A PDE approach to the N-body problem with strong force

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Outline

- General N-body problem
  1. Brief background.
  2. Ground state energy and excited energy.
  3. Dynamic classification: sharp result N=2 and partial results for $N \geq 3$
- Restricted 3body problem: Hill's type lunar problem.
  1. Derivation of the equations of motion
  2. Dynamic classification
- Conclusions and Perspectives

N-body problem

- The N-body problem is a system of ODEs:

$$m_i \ddot{x}_i = \partial_{x_i} U(\mathbf{x}) = -\alpha \sum_{j \neq i} \frac{m_i m_j (x_i - x_j)}{|x_i - x_j|^{\alpha+2}}, \quad i = 1, \cdots, N.$$

- Each body has mass m_i, position $x_i \in \mathbb{R}^3$, and velocity $\dot{x}_i$.
- The self-potential

$$U(\mathbf{x}) = \sum_{i<j} \frac{m_i m_j}{|x_i - x_j|^{\alpha}}, \quad \alpha > 0$$

  - $\alpha = 1$: Newtonian gravitation;
  - $\alpha \geq 2$: "strong force": Lennard-Jones potential which models interaction between a pair of neutral atoms or molecules $U_{LJ}(r) = -\frac{A}{r^6} + \frac{B}{r^{12}}$, $A, B > 0$.

Conservation of N-body problem

- The N-body problem enjoys conservation of energy

$$E(\mathbf{x}, \dot{\mathbf{x}}) := \frac{1}{2}\sum_{i=1}^{N} m_i |\dot{x}_i|^2 - U(\mathbf{x}) \tag{1}$$

- Angular momentum

$$A(\mathbf{x}, \dot{\mathbf{x}}) := \sum_{i=1}^{N} m_i x_i \times \dot{x}_i \tag{2}$$

- Linear momentum

$$M(\mathbf{x}, \dot{\mathbf{x}}) := \sum_{i=1}^{N} m_i \dot{x}_i \tag{3}$$

Usually fix center of mass: $\sum_{i=1}^{N} m_i x_i = 0$

Global existence and singularity

- U is a real-analytic function on $(\mathbb{R}^3)^N \setminus \Delta$:

$$\Delta_{ij} = \{\mathbf{x} = (x_1, \cdots, x_N) \in (\mathbb{R}^3)^N | x_i = x_j\},$$
$$\Delta = \bigcup_{i<j} \Delta_{ij}.$$

- Given $\mathbf{x}(0) \in (\mathbb{R}^3)^N \setminus \Delta$, $\dot{\mathbf{x}}(0) \in (\mathbb{R}^3)^N$, there exists a unique solution $\mathbf{x}(t)$ defined on $[0, \sigma)$, where σ is maximal.
- If $\sigma < \infty$, $\mathbf{x}(t)$ is *singular* at σ;
- If $\sigma = \infty$, $\mathbf{x}(t)$ *exists globally.*

Singularity of the N-body problem

Theorem (Painlevé, 1895)

If $\mathbf{x}(t)$ experiences a singularity at $t = \sigma$, then

$$d(\mathbf{x}(t), \Delta) \to 0, \quad \textit{as } t \to \sigma.$$

- if $\mathbf{x}(t)$ approaches a finite point in Δ, σ is **collision singularity**;
- otherwise, σ is **non-collision singularity**.
  $\alpha = 1$, $N = 5$, first non-collision singularity example by Xia (1992)
- When $\alpha > 2$, only collision singularity.

Saari's Improbability Theorem $0 < \alpha < 2$

Theorem (Saari, 1971-1973)

The set of initial conditions for Newtonian N-body problem leading to collisions has Lebesgue measure zero in the phase space.

- Fleischer and Knauf (2018) extended Saari's improbability theorem to $0 < \alpha < 2$.
- Saari and Xia (1996): it is very likely that the total singularity set has zero Lebesgue measure.

Saari's Improbability Theorem $0 < \alpha < 2$

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- Saari and Xia (1996): it is very likely that the total singularity set has zero Lebesgue measure.
- $\alpha \geq 2$, collision set has positive Lebesgue measure.

