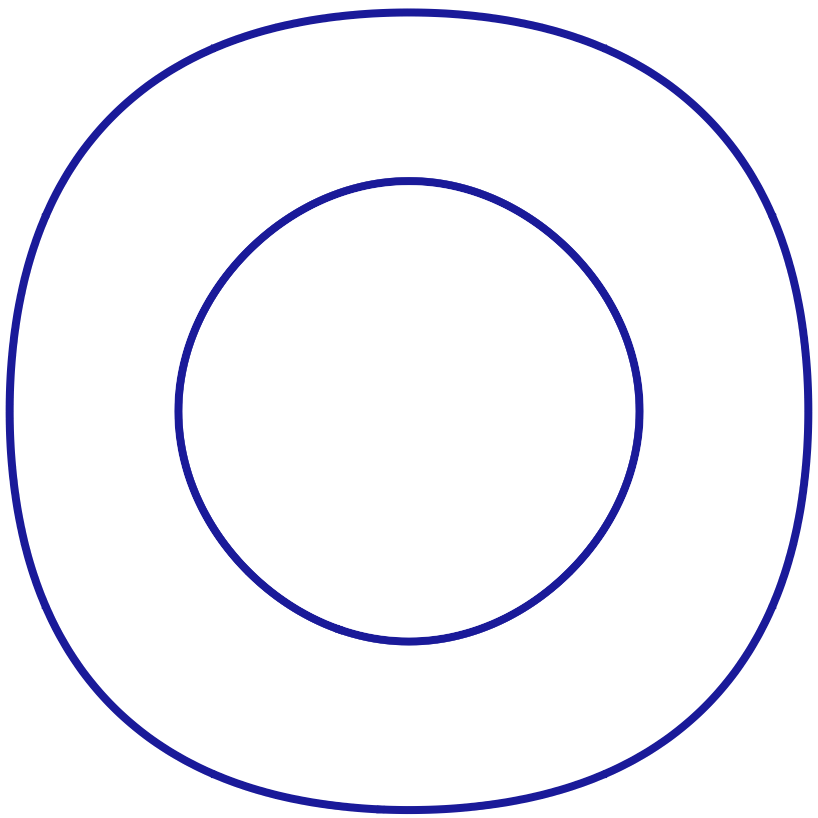
Duality for hyperbolic curves

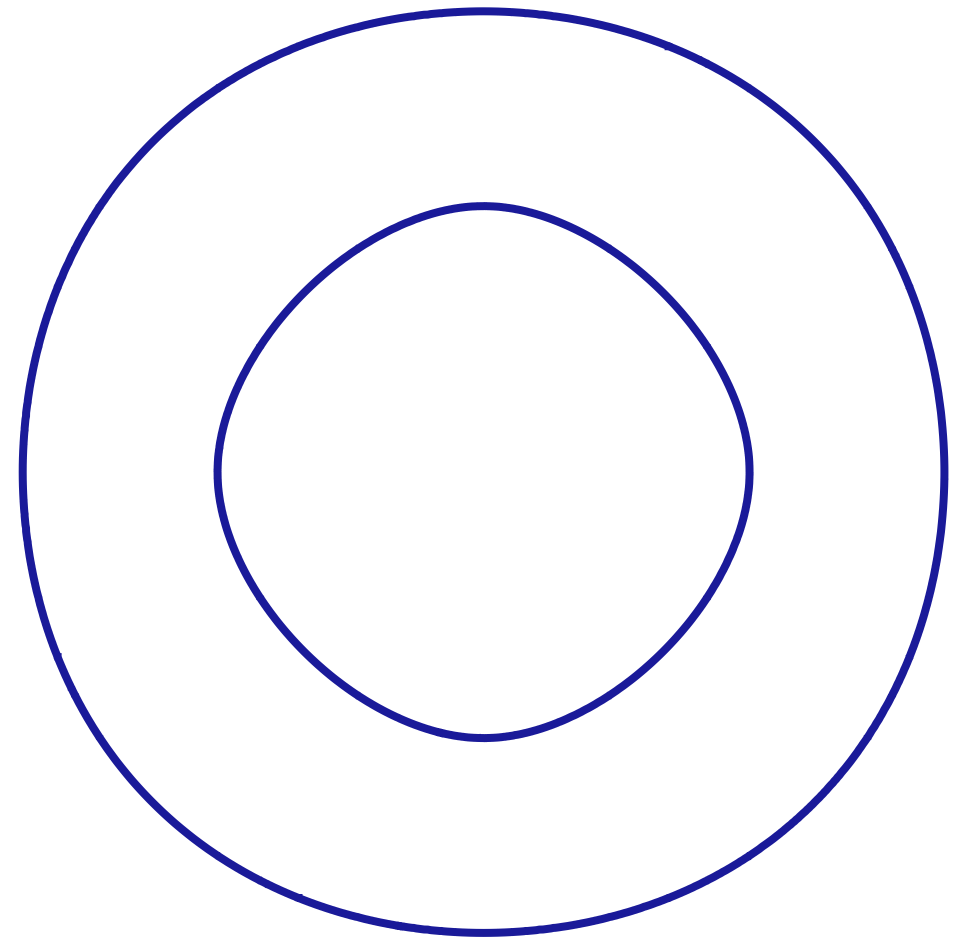
Theorem. *(Kippenhahn for hyperbolic curves)*
Let $p \in \mathbb{R}[x_0, x_1, x_2]$ *be hyperbolic with respect to* $e = (1, 0, 0)$.
The convex dual of the hyperbolicity region $\Lambda_+(f, e) \cap \{x_0 = 1\}$ *is the convex hull of the dual curve of* $\{f = 0\}$ *in the dual plane* $\{u_0 = 1\}$.



Problem: What about isolated real points (nodes) of the dual curve?

