# Time domain interpretation of sum rules in electromagnetism

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Thanks to the organizers of the BIRS Herglotz-Nevanlinna workshop!


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Welcome to EuCAP 2020 in Copenhagen, March 15-20!
Firm deadline Oct 18, https://www.eucap.org
Daniel Sjöberg, Vice Chair of EuCAP 2020

# Overview

# Introduction

# Motivation

We have had success with deriving sum rules and interpreting them in the frequency domain. One example is for the transmission coefficient of a low pass slab:

$$\int_0^\infty \frac{\mathrm{Re}\{1 - T(\omega)\}}{\omega^2}\,\mathrm{d}\omega = \frac{\pi\gamma}{4Ac}$$

where $\mathrm{Re}\{2A(1-T)\} = \sigma_{\mathrm{ext}}$ is the extinction cross section of the slab.

But not all sum rules are easy to interpret. For instance, we can derive the following for the reflection coefficient of a PEC backed slab:

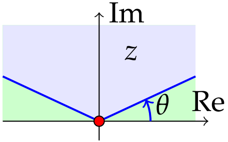
$$\int_0^\infty \frac{\mathrm{Re}\{1 + R(\omega)\}}{\omega^2}\,\mathrm{d}\omega = \pi\left(1 + \frac{\gamma_{\mathrm{m}}}{2Ad}\right)\frac{d}{c} = \pi\mu_{\mathrm{s}}\frac{d}{c}$$

But what is the physical relevance of $\mathrm{Re}\{1 + R(\omega)\}$? If we do not find something in the frequency domain, then maybe in time domain?

# Integral identities for Herglotz functions

Consider Herglotz functions with the symmetry $h(z) = -h^*(-z^*)$ (real-valued in the time domain), having asymptotic expansions ($N_0 \geq 0$ and $N_\infty \geq 0$)

$$\begin{cases} h(z) = \sum_{n=0}^{N_0} a_{2n-1} z^{2n-1} + \mathrm{o}(z^{2N_0-1}) & \text{as } z \hat{\to} 0 \\ h(z) = \sum_{n=0}^{N_\infty} b_{1-2n} z^{1-2n} + \mathrm{o}(z^{1-2N_\infty}) & \text{as } z \hat{\to} \infty \end{cases}$$




where $\hat{\to}$ denotes limits in the Stoltz domain $0 < \theta \leq \arg(z) \leq \pi - \theta$. They satisfy the identities ($1 - N_\infty \leq n \leq N_0$)

$$\lim_{\varepsilon \to 0^+} \lim_{y \to 0^+} \frac{2}{\pi} \int_\varepsilon^{\frac{1}{\varepsilon}} \frac{\mathrm{Im}\, h(x + \mathrm{i}y)}{x^{2n}} \, \mathrm{d}x = a_{2n-1} - b_{2n-1} = \begin{cases} -b_{2n-1} & n < 0 \\ a_{-1} - b_{-1} & n = 0 \\ a_1 - b_1 & n = 1 \\ a_{2n-1} & n > 1 \end{cases}$$

Bernland, Luger, Gustafsson, *Sum rules and constraints on passive systems*, J. Phys. A: Math. Theor., 2011.

Warm thanks to Mats, Annemarie, Sven, Lars, Yevhen, Mitja for all the collaborations!

# Physical bounds

Given that $\operatorname{Im} h(x)/x^{2n} = P(x) \geq 0$, we can estimate the integrals as

$$\int_0^{\infty} P(x)\,\mathrm{d}x \geq \int_{x_1}^{x_2} P(x)\,\mathrm{d}x \geq (x_2 - x_1) \min_{x\in[x_1,x_2]} P(x)$$

This implies

$$\boxed{(x_2 - x_1) \min_{x\in[x_1,x_2]} P(x) \leq \frac{\pi}{2}(a_1 - b_1)}$$

With the interpretations

- $x_2 - x_1$ = bandwidth (in frequency or wavelength)
- $\min_{x\in[x_1,x_2]} P(x)$ = performance level (application specific)

we see that a physical interpretation of the sum rule is that

> *the product of bandwidth and performance level is bounded from above by low- and high-frequency asymptotics, independent of specific behavior in between!*

Further interpretation can be possible when $h$ and the application are specified.

# Notation

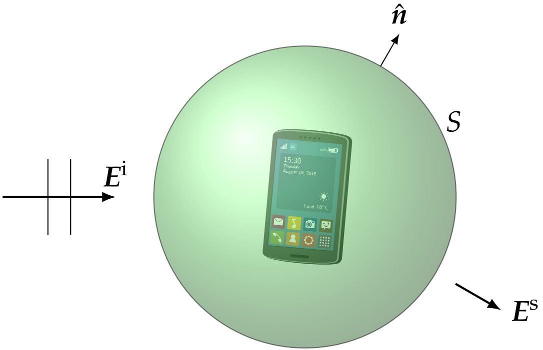
In this presentation, notation is abused in at least the following ways:

- $h$ is used to denote both a Herglotz function in frequency domain, and a time domain impulse response.
- The presentation is based more on positive real functions than Herglotz functions:
  - Time convention $\mathrm{e}^{\mathrm{j}\omega t}$ is used instead of $\mathrm{e}^{-\mathrm{i}\omega t}$.
  - In the stated sum rules, we typically write $\mathrm{Re}\{H(\omega)\}$ instead of $\mathrm{Im}\{\mathrm{i}H(\omega)\}$, where $H(\omega)$ is a transfer function.
- The word “slab” should be interpreted as a planar structure with internal microstructure, although the examples at the end will be for homogeneous slabs due to computational simplicity.

# Scattering problems

# Scattering problems

A plane wave $\boldsymbol{E}^{\mathrm{i}}(\boldsymbol{r}) = \boldsymbol{E}_0 \mathrm{e}^{-\mathrm{j}\boldsymbol{k}\cdot\boldsymbol{r}}$, $\boldsymbol{H}^{\mathrm{i}}(\boldsymbol{r}) = \frac{1}{\eta_0}\hat{\boldsymbol{k}} \times \boldsymbol{E}^{\mathrm{i}}(\boldsymbol{r})$, impinges on a scattering object enclosed by a surface $S$.