Global existence and singularity

- Our goal: characterize the set of initial conditions yielding global solutions or singular solutions under some energy threshold constraints.
- The idea was motivated from PDE.

Motivation from PDE

- Nonlinear dispersive equations, e.g. Klein-Gordon, NLS.
- *scattering, blow-up, solitary waves*
- Global dynamics from initial data: energy below ground state, by the sign of a threshold functional K:
  - $K(\textit{initial data}) \geq 0 \Rightarrow$ scattering of the solution;
  - $K(\textit{initial data}) < 0 \Rightarrow$ finite time blow-up of the solution.
- Extensions to slightly above ground state. Below first excited energy, etc.
- Kenig-Merle, Payne-Sattinger, Shatah, Duyckaerts-Merle, Ibrahim-Masmoudi-Nakanishi, Nakanishi-Schlag, Akahori-Ibrahim-Kikuchi-Nawa and many others...

Ground state for N-body problem

- The Lagrange-Jacobi identity for $I(\mathbf{x}) := \sum_{i=1}^{N} m_i |x_i|^2$,

$$\frac{d^2}{dt^2} I(\mathbf{x}(t)) = 4[E(\mathbf{x}, \dot{\mathbf{x}}) - (\frac{\alpha}{2} - 1)U(\mathbf{x})]$$

Definition (Ground state energy)

Let $V(\mathbf{x}, \dot{\mathbf{x}}) := E(\mathbf{x}, \dot{\mathbf{x}}) - (\alpha/2 - 1)U(\mathbf{x})$,

$$E^\star := \inf\{E(\mathbf{x}, \dot{\mathbf{x}}) | V(\mathbf{x}, \dot{\mathbf{x}}) = 0\}.$$

- when $\alpha \geq 2$, $E^\star = 0$
- when all bodies are at infinity with zero velocity $\Rightarrow$ the *ground state.*

Singularity below the ground state for $\alpha \geq 2$

- If $E = E(\mathbf{x}(0), \dot{\mathbf{x}}(0)) < E^{\star} = 0$, then

$$\frac{d^2}{dt^2} I(\mathbf{x}(t)) \leq 4E < 0$$

$$\Rightarrow I(t) \leq 2Et^2 + \dot{I}(0)t + I(0)$$

- When $\alpha \geq 2$, every solution below the ground state energy is singular.
- We want to go beyond the zero energy.

Relative equilibrium

- A solution $\mathbf{x}(t) = (x_1(t), \cdots, x_N(t))$ of the N-body problem is called a *relative equilibrium* if there exists $O(t) \in SO(3)$ such that
$$x_i(t) = O(t)x_i(0),$$
for all $i = 1, \cdots, N$.
- normal form of $O(t)$ is

$$\exp(\omega \tilde{J} t) = \begin{pmatrix} \cos(\omega t) & \sin(\omega t) & 0 \\ -\sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{J} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Relative equilibrium and Central configuration

- A R.E. with frequency ω and initial configuration $\mathbf{q}$ satisfies

$$\nabla(\frac{\omega^2}{2} I(\mathbf{q}) + U(\mathbf{q})) = 0. \tag{4}$$

- Effective potential

$$U_{\text{eff}}(\mathbf{x}) := -(\frac{\omega^2}{2} I(\mathbf{x}) + U(\mathbf{x})).$$

- Critical points of U_{eff} are known as central configurations.
- Let

$$K_\omega(\mathbf{x}) := -\mathbf{x} \cdot \nabla U_{\text{eff}}(\mathbf{x}) = \omega^2 I(\mathbf{x}) - \alpha U(\mathbf{x}).$$

here,

$$K_\omega(\mathbf{x}) = -\frac{d}{d\lambda}(U_{\text{eff}}(\lambda \mathbf{x}))|_{\lambda=1}.$$

Excited energy

- The energy of a ω-relative equilibrium is

$$E_\omega(\mathbf{q}) := \frac{\omega^2}{2} I(\mathbf{q}) - U(\mathbf{q}).$$

Definition (Excited energy)

$$E^*(\omega) := \inf\{E_\omega(\mathbf{x}) : K_\omega(\mathbf{x}) = 0\}.$$

- When $\alpha > 2$, $E^*(\omega)$ is strictly positive.
- $E^*(\omega)$ is achieved by central configuration.