Duality for hyperbolic curves

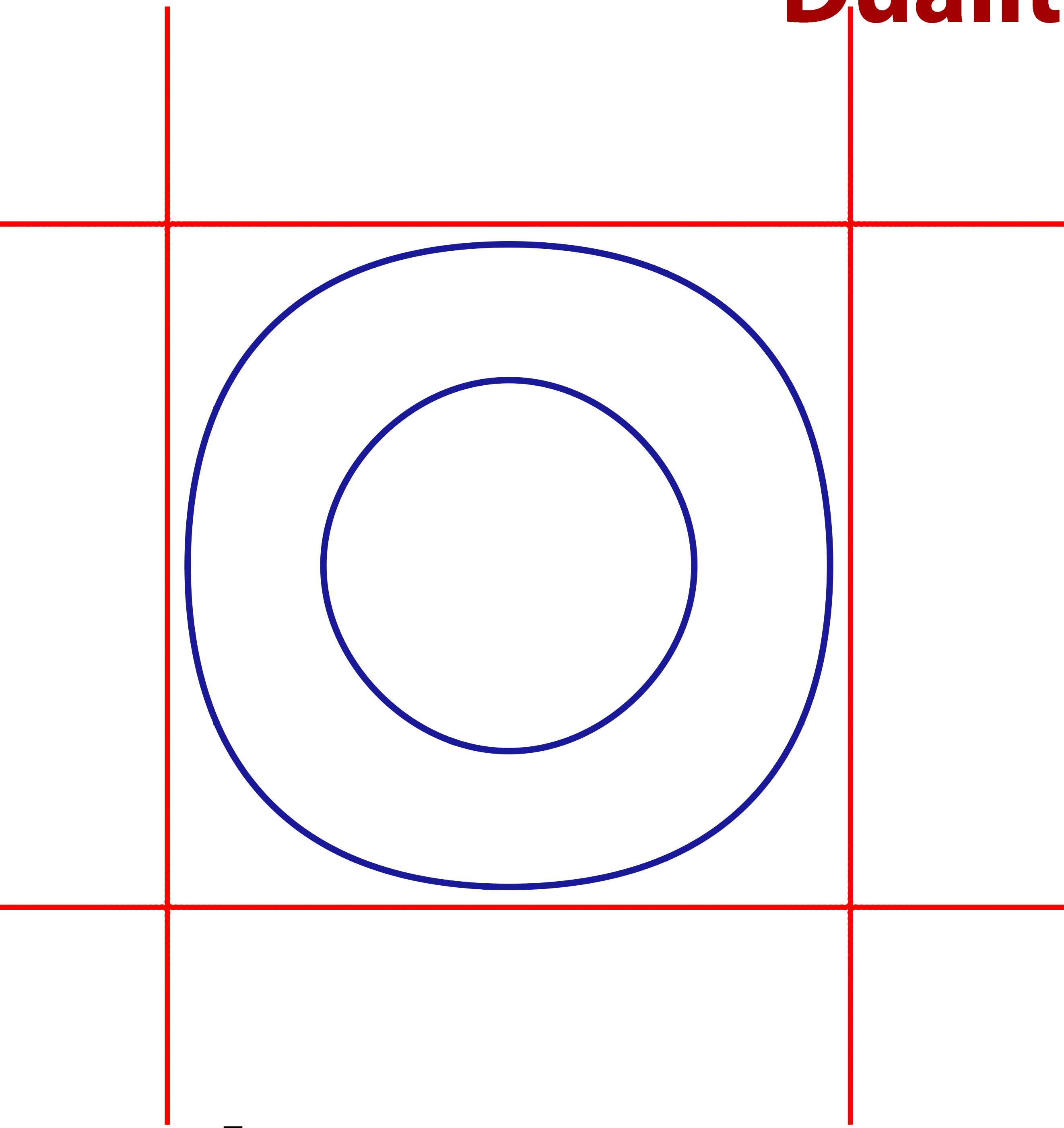


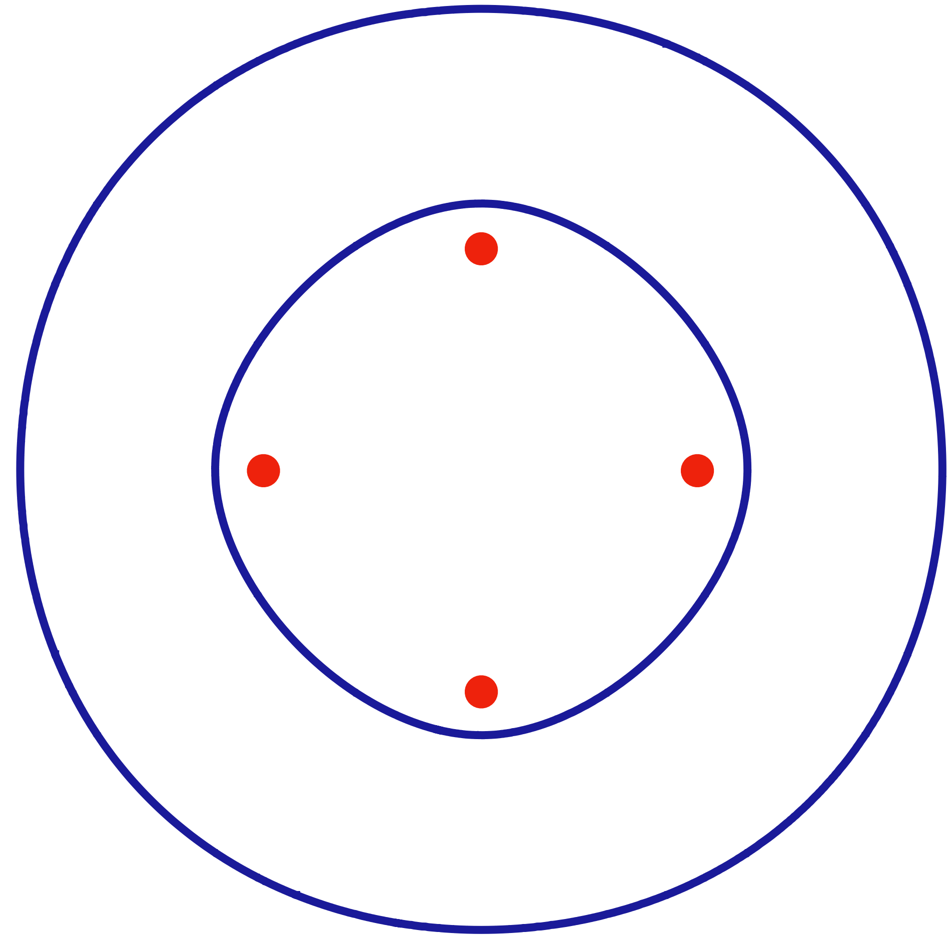


$$x_1^4 + x_2^4 + \frac{7}{4}x_1^2x_2^2 - 4x_0^2x_1^2 - 4x_0^2x_2^2 + 3x_0^4 = 0$$