The interaction with the scatterer results in absorption and scattering. The extincted power is $P_{\mathrm{ext}} = P_{\mathrm{abs}} + P_{\mathrm{sca}}$, and the extinction cross section is $\sigma_{\mathrm{ext}} = \frac{P_{\mathrm{ext}}}{|\boldsymbol{E}_0|^2/(2\eta_0)}$.

# Sum rules in scattering theory

Using the optical theorem, the extinction cross section can be written

$$1\mathrm{D}: \quad \sigma_{\mathrm{ext}} = \mathrm{Re}\{-2A(T-1)\} \qquad T = \text{transmission coefficient}$$

$$2\mathrm{D}: \quad \sigma_{\mathrm{ext}} = \mathrm{Re}\left\{-\frac{4b}{\mathrm{j}k}\frac{\boldsymbol{E}_0^*\cdot\boldsymbol{f}}{|\boldsymbol{E}_0|^2}\right\} \qquad \boldsymbol{f} = \frac{\mathrm{j}k}{4b}\hat{\boldsymbol{k}}\times\int_0^b\oint_C\left[\hat{\boldsymbol{k}}\times(\hat{\boldsymbol{n}}\times\eta_0\boldsymbol{H}^{\mathrm{s}})+\boldsymbol{E}^{\mathrm{s}}\times\hat{\boldsymbol{n}}\right]\mathrm{e}^{\mathrm{j}\boldsymbol{k}\cdot\boldsymbol{r}}\,\mathrm{d}\ell\,\mathrm{d}z$$

$$3\mathrm{D}: \quad \sigma_{\mathrm{ext}} = \mathrm{Re}\left\{-\frac{4\pi}{\mathrm{j}k}\frac{\boldsymbol{E}_0^*\cdot\boldsymbol{F}}{|\boldsymbol{E}_0|^2}\right\} \qquad \boldsymbol{F} = \frac{\mathrm{j}k}{4\pi}\hat{\boldsymbol{k}}\times\int_S\left[\hat{\boldsymbol{k}}\times(\hat{\boldsymbol{n}}\times\eta_0\boldsymbol{H}^{\mathrm{s}})+\boldsymbol{E}^{\mathrm{s}}\times\hat{\boldsymbol{n}}\right]\mathrm{e}^{\mathrm{j}\boldsymbol{k}\cdot\boldsymbol{r}}\,\mathrm{d}S$$

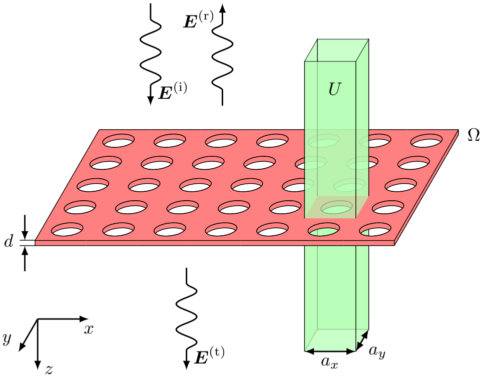
Thus, the extinction cross section (proportional to power, square of field strength) can be evaluated using forward scattering coefficients (proportional to field strength, being transfer functions). This enables the derivation of a forward scattering sum rule:

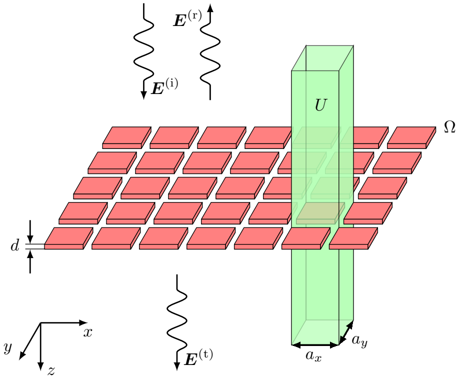
$$\boxed{\int_0^\infty \frac{\sigma_{\mathrm{ext}}(\omega)}{\omega^2}\,\mathrm{d}\omega = \frac{\pi\gamma}{4Ac}}$$

where $\sigma_{\mathrm{ext}} = \sigma_{\mathrm{abs}} + \sigma_{\mathrm{sca}}$ and $\gamma$ is the static polarizability of the scatterer.

# Scattering problem, periodic surfaces

| Band pass (connected metal) | Low pass (disconnected metal) |
|---|---|




$$\begin{cases} \nabla \times \boldsymbol{E} + jkc\mu \boldsymbol{H} = \boldsymbol{0} \\ \nabla \times \boldsymbol{H} - jkc\epsilon \boldsymbol{E} = \boldsymbol{0} \end{cases} \quad \begin{cases} \hat{\boldsymbol{n}} \times \boldsymbol{E} = \boldsymbol{0} \text{ on } \partial\Omega,\ \boldsymbol{E}, \boldsymbol{H} \text{ periodic in } xy \\ \text{input/output Floquet ports at } z = \pm\infty \end{cases}$$