Dichotomy below the excited energy

Theorem (Dichotomy below the excited energy)

For $\alpha > 2$, let $\mathbf{x}(t)$ be a solution of the N-body problem, if there exists $t^* > 0$ so that for $t > t^*$,

- $\mathbf{x}(t)$ *stays in* $\mathcal{K}^+(\omega)$, *then* $\mathbf{x}(t)$ *exists globally;*
- $\mathbf{x}(t)$ *stays in* $\mathcal{K}^-(\omega)$, *then* $\mathbf{x}(t)$ *has a singularity. Moreover, all singularities are collision singularities.*

- $$\mathcal{K}^+(\omega) = \{(\mathbf{x}, \dot{\mathbf{x}}) : E(\mathbf{x}, \dot{\mathbf{x}}) < E^*(\omega), K_\omega(\mathbf{x}) \geq 0\},$$
  $$\mathcal{K}^-(\omega) = \{(\mathbf{x}, \dot{\mathbf{x}}) : E(\mathbf{x}, \dot{\mathbf{x}}) < E^*(\omega), K_\omega(\mathbf{x}) < 0\}.$$
- The problem is that K_ω is not sign-definite, and it may change the sign infinitely many times.

Dichotomy for the 2-body problem

Theorem (Dichotomy for the 2-body problem)

Let $m_1 + m_2 = 1$, *and* $m_1 x_1 + m_2 x_2 = 0$,

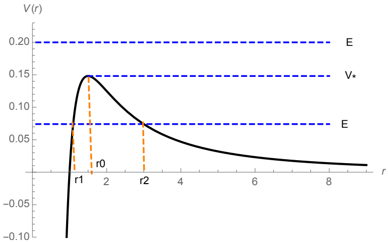
$$\mathcal{K}^+(\omega) = \{(\mathbf{x}, \dot{\mathbf{x}}) : E(\mathbf{x}, \dot{\mathbf{x}}) < E^*(\omega), |A(\mathbf{x}, \dot{\mathbf{x}})| \geq A^*(\omega), K_\omega(\mathbf{x}) \geq 0\}$$
$$\mathcal{K}^-(\omega) = \{(\mathbf{x}, \dot{\mathbf{x}}) : E(\mathbf{x}, \dot{\mathbf{x}}) < E^*(\omega), |A(\mathbf{x}, \dot{\mathbf{x}})| \geq A^*(\omega), K_\omega(\mathbf{x}) < 0\}$$

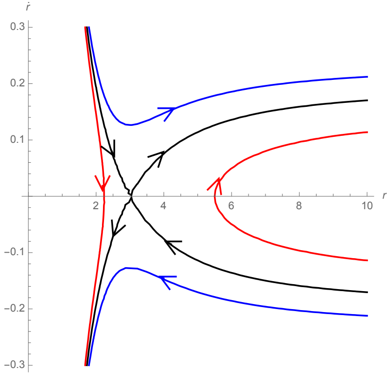
then $\mathcal{K}^\pm(\omega)$ *are invariant. Solutions in* $\mathcal{K}^+(\omega)$ *exist globally and solutions in* $\mathcal{K}^-(\omega)$ *experiences a singularity.*

- $E^*(\omega) = m_1 m_2 \alpha^{\frac{2}{2-\alpha}} (\frac{1}{2} - \frac{1}{\alpha})(\alpha^{\frac{2}{2+\alpha}} \omega^{\frac{\alpha-2}{\alpha+2}})^{\frac{2\alpha}{\alpha-2}}$
- $A^*(\omega) = m_1 m_2 \alpha^{\frac{2}{2+\alpha}} \omega^{\frac{\alpha-2}{\alpha+2}}$