$$\begin{aligned} &12288u_1^{12} + 89088u_1^{10}u_2^2 - 4096u_1^{10}u_0^2 + 248064u_1^8u_2^4 - 150784u_1^8u_2^2u_0^2 - 14976u_1^8u_0^4 + 340800u_1^6u_2^6 - 410560u_1^6u_2^4u_0^2 \\ &+ 137328u_1^6u_2^2u_0^4 + 4800u_1^6u_0^6 + 248064u_1^4u_2^8 - 410560u_1^4u_2^6u_0^2 + 283881u_1^4u_2^4u_0^4 - 85260u_1^4u_2^2u_0^6 + 3619u_1^4u_0^8 \\ &+ 89088u_1^2u_2^{10} - 150784u_1^2u_2^8u_0^2 + 137328u_1^2u_2^6u_0^4 - 85260u_1^2u_2^4u_0^6 + 23152u_1^2u_2^2u_0^8 - 1860u_1^2u_0^{10} + 12288u_2^{12} \\ &- 4096u_2^{10}u_0^2 - 14976u_2^8u_0^4 + 4800u_2^6u_0^6 + 3619u_2^4u_0^8 - 1860u_2^2u_0^{10} + 225u_0^{12} \quad = \quad 0 \end{aligned}$$

Duality for hyperbolic curves





$$x_1^4 + x_2^4 + \frac{7}{4}x_1^2x_2^2 - 4x_0^2x_1^2 - 4x_0^2x_2^2 + 3x_0^4 = 0$$

$$\begin{aligned}&12288u_1^{12} + 89088u_1^{10}u_2^2 - 4096u_1^{10}u_0^2 + 248064u_1^8u_2^4 - 150784u_1^8u_2^2u_0^2 - 14976u_1^8u_0^4 + 340800u_1^6u_2^6 - 410560u_1^6u_2^4u_0^2\\ &+ 137328u_1^6u_2^2u_0^4 + 4800u_1^6u_0^6 + 248064u_1^4u_2^8 - 410560u_1^4u_2^6u_0^2 + 283881u_1^4u_2^4u_0^4 - 85260u_1^4u_2^2u_0^6 + 3619u_1^4u_0^8\\ &+ 89088u_1^2u_2^{10} - 150784u_1^2u_2^8u_0^2 + 137328u_1^2u_2^6u_0^4 - 85260u_1^2u_2^4u_0^6 + 23152u_1^2u_2^2u_0^8 - 1860u_1^2u_0^{10} + 12288u_2^{12}\\ &- 4096u_2^{10}u_0^2 - 14976u_2^8u_0^4 + 4800u_2^6u_0^6 + 3619u_2^4u_0^8 - 1860u_2^2u_0^{10} + 225u_0^{12} \;=\; 0\end{aligned}$$

The joint numerical range

Let $A_1, \ldots, A_n$ be Hermitian $d \times d$-matrices.
The **joint numerical range** of $A_1, \ldots, A_n$ is the set

$$W(A_1, \ldots, A_n) = \left\{ \left(\overline{x}^T A_1 x, \ldots, \overline{x}^T A_n x \right) \;\middle|\; x \in \mathbb{C}^n \text{ with } \|x| = 1 \right\} \subset \mathbb{R}^n$$

The joint numerical range

Let $A_1, \ldots, A_n$ be Hermitian $d \times d$-matrices.
The **joint numerical range** of $A_1, \ldots, A_n$ is the set

$$W(A_1, \ldots, A_n) = \left\{ \left(\overline{x}^T A_1 x, \ldots, \overline{x}^T A_n x \right) \;\middle|\; x \in \mathbb{C}^n \text{ with } \|x| = 1 \right\} \subset \mathbb{R}^n$$

The joint numerical range is **not convex** in general (studied by Li&Poon 2000).

The joint numerical range

Let $A_1, \ldots, A_n$ be Hermitian $d \times d$-matrices.
The **joint numerical range** of $A_1, \ldots, A_n$ is the set

$$W(A_1, \ldots, A_n) = \left\{ \left(\overline{x}^T A_1 x, \ldots, \overline{x}^T A_n x \right) \;\middle|\; x \in \mathbb{C}^n \text{ with } \|x| = 1 \right\} \subset \mathbb{R}^n$$

The joint numerical range is **not convex** in general (studied by Li&Poon 2000).

The convex hull can be described as

$$\operatorname{conv} W(A_1, \ldots, A_n) = \left\{ \left(\langle A_1, X \rangle, \ldots, \langle A_n, X \rangle \right) \;\middle|\; X \geq 0,\ \operatorname{trace}(X) = 1 \right\}$$

where $\langle A, B \rangle = \operatorname{trace}(A\overline{B^T})$ and $X \geq 0$ means that X is Hermitian and positive semidefinite.

The joint numerical range

Let $A_1, \ldots, A_n$ be Hermitian $d \times d$-matrices.
The **joint numerical range** of $A_1, \ldots, A_n$ is the set

$$W(A_1, \ldots, A_n) = \left\{ \left(\overline{x}^T A_1 x, \ldots, \overline{x}^T A_n x \right) \;\middle|\; x \in \mathbb{C}^n \text{ with } \|x| = 1 \right\} \subset \mathbb{R}^n$$

The joint numerical range is **not convex** in general (studied by Li&Poon 2000).