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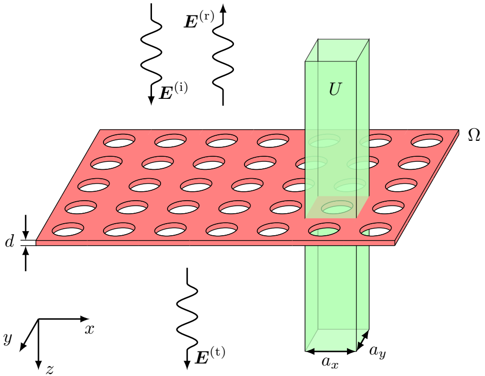
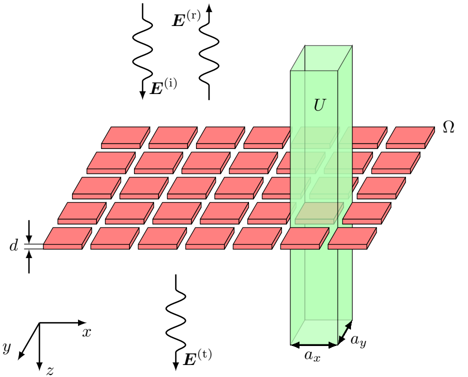
|   |   |
| $\begin{cases} T = -\dfrac{\mathrm{j}k}{2A}(\gamma_\mathrm{e} + \gamma_\mathrm{m}) + \mathrm{O}((kd)^2) \\ \Gamma = -1 - \dfrac{\mathrm{j}k}{2A}(\gamma_\mathrm{e} - \gamma_\mathrm{m}) + \mathrm{O}((kd)^2) \end{cases}$ | $\begin{cases} T = 1 - \dfrac{\mathrm{j}k}{2A}(\gamma_\mathrm{e} + \gamma_\mathrm{m}) + \mathrm{O}((kd)^2) \\ \Gamma = -\dfrac{\mathrm{j}k}{2A}(\gamma_\mathrm{e} - \gamma_\mathrm{m}) + \mathrm{O}((kd)^2) \end{cases}$ |

# How to compute the polarizabilities

The electric polarizability $\gamma_\mathrm{e}$ is computed from the static Maxwell's equations:

$$\nabla \times \boldsymbol{E} = \boldsymbol{0}, \quad \nabla \cdot \boldsymbol{D} = 0, \quad \boldsymbol{D}(\boldsymbol{r}) = \boldsymbol{\epsilon}(\boldsymbol{r}) \cdot \boldsymbol{E}(\boldsymbol{r})$$

The excitation is from a uniform field $\boldsymbol{E}_0$,

$$\boldsymbol{E}(\boldsymbol{r}) \to \boldsymbol{E}_0, \quad |\boldsymbol{r}| \to \infty$$

For periodic structures periodic boundary conditions may apply in one or two dimensions. The polarizability tensor $\bar{\bar{\gamma}}_\mathrm{e}$ is defined from the dipole moment

$$\boldsymbol{p} = \int_U (\boldsymbol{\epsilon}(\boldsymbol{r}) - \epsilon_0 \mathbf{I}) \cdot \boldsymbol{E}(\boldsymbol{r}) \, \mathrm{d}V = \epsilon_0 \bar{\bar{\gamma}}_\mathrm{e} \cdot \boldsymbol{E}_0$$

and the scalar $\gamma_\mathrm{e}$ is the diagonal element

$$\gamma_\mathrm{e} = \frac{\boldsymbol{E}_0 \cdot \bar{\bar{\gamma}}_\mathrm{e} \cdot \boldsymbol{E}_0}{|\boldsymbol{E}_0|^2}$$

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This is very similar to classical homogenization theory, which has periodic boundary conditions in all dimensions.

# How to estimate the polarizability

The electrostatic problem can be solved using two different potentials (with $\boldsymbol{D}_0 = \epsilon_0 \boldsymbol{E}_0$):

$$\begin{aligned} \nabla \times \boldsymbol{E} = \boldsymbol{0} \quad &\Rightarrow \quad \boldsymbol{E} = \boldsymbol{E}_0 - \nabla \varphi \\ \nabla \cdot \boldsymbol{D} = 0 \quad &\Rightarrow \quad \boldsymbol{D} = \boldsymbol{D}_0 + \nabla \times \boldsymbol{F} \end{aligned}$$

This provides two different expressions for the energy:

$$\begin{aligned} J(\varphi, \boldsymbol{E}_0) &= \int \left[ (\boldsymbol{E}_0 - \nabla \varphi) \cdot \boldsymbol{\epsilon}(\boldsymbol{r}) \cdot (\boldsymbol{E}_0 - \nabla \varphi) - \epsilon_0 |\boldsymbol{E}_0|^2 \right] \mathrm{d}V \\ K(\boldsymbol{F}, \boldsymbol{D}_0) &= \int \left[ (\boldsymbol{D}_0 + \nabla \times \boldsymbol{F}) \cdot \boldsymbol{\epsilon}(\boldsymbol{r})^{-1} \cdot (\boldsymbol{D}_0 + \nabla \times \boldsymbol{F}) - \epsilon_0^{-1} |\boldsymbol{D}_0|^2 \right] \mathrm{d}V \end{aligned}$$

It can be shown that for all test functions $\varphi$ and $\boldsymbol{F}$ (with correct boundary conditions and unit background energy density $\epsilon_0 |\boldsymbol{E}_0|^2 = \epsilon_0^{-1} |\boldsymbol{D}_0|^2 = 1$)

$$\boxed{-K(\boldsymbol{F}, \boldsymbol{D}_0) \leq \gamma_{\mathrm{e}} \leq J(\varphi, \boldsymbol{E}_0)}$$

Equality is obtained for the unique minimizing potentials $\varphi_0$ and $\boldsymbol{F}_0$, which are the solutions to the electrostatic equations.

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Equality is obtained for the unique minimizing potentials $\varphi_0$ and $\boldsymbol{F}_0$, which are the solutions to the electrostatic equations. If we can **guess** $\varphi$ and $\boldsymbol{F}$ based on limited information, we can bound $\gamma_{\mathrm{e}}$!

# Application to absorption

### Low pass case

Since $\sigma^{\mathrm{ext}} = \sigma^{\mathrm{sca}} + \sigma^{\mathrm{abs}} \geq \sigma^{\mathrm{abs}}$, we have

$$(\lambda_2 - \lambda_1)\frac{\sigma^{\mathrm{abs}}_{\mathrm{min}}}{A} \leq \pi^2 \frac{\gamma_{\mathrm{e}} + \gamma_{\mathrm{m}}}{A}$$

where $\sigma^{\mathrm{abs}}_{\mathrm{min}}/A$ is the minimum allowed absorption cross section per unit cell area in the band.