The two-body problem and Kepler problem

Let $x = x_1 - x_2$, the Kepler problem for $\alpha > 2$

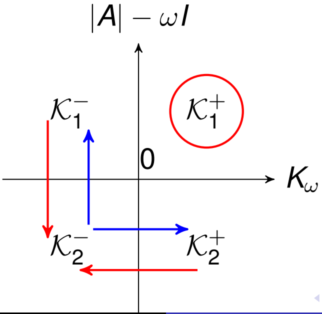




Refinement of characterization for $N \geq 3$

- Let $\mathcal{K} = \{(\mathbf{x}, \dot{\mathbf{x}}) : E(\mathbf{x}, \dot{\mathbf{x}}) < E^*(\omega), |A(\mathbf{x}, \dot{\mathbf{x}})| \neq 0\}$

$$\mathcal{K}_1^+ = \{(\mathbf{x}, \dot{\mathbf{x}}) \in \mathcal{K} : |A(\mathbf{x}, \dot{\mathbf{x}})| \geq \omega I(\mathbf{x}), K_\omega(\mathbf{x}) \geq 0\}$$
$$\mathcal{K}_1^- = \{(\mathbf{x}, \dot{\mathbf{x}}) \in \mathcal{K} : |A(\mathbf{x}, \dot{\mathbf{x}})| \geq \omega I(\mathbf{x}), K_\omega(\mathbf{x}) < 0\}$$
$$\mathcal{K}_2^+ = \{(\mathbf{x}, \dot{\mathbf{x}}) \in \mathcal{K} : |A(\mathbf{x}, \dot{\mathbf{x}})| < \omega I(\mathbf{x}), K_\omega(\mathbf{x}) \geq 0\}$$
$$\mathcal{K}_2^- = \{(\mathbf{x}, \dot{\mathbf{x}}) \in \mathcal{K} : |A(\mathbf{x}, \dot{\mathbf{x}})| < \omega I(\mathbf{x}), K_\omega(\mathbf{x}) < 0\}$$



Theorem (Refinement of characterization for $N \geq 3$)

(a) $\mathcal{K}_1^+$ *is empty.*

(b) *If* $\mathbf{x}(t)$ *starts in* $\mathcal{K}_2^-$, *and enters* $\mathcal{K}_1^-$, *then it stays in* $\mathcal{K}_1^-$ *and experiences a collision singularity.*

(c) *If* $\mathbf{x}(t)$ *starts in* $\mathcal{K}_2^-$, *and never enters* $\mathcal{K}_1^-$, *then it stays in* $\mathcal{K}_2^+ \cup \mathcal{K}_2^-$.

  (c1) *If there exists time* t_1, *so that* $\mathbf{x}(t)$ *stays in* $\mathcal{K}_2^-$ *after* t_1, *then it experiences a collision;*

  (c2) *If there exists time* t_1, *so that* $\mathbf{x}(t)$ *stays in* $\mathcal{K}_2^+$ *after* t_2, *then it exists globally;*

  (c3) *If there are infinitely many transitions between* $\mathcal{K}_2^+$ *and* $\mathcal{K}_2^-$, *then it exists globally.*

(d) *If* $\mathbf{x}(t)$ *starts in* $\mathcal{K}_2^+(resp.\ \mathcal{K}_1^-)$, *and stays in* $\mathcal{K}_2^+(resp.\ \mathcal{K}_1^-)$, *then it exists globally (resp. experiences a collision).*

(e) *If* $\mathbf{x}(t)$ *starts in* $\mathcal{K}_2^+(resp.\ \mathcal{K}_1^-)$, *and enters* $\mathcal{K}_2^-$, *then see (b)(c).*

Non-invariance of $K^{\pm}(\omega)$ for $N \geq 3$: Example 1

Example (Example for the non-invariance of $\mathcal{K}^{+}(\omega)$)

$$\mathcal{K}^{+}(\omega) = \{(\mathbf{x}, \dot{\mathbf{x}}) : E(\mathbf{x}, \dot{\mathbf{x}}) < E^{*}(\omega), K_{\omega}(\mathbf{x}) \geq 0\},$$

$$K_{\omega}(\mathbf{x}) = \frac{\omega^2}{M} \sum_{i<j} m_i m_j r_{ij}^2 - \alpha \sum_{i<j} \frac{m_i m_j}{r_{ij}^{\alpha}}.$$

Homothetic motion: take an equilateral triangle configuration $\mathbf{x}^0$ with initial velocity $\dot{\mathbf{x}}^0 = \mathbf{0}$ and $(\sqrt{3}|x_i^0|)^{2+\alpha} \geq \frac{\alpha M}{\omega^2}$ for $i = 1, 2, 3$. ($(\mathbf{x}^0, \mathbf{0}) \in \mathcal{K}^{+}(\omega)$).