The convex hull can be described as

$$\operatorname{conv} W(A_1, \ldots, A_n) = \left\{ \left(\langle A_1, X \rangle, \ldots, \langle A_n, X \rangle \right) \;\middle|\; X \succeq 0, \ \operatorname{trace}(X) = 1 \right\}$$

where $\langle A, B \rangle = \operatorname{trace}(A\overline{B^T})$ and $X \succeq 0$ means that X is Hermitian and positive semidefinite.

The set $\operatorname{conv} W(A_1, \ldots, A_n)$ is again the convex dual of the spectrahedron

$$\left\{ x \in \mathbb{R}^n \;\middle|\; I_d + x_1 A_1 + \cdots + x_n A_n \succeq 0 \right\}$$

Projective duality in higher dimensions

Let $V \subset \mathbb{P}^n$ be a projective variety. The **dual variety** of V (over $\mathbb{C}$) is

$$V^* = \overline{\left\{ u \in (\mathbb{P}^n)^* \mid \exists p \in V_{\text{reg}} \colon T_p(V) \subset \left\{ \sum u_i x_i = 0 \right\} \right\}}.$$

In words: V^* parametrizes all hyperplanes tangent to V at regular points.

Projective duality in higher dimensions

Let $V \subset \mathbb{P}^n$ be a projective variety. The **dual variety** of V (over $\mathbb{C}$) is

$$V^* = \overline{\left\{ u \in (\mathbb{P}^n)^* \mid \exists p \in V_{\text{reg}}\colon T_p(V) \subset \left\{ \sum u_i x_i = 0 \right\} \right\}}.$$

In words: V^* parametrizes all hyperplanes tangent to V at regular points.

Facts:

(1) If V is irreducible, then **biduality** holds: $(V^*)^* = V$.

(2) If $V = \{f = 0\}$ is a **generic** hypersurface of degree d, then V^* is a hypersurface of degree $d(d-1)^{n-1}$.

Projective duality in higher dimensions

Let $V \subset \mathbb{P}^n$ be a projective variety. The **dual variety** of V (over $\mathbb{C}$) is

$$V^* = \overline{\left\{ u \in (\mathbb{P}^n)^* \mid \exists p \in V_{\text{reg}}\colon T_p(V) \subset \left\{\sum u_i x_i = 0\right\} \right\}}.$$

In words: V^* parametrizes all hyperplanes tangent to V at regular points.

Facts:

(1) If V is irreducible, then **biduality** holds: $(V^*)^* = V$.

(2) If $V = \{f = 0\}$ is a **generic** hypersurface of degree d, then V^* is a hypersurface of degree $d(d-1)^{n-1}$.

(3) Determinantal hypersurfaces are **not generic** in this sense, for $n \geqslant 3$: The variety V^* is usually **not** a hypersurface.

Projective duality in higher dimensions

Let $V \subset \mathbb{P}^n$ be a projective variety. The **dual variety** of V (over $\mathbb{C}$) is

$$V^* = \overline{\left\{ u \in (\mathbb{P}^n)^* \mid \exists p \in V_{\text{reg}} \colon T_p(V) \subset \left\{ \sum u_i x_i = 0 \right\} \right\}}.$$

In words: V^* parametrizes all hyperplanes tangent to V at regular points.

Facts:

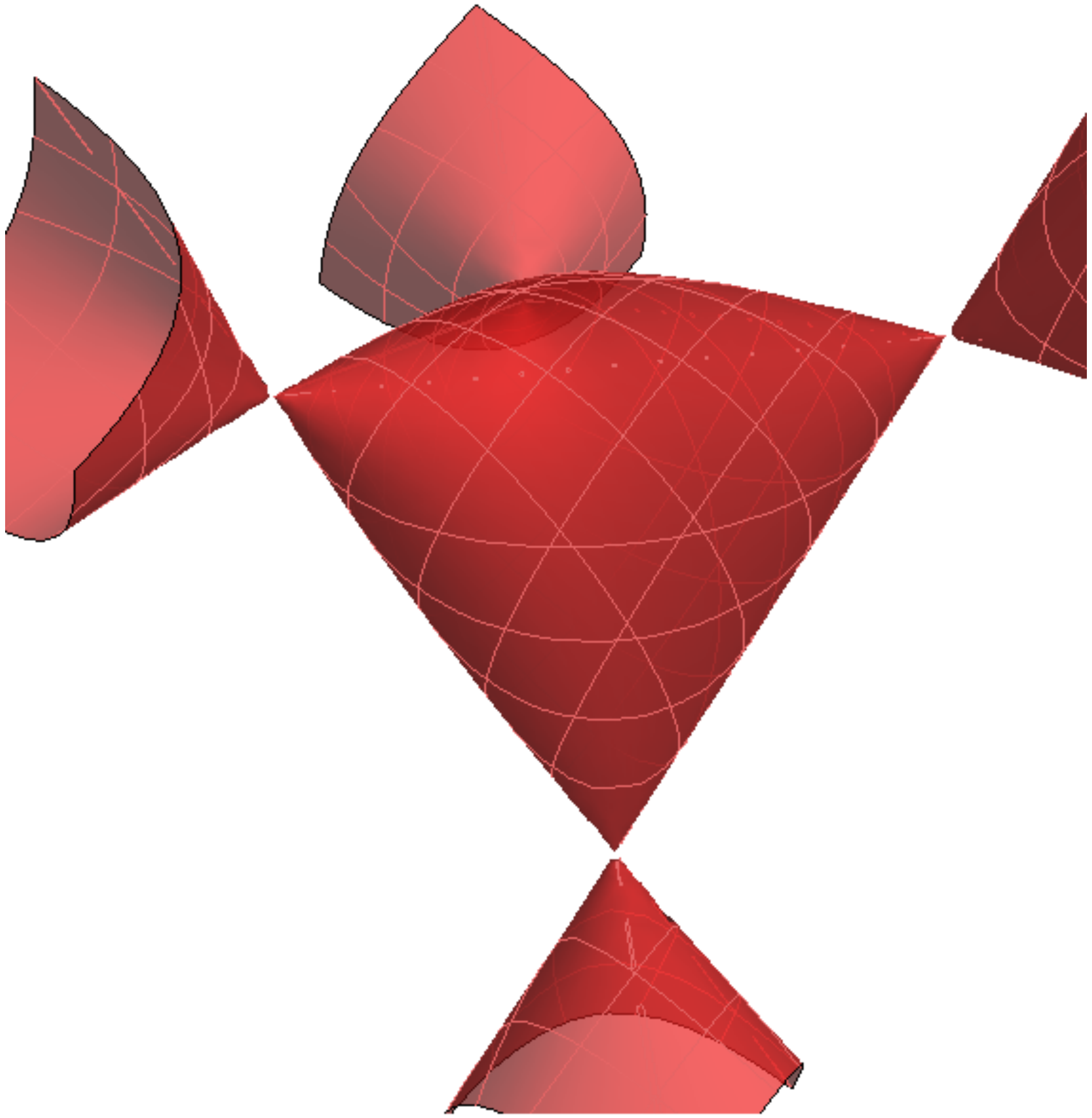
(1) If V is irreducible, then **biduality** holds: $(V^*)^* = V$.