### Ground plane backing

Rozanov showed in 2000, using the same analytical properties, that

$$(\lambda_2 - \lambda_1)\Gamma_0 \leq 172\mu_{\mathrm{s}}d$$

where $\Gamma_0 = \min_{\lambda\in(\lambda_1,\lambda_2)} |\Gamma_{\mathrm{dB}}(\lambda)|$ is the minimum allowed return loss in dB. Logarithmic metric instead of linear.

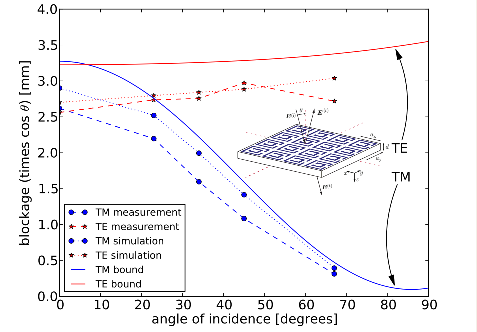
The result is that the product of bandwidth and absorption performance is bounded by the polarizability per unit area.

# Other applications

## Transmission blockage

Low pass structure:

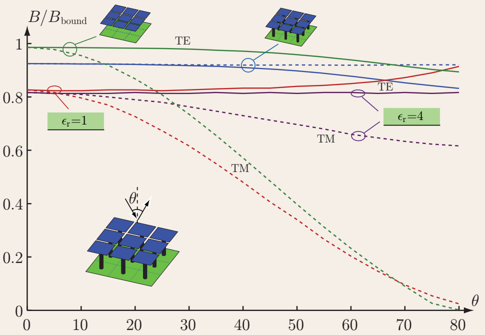
$$(\lambda_2-\lambda_1)\ln\frac{1}{T_0}\leq\pi^2\frac{\gamma_e+\gamma_m}{2A}$$




## High impedance surfaces

PEC ground plane backing, $Z_s \geq 2Z_0$:

$$\frac{\lambda_2-\lambda_1}{d}\leq\pi$$



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Let $\sigma_{\mathrm{ext}} = \mathrm{Re}\{H\}$, where $H(\omega)$ is a transfer function corresponding to a real, causal impulse response $h(t) = [\mathcal{F}^{-1}H](t)$. Since $\mathrm{Im}\{H(\omega)\}$ is odd in $\omega$, we have

$$\int_0^\infty \frac{\mathrm{Re}\{H(\omega)\}}{\omega^2}\,\mathrm{d}\omega = \frac{1}{2}\int_{-\infty}^\infty \frac{\mathrm{Re}\{H(\omega)\}}{\omega^2}\,\mathrm{d}\omega = \frac{1}{2}\int_{-\infty}^\infty \frac{H(\omega)}{\omega^2}\mathrm{e}^{\mathrm{j}\omega t}\,\mathrm{d}\omega\bigg|_{t=0}$$

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$$= \frac{1}{2}\left[2\pi(\mathcal{F}^{-1}H(\omega)) * (\mathcal{F}^{-1}\frac{1}{\omega^2})\right]_{t=0} = \pi\int_{-\infty}^\infty h(t')\left(-\frac{1}{2}|t-t'|\right)\mathrm{d}t'\Big|_{t=0}$$

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Summarizing, we have

$$\boxed{\int_0^\infty \frac{\text{Re}\{H(\omega)\}}{\omega^2}\,\mathrm{d}\omega = -\frac{\pi}{2}\int_0^\infty th(t)\,\mathrm{d}t}$$

Generalizing, it can be shown that sum rules with weight factors $1/\omega^{2n}$ correspond to moments $t^{2n-1}$ of the impulse response.

# Interpretation

The negative of the first moment of the impulse response,

$$\boxed{t_h = -\int_0^\infty t h(t)\,\mathrm{d}t}$$

can be seen as a delay (known as Elmore delay in electronic circuits).

- Delay is an important quantity in filters, communication channels, logic circuits etc.
- When $h(t) \leq 0$, we have $t_h = \int_0^\infty t|h(t)|\,\mathrm{d}t$ and the delay interpretation is clear.
- When $h(t)$ has alternating signs, we have $t_h \leq \int_0^\infty t|h(t)|\,\mathrm{d}t$, and the delay interpretation is less clear, but is sometimes used as a definition.

Thus, the time domain version of the forward scattering sum rules in 1D, 2D, 3D, is

$$t_h = \text{(expression proportional to } \gamma\text{)}$$

meaning

*the delay in forward transmission through any linear, time-invariant, passive scattering system is given by the static polarizability $\gamma$.*

For propagation through rain or fog, $\gamma$ is related to the volume of water in the air.

# Impulse forward scattering

The impulse forward scattering in different dimensions is

$$\text{1D}: \quad h(t) = -2A(\mathcal{T}(t) - \delta(t))$$

$$\text{2D}: \quad h(t) = -\hat{\boldsymbol{k}} \times \int_0^b \oint_C \left[\hat{\boldsymbol{k}} \times (\hat{\boldsymbol{n}} \times \eta_0 \boldsymbol{H}^{\mathrm{s}}(\boldsymbol{r}, t + \hat{\boldsymbol{k}} \cdot \boldsymbol{r}/c)) + \boldsymbol{E}^{\mathrm{s}}(\boldsymbol{r}, t + \hat{\boldsymbol{k}} \cdot \boldsymbol{r}/c) \times \hat{\boldsymbol{n}}\right] \mathrm{d}\ell \, \mathrm{d}z$$

$$\text{3D}: \quad h(t) = -\hat{\boldsymbol{k}} \times \int_S \left[\hat{\boldsymbol{k}} \times (\hat{\boldsymbol{n}} \times \eta_0 \boldsymbol{H}^{\mathrm{s}}(\boldsymbol{r}, t + \hat{\boldsymbol{k}} \cdot \boldsymbol{r}/c)) + \boldsymbol{E}^{\mathrm{s}}(\boldsymbol{r}, t + \hat{\boldsymbol{k}} \cdot \boldsymbol{r}/c) \times \hat{\boldsymbol{n}}\right] \mathrm{d}S$$