By the attracting forces of the 3 bodies, all of which point to the center of mass (the origin), the 3 bodies will encounter a total collision in finite time.

Non-invariance of $K^{\pm}(\omega)$ for $N \geq 3$: Example 2

Example (Example for the non-invariance of $\mathcal{K}^{-}(\omega)$)

Similarly, take an equilateral triangle configuration $\mathbf{x}^0$ and initial velocity $\dot{\mathbf{x}}^0 = v\mathbf{x}^0$, where $v > 0$. We can choose $(\mathbf{x}^0, \dot{\mathbf{x}}^0) \in \mathcal{K}^{-}(\omega)$ and $E(\mathbf{x}^0, \dot{\mathbf{x}}^0) > 0$. Since

$$E(\mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2}\sum_{i=1}^{3} m_i|\dot{x}_i|^2 + U(\mathbf{x}), \tag{5}$$

is conserved and $U(\mathbf{x}) < 0$, the three bodies will keep going away ($|\dot{\mathbf{x}}| \neq 0$) and never come back, thus enter the set $\mathcal{K}^{+}(\omega)$.

Defining the Hill's lunar problem

- A model for "earth", "moon", "sun"
- Consider a uniform rotating frame with frequency one with reference to a fixed inertial frame.
- Use Jacobi coordinates and make appropriate assumptions on the masses and the distances, one gets the Hill's Lunar Problem. (cf. Hill (1878), Meyer-Schmidt (1982))

Defining the Hill's lunar problem: Cntd.

The planar Hill's equation with homogenous gravitational potential is given by

$$\begin{cases} \ddot{x} - 2\dot{y} & = -V_x \\ \ddot{y} + 2\dot{x} & = -V_y, \end{cases} \tag{6}$$

where

$$V(x,y) = -\frac{\alpha+2}{2}x^2 - \frac{\alpha+2}{r^\alpha}, \quad r = \sqrt{x^2+y^2}, \quad \alpha > 0 \tag{7}$$

is known as the effective potential.

- (x, y) can be thought of as the position of the moon.
- First integral: the energy

$$E(x,y,\dot{x},\dot{y}) := \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + V(x,y). \tag{8}$$

Contour plot of $V(x, y)$

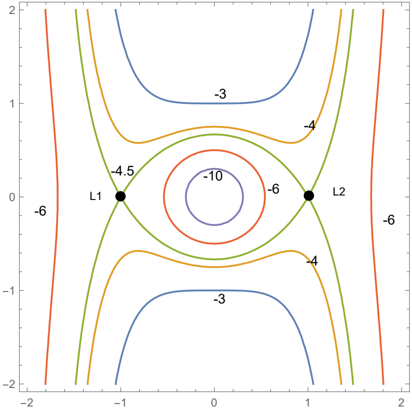


Figure: The contour plot of $V(x, y)$ with $\alpha = 1$. $V(x, y)$ has two critical points $L_1 := (-\alpha^{\frac{1}{\alpha+2}}, 0)$ and $L_2 := (\alpha^{\frac{1}{\alpha+2}}, 0)$.

Defining the ground state

Let $I := \frac{1}{2}(x^2 + y^2)$ be the moment of inertia. Then

$$\frac{d^2 I}{dt^2} = \dot{x}^2 + \dot{y}^2 + 2(x\dot{y} - \dot{x}y) - xV_x - yV_y. \quad (9)$$

Let

$$K(x, y, \dot{x}, \dot{y}) := \dot{x}^2 + \dot{y}^2 + 2(x\dot{y} - \dot{x}y) - xV_x - yV_y, \quad (10)$$

and

$$W(x, y) := -xV_x - yV_y = (\alpha + 2)x^2 - \frac{\alpha + 2}{r^\alpha}, \quad (11)$$

Defining the ground state

Consider the following variational problem in $\mathbb{R}^4$:

$$\inf\{E(x, y, \dot{x}, \dot{y}) | W(x, y) = 0\}. \tag{12}$$

Lemma

When $\alpha \geq 2$, we have

$$\begin{aligned} \inf\{E|W = 0\} &= \inf\{E|K = 0, W = 0\} \\ &= \inf\{E|K \geq 0, W \leq 0\} \\ &= E(L_i, 0) := E^* \end{aligned}$$

Let $Q = (\alpha^{\frac{1}{\alpha+2}}, 0, 0, 0)$, define $\pm Q$ to be the ground states.