(2) If $V = \{f = 0\}$ is a **generic** hypersurface of degree d, then V^* is a hypersurface of degree $d(d-1)^{n-1}$.

(3) Determinantal hypersurfaces are **not generic** in this sense, for $n \geqslant 3$: The variety V^* is usually **not** a hypersurface.

Examples. For the general determinantal hypersurface $\{\det(X) = 0\}$ in the space $\mathbb{P}^{\binom{d+1}{2}}$ of all symmetric $d \times d$-matrices, the dual variety is the set of all symmetric matrices of rank 1 (the **Veronese variety**).

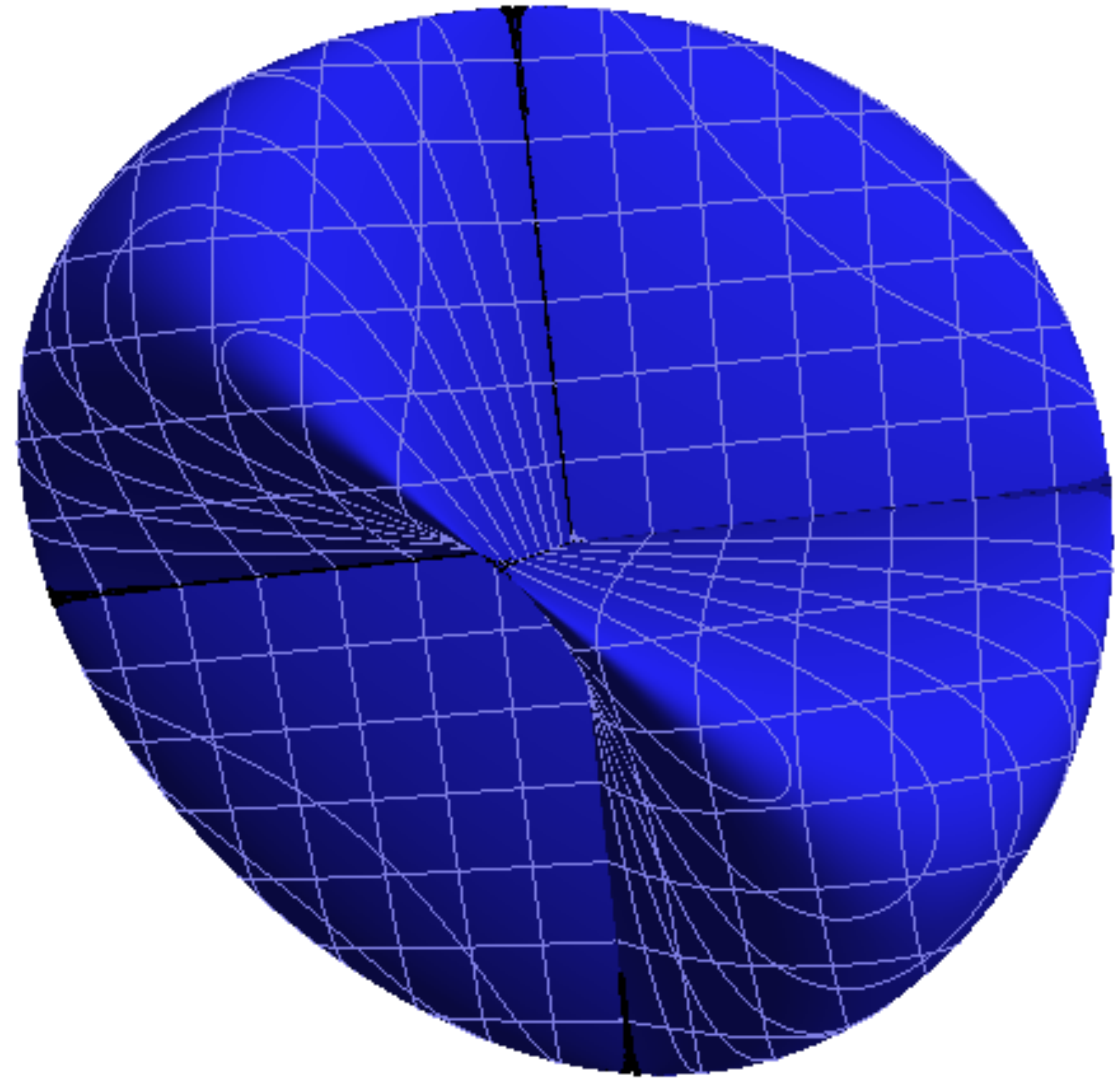
Famous example



Cayley's cubic

$$2x_1x_2x_3 - x_0x_1^2 - x_0x_2^2 - x_0x_3^2 + x_0^3$$

$$= \det\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_0 & x_3 \\ x_2 & x_3 & x_0 \end{pmatrix} = 0$$