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Consider the 1D case for simplicity:

$$\frac{\pi\gamma}{4Ac} = \int_0^\infty \frac{\mathrm{Re}\{1 - T(\omega)\}}{\omega^2} \, \mathrm{d}\omega = \frac{\pi}{2} \int_0^\infty t(\mathcal{T}(t) - \delta(t)) \, \mathrm{d}t = \frac{\pi}{2} \int_0^\infty t\mathcal{T}(t) \, \mathrm{d}t = \frac{\pi}{2} t_{-\mathcal{T}}$$

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$$2\text{D}: \quad h(t) = -\hat{\boldsymbol{k}} \times \int_0^b \oint_C \left[\hat{\boldsymbol{k}} \times (\hat{\boldsymbol{n}} \times \eta_0 \boldsymbol{H}^{\mathrm{s}}(\boldsymbol{r}, t + \hat{\boldsymbol{k}} \cdot \boldsymbol{r}/c)) + \boldsymbol{E}^{\mathrm{s}}(\boldsymbol{r}, t + \hat{\boldsymbol{k}} \cdot \boldsymbol{r}/c) \times \hat{\boldsymbol{n}}\right] \mathrm{d}\ell \, \mathrm{d}z$$

$$3\text{D}: \quad h(t) = -\hat{\boldsymbol{k}} \times \int_S \left[\hat{\boldsymbol{k}} \times (\hat{\boldsymbol{n}} \times \eta_0 \boldsymbol{H}^{\mathrm{s}}(\boldsymbol{r}, t + \hat{\boldsymbol{k}} \cdot \boldsymbol{r}/c)) + \boldsymbol{E}^{\mathrm{s}}(\boldsymbol{r}, t + \hat{\boldsymbol{k}} \cdot \boldsymbol{r}/c) \times \hat{\boldsymbol{n}}\right] \mathrm{d}S$$

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For a non-dispersive slab with refractive index $n$ and thickness $d$ we have

$$\boxed{t_{-\mathcal{T}}} = \frac{\gamma}{2Ac} = \frac{Ad(n^2 - 1)}{2Ac} = \frac{d(n-1)(n+1)}{2c} = t_{\mathrm{d}} \frac{n+1}{2} = \boxed{t_{\mathrm{d}} + t_{\mathrm{d}} \frac{n-1}{2}}$$

where $t_{\mathrm{d}} = (n-1)d/c$ is the one-pass delay through the slab. The delay as measured by $t_{-\mathcal{T}}$ takes into account additional multiple reflections inside the slab.

# Power series of the forward scattering

To indicate the generalization to higher moments, consider the 3D forward scattering:

$$H(\omega) = -\frac{\boldsymbol{E}_0^*}{|\boldsymbol{E}_0|^2} \cdot \hat{\boldsymbol{k}} \times \int_S \left[\hat{\boldsymbol{k}} \times (\hat{\boldsymbol{n}} \times \eta_0 \boldsymbol{H}^{\mathrm{s}}) + \boldsymbol{E}^{\mathrm{s}} \times \hat{\boldsymbol{n}}\right] \mathrm{e}^{\mathrm{j}\boldsymbol{k}\cdot\boldsymbol{r}} \,\mathrm{d}S$$

With the power series ($a$ being the radius of a sphere enclosing the scatterer)

$$\boldsymbol{E}^{\mathrm{s}} = \sum_{n=0}^{\infty} (\mathrm{j}ka)^n \boldsymbol{E}_n^{\mathrm{s}} \qquad \boldsymbol{H}^{\mathrm{s}} = \sum_{n=0}^{\infty} (\mathrm{j}ka)^n \boldsymbol{H}_n^{\mathrm{s}} \qquad \mathrm{e}^{\mathrm{j}\boldsymbol{k}\cdot\boldsymbol{r}} = \sum_{n=0}^{\infty} (\mathrm{j}ka)^n \frac{(\hat{\boldsymbol{k}} \cdot \boldsymbol{r}/a)^n}{n!}$$

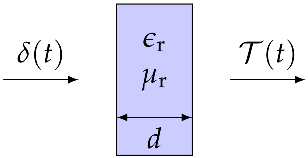
we have

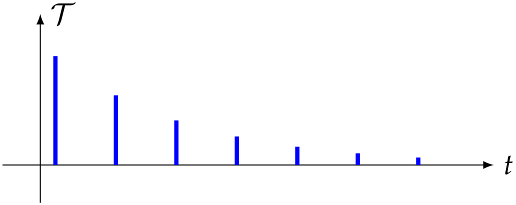
$$\begin{aligned} H(\omega) &= -\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (\mathrm{j}ka)^{m+n} \frac{\boldsymbol{E}_0^*}{|\boldsymbol{E}_0|^2} \cdot \hat{\boldsymbol{k}} \times \int_S \left[\hat{\boldsymbol{k}} \times (\hat{\boldsymbol{n}} \times \eta_0 \boldsymbol{H}_n^{\mathrm{s}}) + \boldsymbol{E}_n^{\mathrm{s}} \times \hat{\boldsymbol{n}}\right] \frac{(\hat{\boldsymbol{k}} \cdot \boldsymbol{r}/a)^m}{m!} \,\mathrm{d}S \\ &= -\sum_{n=0}^{\infty} (\mathrm{j}ka)^n \sum_{m=0}^{n} \frac{\boldsymbol{E}_0^*}{|\boldsymbol{E}_0|^2} \cdot \hat{\boldsymbol{k}} \times \int_S \left[\hat{\boldsymbol{k}} \times (\hat{\boldsymbol{n}} \times \eta_0 \boldsymbol{H}_m^{\mathrm{s}}) + \boldsymbol{E}_m^{\mathrm{s}} \times \hat{\boldsymbol{n}}\right] \frac{(\hat{\boldsymbol{k}} \cdot \boldsymbol{r}/a)^{n-m}}{(n-m)!} \,\mathrm{d}S \end{aligned}$$

The expansion terms (moments of the scattered field distribution) correspond to the derivatives $H^{(n)}|_{\omega=0}$, or the moments $\int_0^\infty t^n h(t)\,\mathrm{d}t$.