Dichotomy below the ground state

Define $\mathcal{K} = \{\Gamma = (x, y, \dot{x}, \dot{y}) | E(\Gamma) < E^*\}$ and set

$$\begin{aligned} \mathcal{K}_+ &= \{\Gamma \in \mathcal{K} | W(\Gamma) > 0\} \\ \mathcal{K}_- &= \{\Gamma \in \mathcal{K} | W(\Gamma) \leq 0\} \end{aligned} \tag{13}$$

Theorem (Dichotomy below the ground state)

For the Hill's lunar problem with $\alpha \geq 2$ the sets $\mathcal{K}_+$ and $\mathcal{K}_-$ are invariant. Solutions in $\mathcal{K}_+$ exist globally and solutions in $\mathcal{K}_2$ are singular.

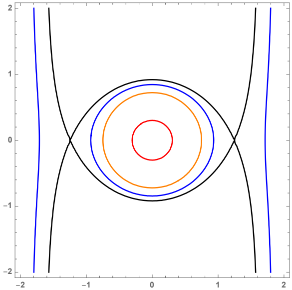


Figure: Level curves of $V(x,y) \leq E^*$

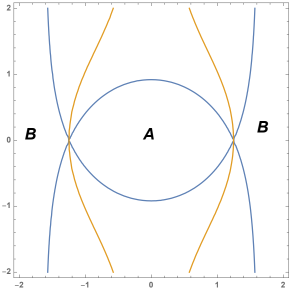
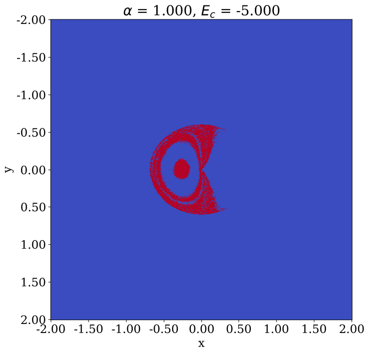


Figure: $V = E^*$ (blue) and $W = 0$ (orange)

Numerical simulations for different α



Red indicate the fate is collision. Both energies are below E^*.

Trichotomy at the ground state energy threshold

Let

$$\begin{aligned} \mathcal{K}_+ &= \{\Gamma \in (x, y, \dot{x}, \dot{y}) | E(\Gamma) = E^*, W(\Gamma) > 0\} \\ \mathcal{K}_- &= \{\Gamma \in (x, y, \dot{x}, \dot{y}) | E(\Gamma) = E^*, W(\Gamma) \leq 0\} \end{aligned} \tag{14}$$

Theorem

The sets $\mathcal{K}_+$ and $\mathcal{K}_-$ are invariant. Moreover,

- *Solutions in $\mathcal{K}_+$ exist for all time.*
- *Solutions in $\mathcal{K}_-$ either have a finite time collision or approach the ground state as $t \to \infty$.*

Above the ground state

Symplectic coordinates $q = (x, y)$ and
$p = (p_x, p_y) = (\dot{x} - y, \dot{y} + x)$, the Hamiltonian, i.e. the energy is

$$E(x, y, p_x, p_y) = \frac{1}{2}[(p_x + y)^2 + (p_y - x)^2] + V(x, y).$$

The Hill's equations (6) in Symplectic canonical form is

$$\dot{q} = \frac{\partial E}{\partial p}, \quad \dot{p} = -\frac{\partial E}{\partial q}. \tag{15}$$

That is,

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = J \nabla E, \quad J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}.$$

- The eigenvalues of the linearized operator $A := J\nabla^2 E(Q)$ are $\pm k$, $\pm i\omega$, decompose $\mathbb{R}^4 = E^u \bigoplus E^s \bigoplus E^c$.