Steiner's quartic

$$u_1^2u_2^2 - u_1^2u_3^2 - u_2^2u_3^2 - 2u_0u_1u_2u_3 = 0$$

Example

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(Chien and Nakazato 2010)

$$\begin{aligned} p(u_0, u_1, u_2, u_3) &= \det(u_0 \operatorname{id} + u_1 A_1 + u_2 A_2 + u_3 A_3) \\ &= u_0^3 + u_0^2 u_3 - 2u_0 u_1^2 - u_0 u_2^2 - u_1^3 - u_1^2 u_3 + u_1 u_2^2 \end{aligned}$$

The projective dual is a surface defined by

$$\begin{aligned} q(x_0, x_1, x_2, x_3) = 4x_0^2 x_3^2 + 8x_0 x_1 x_3^2 - 4x_0 x_2^2 x_3 - 24x_0 x_3^3 + 4x_1^2 x_3^2 \\ - 4x_1 x_2^2 x_3 - 8x_1 x_3^3 + x_2^4 + 8x_2^2 x_3^2 + 20x_3^4. \end{aligned}$$

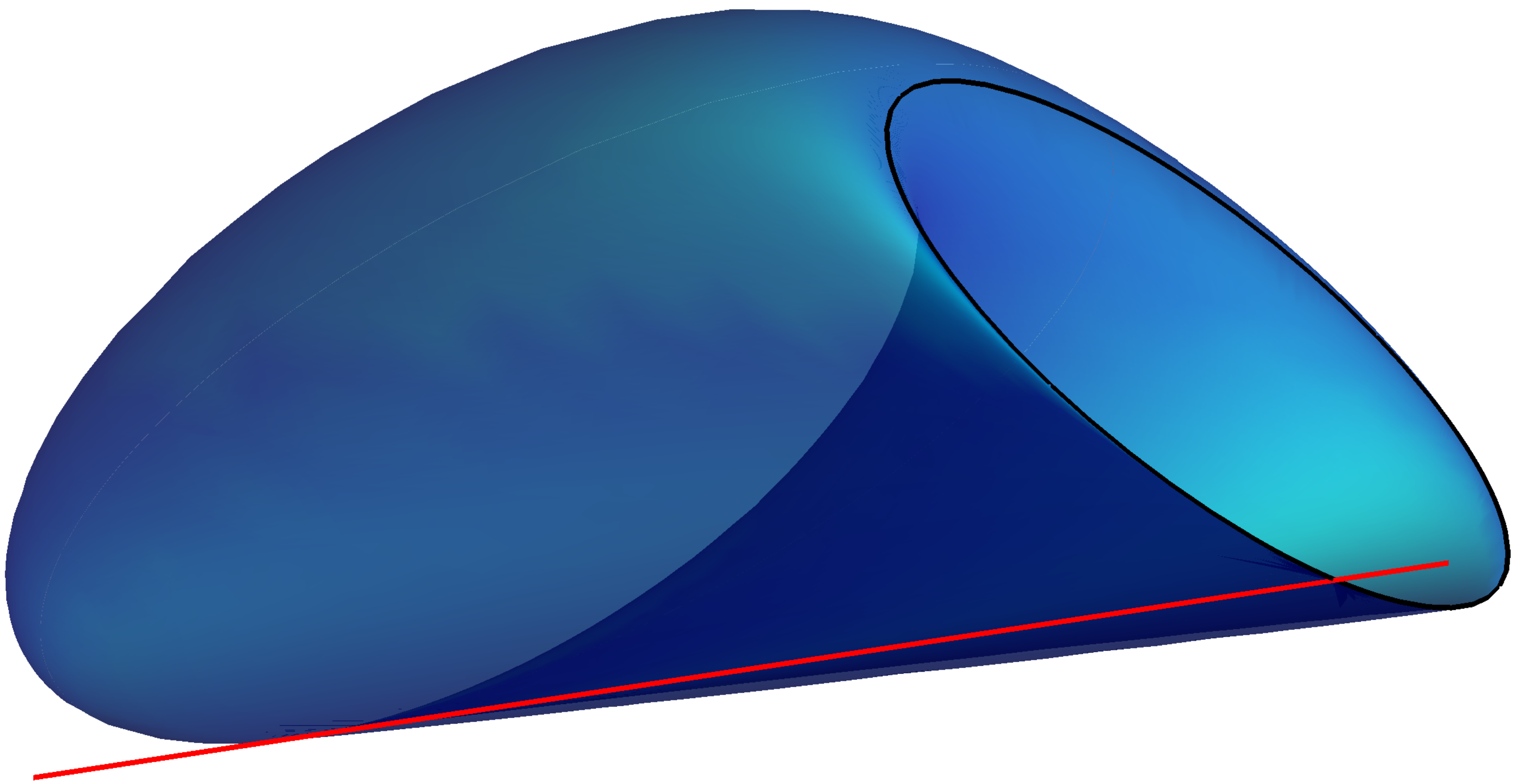
Its singular locus is $\{(x_0, x_1, x_2, x_3) \in \mathbb{P}^3 : x_2 = x_3 = 0\}$