# Examples

# Transmission through a non-dispersive slab




Multiple reflections inside the slab gives a sequence of exponentially decaying delta pulses as impulse response:

$$\mathcal{T}(t) = \sum_{n=0}^{\infty}(1-\rho^2)\rho^{2n}\delta(t - n2t_0 - t_0 + t_b)$$

$$t_0 = \sqrt{\epsilon_r\mu_r}\frac{d}{c} \qquad t_b = \frac{d}{c} \qquad \rho = \frac{\eta_r - 1}{\eta_r + 1} \qquad \eta_r = \sqrt{\frac{\mu_r}{\epsilon_r}}$$

The sum rule is $\int_0^{\infty}\frac{\mathrm{Re}\{1-T(\omega)\}}{\omega^2}\,\mathrm{d}\omega = \frac{\pi\gamma}{4Ac}$, with $\gamma = Ad(\epsilon_r - 1 + \mu_r - 1)$.

# Verifying the sum rule

Given $\mathcal{T}(t) = \sum_{n=0}^{\infty}(1-\rho^2)\rho^{2n}\delta(t - n2t_0 - t_0 + t_\mathrm{b})$, the first moment is

$$\int_0^\infty t\mathcal{T}(t)\,\mathrm{d}t = \sum_{n=0}^{\infty}(1-\rho^2)\rho^{2n}(n2t_0 + t_0 - t_\mathrm{b})$$

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$$\int_0^\infty t\mathcal{T}(t)\,\mathrm{d}t = \sum_{n=0}^{\infty}(1-\rho^2)\rho^{2n}(n2t_0 + t_0 - t_\mathrm{b})$$
$$= (1-\rho^2)\left(2t_0\sum_{n=0}^{\infty} n\rho^{2n} + (t_0 - t_\mathrm{b})\sum_{n=0}^{\infty}\rho^{2n}\right)$$

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Given $\mathcal{T}(t) = \sum_{n=0}^{\infty}(1-\rho^2)\rho^{2n}\delta(t - n2t_0 - t_0 + t_{\mathrm{b}})$, the first moment is

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\int_0^{\infty} t\mathcal{T}(t)\,\mathrm{d}t &= \sum_{n=0}^{\infty}(1-\rho^2)\rho^{2n}(n2t_0 + t_0 - t_{\mathrm{b}}) \\
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&= (1-\rho^2)\left(2t_0\frac{\rho^2}{(1-\rho^2)^2} + (t_0 - t_{\mathrm{b}})\frac{1}{1-\rho^2}\right)
\end{aligned}$$

# Verifying the sum rule

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&= 2t_0\frac{\rho^2}{1-\rho^2} + t_0 - t_\mathrm{b}
\end{aligned}$$

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&= 2t_0\frac{\rho^2}{1-\rho^2} + t_0 - t_b = \quad \cdots \quad =
\end{aligned}$$

# Verifying the sum rule

Given $\mathcal{T}(t) = \sum_{n=0}^{\infty}(1-\rho^2)\rho^{2n}\delta(t - n2t_0 - t_0 + t_\mathrm{b})$, the first moment is

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&= 2t_0\frac{\rho^2}{1-\rho^2} + t_0 - t_\mathrm{b} = \quad \cdots \quad = \frac{d}{c}\frac{\epsilon_\mathrm{r} + \mu_\mathrm{r} - 2}{2}
\end{aligned}$$

# Verifying the sum rule

Given $\mathcal{T}(t) = \sum_{n=0}^{\infty}(1-\rho^2)\rho^{2n}\delta(t - n2t_0 - t_0 + t_\mathrm{b})$, the first moment is

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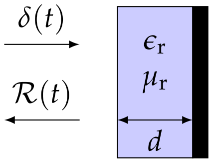
&= 2t_0\frac{\rho^2}{1-\rho^2} + t_0 - t_\mathrm{b} = \quad \cdots \quad = \frac{d}{c}\frac{\epsilon_\mathrm{r} + \mu_\mathrm{r} - 2}{2}
\end{aligned}$$

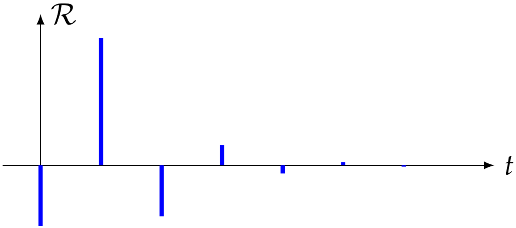
Hence, we have

$$\boxed{\int_0^\infty t\mathcal{T}(t)\,\mathrm{d}t = \frac{d}{c}\frac{\epsilon_\mathrm{r} + \mu_\mathrm{r} - 2}{2} = \frac{1}{2Ac}Ad(\epsilon_\mathrm{r} - 1 + \mu_\mathrm{r} - 1) = \frac{\gamma}{2Ac}}$$

and the sum rule is verified.

# Reflection from a grounded non-dispersive slab



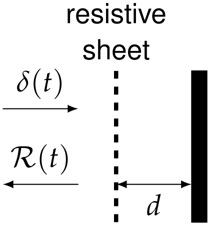

With $\epsilon_r > \mu_r \geq 1$ we have $\rho < 0$ and alternating signs in reflections.

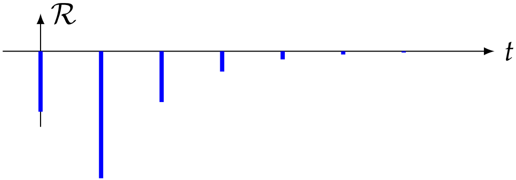
$$\mathcal{R}(t) = \rho\delta(t) - \sum_{n=1}^{\infty}(1-\rho^2)\rho^{n-1}\delta(t - n2t_0)$$

$$\int_0^{\infty} \frac{\mathrm{Re}\{1 + R(\omega)\}}{\omega^2}\,\mathrm{d}\omega = \pi\mu_r\frac{d}{c} \quad \Rightarrow \quad \boxed{-\int_0^{\infty} t\mathcal{R}(t)\,\mathrm{d}t = 2\mu_r\frac{d}{c}}$$

This can be explicitly verified in the same way as the previous sum rule.