$$k = \frac{1}{\sqrt{2}}\sqrt{\sqrt{36 + 36\alpha + 29\alpha^2 + 10\alpha^3 + \alpha^4} + (\alpha^2 + 3\alpha - 2)},$$

and

$$\omega = \frac{1}{\sqrt{2}}\sqrt{\sqrt{36 + 36\alpha + 29\alpha^2 + 10\alpha^3 + \alpha^4} - (\alpha^2 + 3\alpha - 2)}.$$

Ideas

- Solutions on the center-stable manifold remain close to $\pm Q$, "trapped orbits"
- Solutions do not remain close to the ground state for all positive times are ejected from any small neighborhood of it after some positive time, "non-trapped"
  1. Distance function WRT ground states, eigenmode dominance
  2. Ejection Lemma
  3. Variational estimates
  4. One-pass Theorem

Decomposition near the ground state

Write $\psi = Q + X$, where X is the perturbation, decompose X as follows:

$$X = \lambda_+(t)\xi_+ + \lambda_-(t)\xi_- + \gamma(t), \tag{16}$$

where

$$\xi_+ \in E^u, \xi_- \in E^s, \gamma(t) \in E^c, \quad \Omega(\gamma(t), \xi_+) = \Omega(\gamma(t), \xi_-) = 0. \tag{17}$$

One has $\lambda_\pm = \pm\Omega(X, \xi_\mp)$ and we can derive the differential equations for $\lambda_\pm(t)$.

$$\frac{d\lambda_+}{dt}(t) = k\lambda_+(t) + \Omega(N(X), \xi_-), \tag{18}$$

$$\frac{d\lambda_-}{dt}(t) = -k\lambda_-(t) + \Omega(N(X), \xi_+). \tag{19}$$

Linearized energy norm

Lemma

The function $\gamma(t)$ in the decomposition satisfies

$$\Omega(\gamma, A\gamma) \sim |\gamma|^2.$$

$$|X|_E^2 := \frac{k}{2}(\lambda_+^2(t) + \lambda_-^2(t)) + \frac{1}{2}\Omega(\gamma, A\gamma). \tag{20}$$

Lemma

We have $|X(t)| \sim |X(t)|_E$.

Distance function with respect to ground states

There exists $\delta_E > 0$ with the following property: for any solution $\psi = \pm(Q + X)$ and any time $t \in I_{max}(\psi)$ for which $|X(t)|_E \leq 4\delta_E$,

$$|E(\psi(t)) - E(Q) + \frac{k}{2}(\lambda_+(t) + \lambda_-(t))^2 - |X(t)|_E^2| \leq \frac{|X(t)|_E^2}{10}. \quad (21)$$

Let χ be a smooth function on $\mathbb{R}$ such that $\chi(r) = 1$ for $|r| \leq 1$ and $\chi(r) = 0$ for $|r| \geq 2$. We define

$$d_Q(\psi(t)) := \sqrt{|X(t)|_E^2 + \chi(|X(t)|_E/2\delta_E)C(\psi(t))},$$

where

$$C(\psi(t)) := E(\psi(t)) - E(Q) + \frac{k}{2}(\lambda_+(t) + \lambda_-(t))^2 - |X(t)|_E^2.$$

Distance function, eigenmode dominance

Lemma

Assume that there exists an interval I on which

$$\sup_{t\in I} d_Q(\psi(t)) \leq \delta_E.$$

Then, all of the following hold for all $t \in I$:

(i) $\frac{1}{2}|X(t)|_E^2 \leq d_Q(\psi(t))^2 \leq \frac{3}{2}|X|_E^2,$

(ii) $d_Q(\psi(t))^2 = E(\psi(t)) - E(Q) + 2k\lambda_1^2(t),$

(iii) $\frac{d}{dt} d_Q(\psi(t))^2 = 4k^2\lambda_1(t)\lambda_2(t) + 2k\lambda_1(t)\Omega(N(X), \xi_+ + \xi_-).$

(iv) *if $E(\psi) < E^* + \frac{1}{2}d_Q(\psi(t))^2$ holds for all $t \in I$, then $d_Q(\psi(t)) \sim |\lambda_1(t)|$ for all $t \in I$.*

Ejection Lemma

Lemma (Ejection Lemma)