Example

The projective dual is a surface defined by

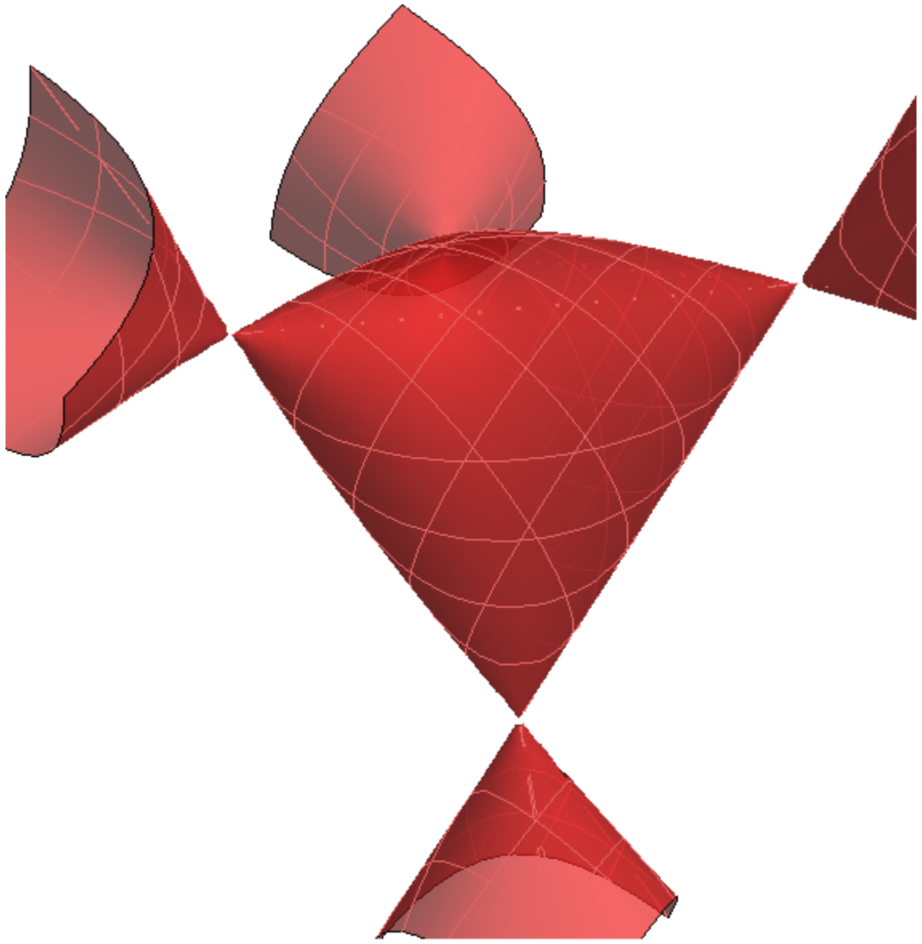
$$q(x_0, x_1, x_2, x_3) = 4x_0^2x_3^2 + 8x_0x_1x_3^2 - 4x_0x_2^2x_3 - 24x_0x_3^3 + 4x_1^2x_3^2 \\ - 4x_1x_2^2x_3 - 8x_1x_3^3 + x_2^4 + 8x_2^2x_3^2 + 20x_3^4.$$

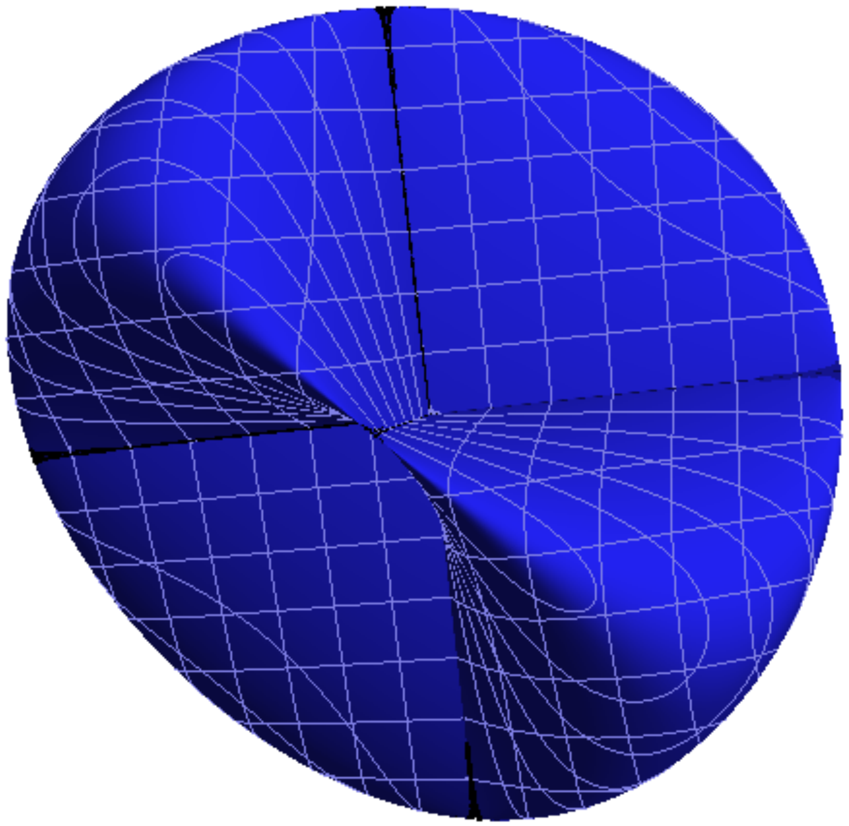
Its singular locus is $\{(x_0, x_1, x_2, x_3) \in \mathbb{P}^3 : x_2 = x_3 = 0\}$