# Reflection from a Salisbury absorber



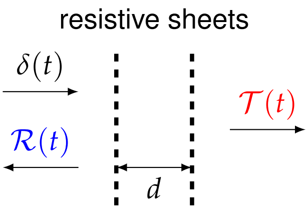

The reflection $\rho = \frac{-1}{1+2/G}$, $G = \sigma d_s \sqrt{\mu_0/\epsilon_0}$, from a resistive sheet has the same sign from both directions, leading to non-alternating signs of reflections.

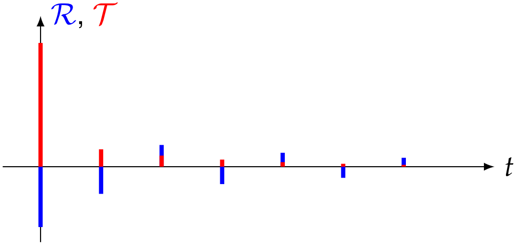
$$\mathcal{R}(t) = \rho\delta(t) - \sum_{n=1}^{\infty} (1+\rho)^2(-\rho)^{n-1}\delta(t - n2t_0)$$

$$\boxed{-\int_0^{\infty} t\mathcal{R}(t)\,\mathrm{d}t = 2\frac{d}{c}}$$

Note the delay only depends on thickness! True for any non-magnetic absorber.

# Fabry-Perot resonator

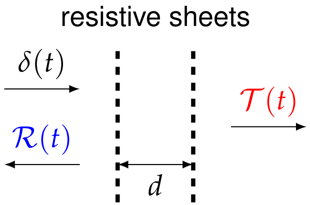


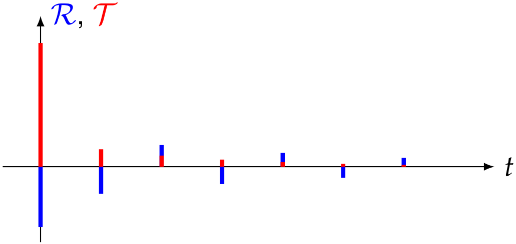




$$\mathcal{T}(t) = \sum_{n=0}^{\infty} (1+\rho)^2 \rho^{2n} \delta(t - n2t_0)$$

$$\mathcal{R}(t) = \rho\delta(t) + \sum_{n=1}^{\infty} (1+\rho)^2 \rho^{2n-1} \delta(t - n2t_0)$$

# Fabry-Perot resonator

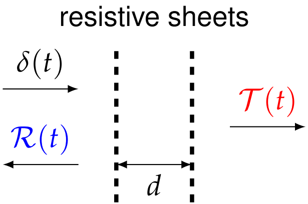


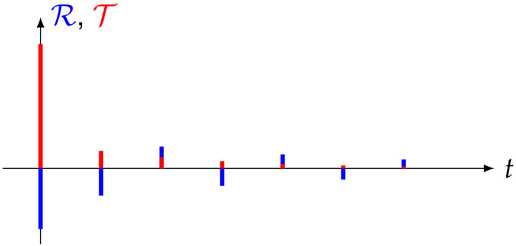




$$\mathcal{T}(t) = \sum_{n=0}^{\infty} (1+\rho)^2 \rho^{2n} \delta(t - n2t_0) \qquad \int_0^{\infty} t\mathcal{T}(t)\,\mathrm{d}t = \frac{t_0}{2}\frac{1}{(1+1/G)^2}$$

$$\mathcal{R}(t) = \rho\delta(t) + \sum_{n=1}^{\infty} (1+\rho)^2 \rho^{2n-1} \delta(t - n2t_0) \qquad -\int_0^{\infty} t\mathcal{R}(t)\,\mathrm{d}t = \frac{t_0}{2}\frac{1+2/G}{(1+1/G)^2}$$

# Fabry-Perot resonator




$$\mathcal{T}(t) = \sum_{n=0}^{\infty} (1+\rho)^2 \rho^{2n} \delta(t - n2t_0) \qquad \int_0^{\infty} t\mathcal{T}(t)\,\mathrm{d}t = \frac{t_0}{2} \frac{1}{(1+1/G)^2}$$

$$\mathcal{R}(t) = \rho\delta(t) + \sum_{n=1}^{\infty} (1+\rho)^2 \rho^{2n-1} \delta(t - n2t_0) \qquad -\int_0^{\infty} t\mathcal{R}(t)\,\mathrm{d}t = \frac{t_0}{2} \frac{1+2/G}{(1+1/G)^2}$$

The Fabry-Perot resonator is neither low pass, nor backed by a ground plane. Hence the previous sum rules do not apply.

# Impulse moments using bandlimited signals

Explicit impulse responses are very rare. In numerical or experimental approaches, the exciting signal is bandlimited. Consider the input signal $x(t)$ and output signal $y(t)$:

$$y(t) = \int_0^\infty h(t-t')x(t')\,\mathrm{d}t'$$

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The negative first moment of the bandlimited signal is

$$t_y = -\int_0^\infty t y(t)\,\mathrm{d}t = -\int_{t=0}^\infty t \int_{t'=0}^\infty h(t-t')x(t')\,\mathrm{d}t'\,\mathrm{d}t$$

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&= -\int_{t'=0}^\infty x(t') \int_{t=0}^\infty t h(t-t')\,\mathrm{d}t\,\mathrm{d}t' = -\int_{t'=0}^\infty x(t') \left[\mathrm{j}\frac{\partial}{\partial\omega} H(\omega)\mathrm{e}^{-\mathrm{j}\omega t'}\right]_{\omega=0} \mathrm{d}t' \\
&= -\int_{t'=0}^\infty x(t') \left[\mathrm{j}H'(0) + t'H(0)\right]\mathrm{d}t' = -\int_{t'=0}^\infty x(t') \left[\int_0^\infty t h(t)\,\mathrm{d}t + t'H(0)\right]\mathrm{d}t'
\end{aligned}$$

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where $X(0) = \int_0^\infty x(t)\,\mathrm{d}t$ is the zeroth moment of $x(t)$.