There exists constants $0 < \delta_X \le \delta_E$ *and* A_*, B_*, C_* *with the property: If* $\psi(t)$ *is a local solution to (15) on* $[0, T]$ *satisfying*

$$R_0 := d_Q(\psi(0)) \le \delta_X, \quad E(\psi) < E^* + \frac{1}{2}R_0^2, \tag{22}$$

then we can extend $\psi(t)$ *as long as* $d_Q(\psi(t)) \le \delta_X$*. Furthermore, if there exists some* $t_0 \in (0, T)$ *such that*

$$d_Q(\psi(t)) \ge R_0, \quad \forall 0 < t < t_0, \tag{23}$$

and let

$$T_X := \inf\{t \in [0, t_0] : d_Q(\psi(t)) = \delta_X\}$$

where $T_X = t_0$ *if* $d_Q(\psi(t)) < \delta_X$ *on* $[0, t_0]$*, then for all* $t \in [0, T_X]$:

Ejection Lemma: Cntd

Lemma (Ejection Lemma: Cntd)

(i) $A_* e^{kt} R_0 \leq d_Q(\psi(t)) \leq B_* e^{kt} R_0$,

(ii) $|X(t)| \sim \mathfrak{s}\lambda_1(t) \sim \mathfrak{s}\lambda_2(t) \sim e^{kt} R_0$,

(iii) $|\lambda_-(t)| + |\gamma(t)| \lesssim R_0 + d_Q(\psi(t))^2$,

where $\mathfrak{s} = 1$ *or* -1. *Moreover,* $d_Q(\psi(t))$ *is increasing on the region* $t \in [0, T_X]$.

Variational estimates

Lemma

For the strong force $\alpha \geq 2$*, for any* $\delta > 0$*, there exist* $\epsilon(\delta), \kappa(\delta) > 0$ *such that for any* $\Gamma \in \mathbb{R}^4$ *satisfying*

$$E(\Gamma) < E^* + \epsilon(\delta), \quad d_Q(\Gamma) \geq \delta, \tag{24}$$

one has either

$$W(\Gamma) \leq -\kappa(\delta) \quad and \quad K(\Gamma) \leq -\kappa(\delta),$$

or

$$W(\Gamma) \geq \kappa(\delta).$$

Variational estimates

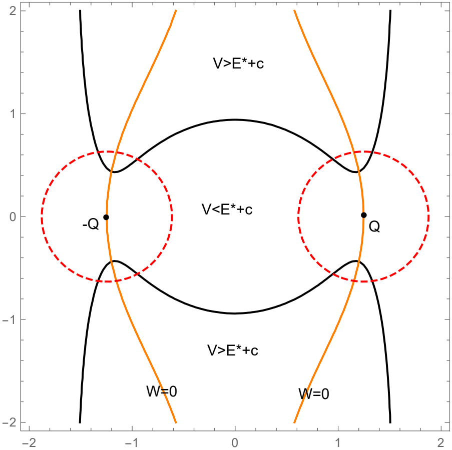


Figure: The black curve is the zero velocity curve for $E(\Gamma) = E^* + c$, i.e. $V(x, y) = E^* + c$. When $c = \epsilon(\delta)$ is small enough, the value of $|W|$ is uniformly away from 0, provided $d_Q(\Gamma) > \delta$.

One-pass Theorem

Conjecture (One-pass theorem)

There exists constants ϵ_*, R_* *with the property: for any* $\epsilon \in (0, \epsilon_*]$, $R \in (\sqrt{2\epsilon}, R_*]$ *and any solution* ψ *of the HLP (15) on an interval* $[0, T_{\max})$ *satisfying*

$$E(\psi) < E^* + \epsilon, \quad d_Q(\psi(0)) < R,$$

define $T_{\text{trap}} := \sup\{t \geq 0 | d_Q(\psi(t)) < R\}$, *then*

1. *if* $T_{\text{trap}} = T_{\max}$, *then* ψ *is "trapped";*
2. *if* $T_{\text{trap}} < T_{\max}$, *then* $d_Q(\psi(t)) \geq R$ *for all* $t \in (T_{\text{trap}}, T_{\max})$.

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   - *global existence*
   - *finite time collision*

Thank you for listening!