How to fix it

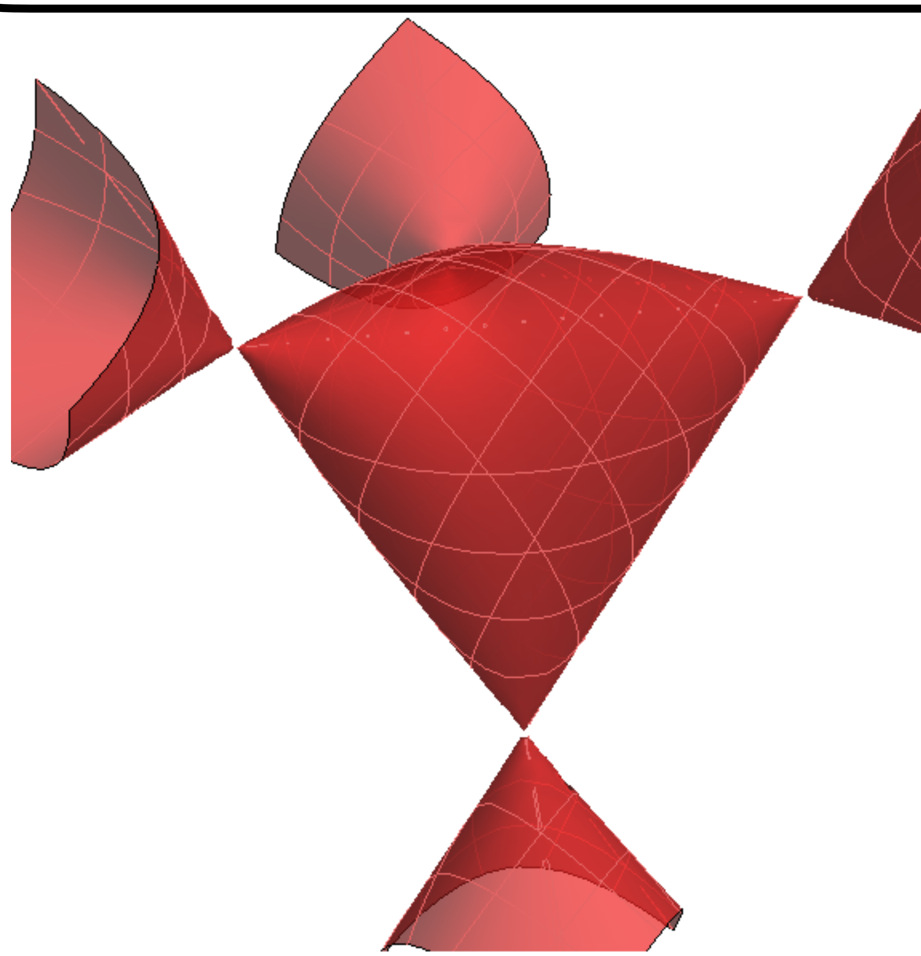
Theorem. *(Sinn 2015/P-Sinn-Weis 2019)*
Let $p \in \mathbb{R}[x_0, \ldots, x_n]$ *be irreducible and hyperbolic with respect to* $e = (1, 0 \ldots, 0)$.
Let $V = \{p = 0\} \subset \mathbb{P}^n$ *and let* V^* *be the dual projective variety.*
The convex dual of the hyperbolicity region $C(p, e) \cap \{x_0 = 1\}$ *is the closure of the convex hull of* $V^*_{\mathrm{reg}}(\mathbb{R}) \cap \{u_0 = 1\}$, *where* $V_{\mathrm{reg}}(\mathbb{R})$ *is the set of regular real points of* V^*.

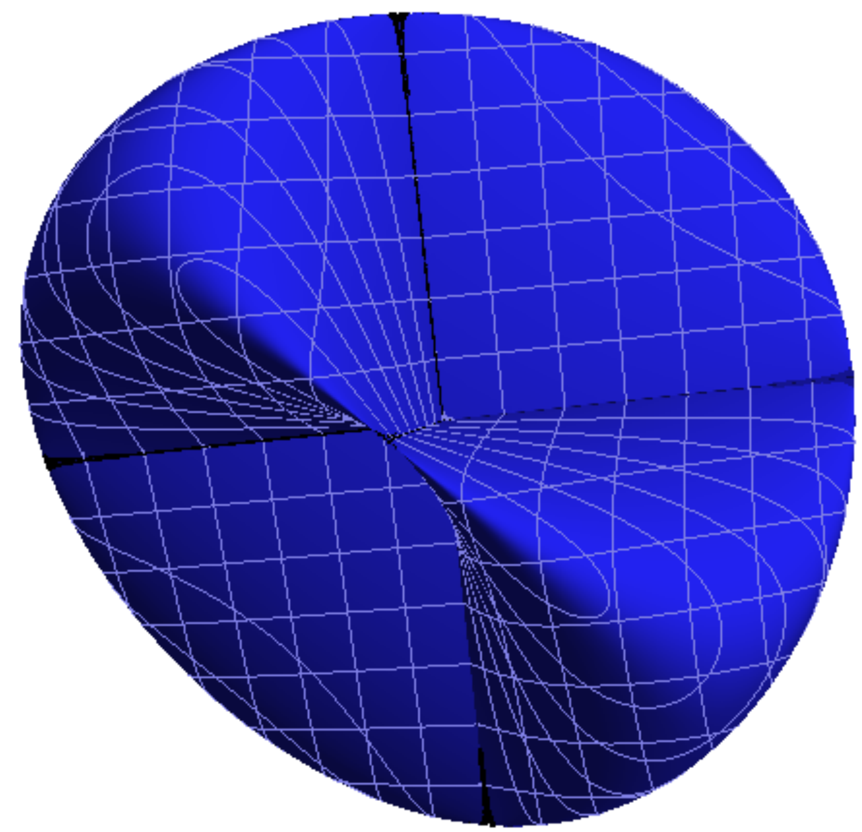




How to fix it

Theorem. *(Sinn 2015/P-Sinn-Weis 2019)*
Let $p \in \mathbb{R}[x_0, \ldots, x_n]$ *be irreducible and hyperbolic with respect to* $e = (1, 0 \ldots, 0)$.
Let $V = \{p = 0\} \subset \mathbb{P}^n$ *and let* V^* *be the dual projective variety.*
The convex dual of the hyperbolicity region $C(p, e) \cap \{x_0 = 1\}$ *is the closure of the convex hull of* $V^*_{\text{reg}}(\mathbb{R}) \cap \{u_0 = 1\}$, *where* $V_{\text{reg}}(\mathbb{R})$ *is the set of regular real points of* V^*.





Corollary. *(PSW 2019) The convex hull of the joint numerical range of Hermitian* $d \times d$ *matrices* $A_1, \ldots, A_n$ *is the closure of the convex hull of the real non-singular part of the dual variety of the hyperbolic hypersurface* $\det(x_0 I_d + x_1 A_1 + \cdots + x_n A_n)$.