# Impulse moments using bandlimited signals

Apply two different input signals $x_1(t)$ and $x_2(t)$:

$$\begin{aligned} t_{y_1} &= t_h X_1(0) + t_{x_1} H(0) \\ t_{y_2} &= t_h X_2(0) + t_{x_2} H(0) \end{aligned}$$

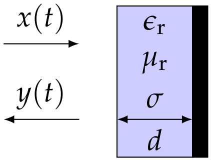
and solve for impulse response moments $H(0)$ and $t_h$ in terms of bandlimited data:

$$\boxed{\begin{aligned} H(0) &= \frac{t_{y_1} X_2(0) - t_{y_2} X_1(0)}{t_{x_1} X_2(0) - t_{x_2} X_1(0)} \\ t_h &= \frac{t_{y_1} t_{x_2} - t_{y_2} t_{x_1}}{t_{x_2} X_1(0) - t_{x_1} X_2(0)} \end{aligned}}$$

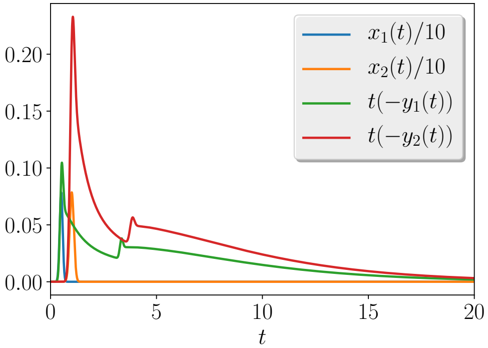
This has been verified by explicit calculations for the Salisbury screen, and implemented numerically as follows. This may prove to be a useful tool also for experimental work, but requires input signals with a DC component.

# Verifying the sum rule for a dispersive slab

Maxwell's equations for a 1D-problem were solved numerically in the time domain using finite differences.




$$\begin{cases} \partial_z E + \mu \partial_t H = 0 \\ \partial_z H + \epsilon \partial_t E + \sigma E = 0 \end{cases}$$



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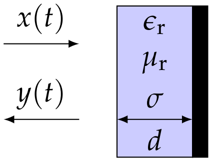


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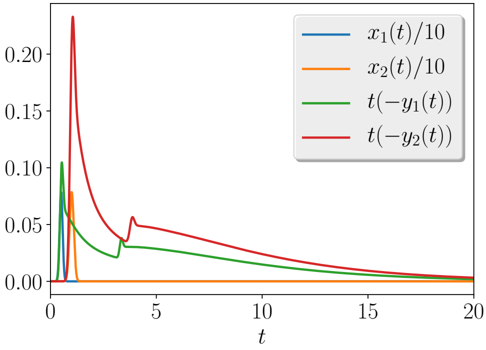


$$t_{\mathcal{R}} = 2\mu_{\mathrm{r}} \frac{d}{c}$$

| | | | | | | |
|---|---|---|---|---|---|---|
| $\epsilon_{\mathrm{r}}$ | 1 | 2 | 1 | 1 | 1 | 2 |
| $\mu_{\mathrm{r}}$ | 1 | 1 | 2 | 1 | 1 | 1 |
| $\sigma$ | 0 | 0 | 0 | 0.01 | 0 | 0.01 |
| $d/c$ | 1 | 1 | 1 | 1 | 2 | 1 |
| $-\int_0^{t_{\max}} t\mathcal{R}(t)\,\mathrm{d}t$ | | | | | | |

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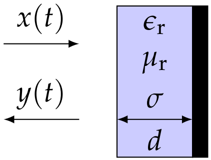
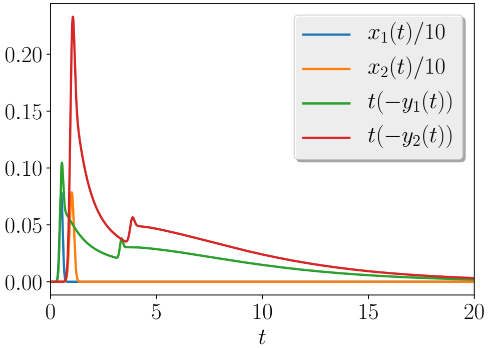



$$\begin{cases} \partial_z E + \mu \partial_t H = 0 \\ \partial_z H + \epsilon \partial_t E + \sigma E = 0 \end{cases}$$




$$\boxed{t_{\mathcal{R}} = 2\mu_{\mathrm{r}}\frac{d}{c}}$$

| | | | | | | |
|---|---|---|---|---|---|---|
| $\epsilon_{\mathrm{r}}$ | 1 | 2 | 1 | 1 | 1 | 2 |
| $\mu_{\mathrm{r}}$ | 1 | 1 | 2 | 1 | 1 | 1 |
| $\sigma$ | 0 | 0 | 0 | 0.01 | 0 | 0.01 |
| $d/c$ | 1 | 1 | 1 | 1 | 2 | 1 |
| $-\int_0^{t_{\max}} t\mathcal{R}(t)\,\mathrm{d}t$ | 1.999 | 2.000 | 4.000 | 1.999 | 3.999 | 2.000 |

# Conclusions

# Conclusions

- The frequency domain sum rules were rewritten in terms of the time domain impulse response.
- The sum rules become restrictions on the moments of impulse response.
- The first moment is (at least sometimes) associated with the average delay through the system. This quantity is of interest in many applications, like filters, communication channels, and electrical networks.
- The static polarizability may be directly linked to physically interesting quantities, like the volume of water in the air.
- A few time domain sum rules were verified by explicit calculations and numerical simulations.
- A means of extracting moments of the impulse response using bandlimited data was demonstrated.

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