Analysis of Markov Modulated Markov Chains

-- A Divide and Conquer Approach to Queueing Problems --

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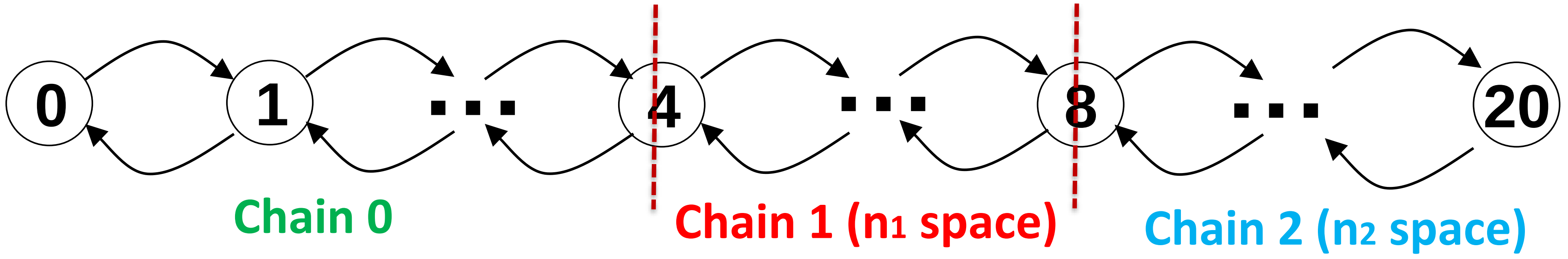
This is a joint work with Alan Scheller-Wolf (CMU) and

Robert Hampshire (University of Michigan)

CanQueue, 8/22/2020

Last Year at CanQueue:
We analyzed service systems with multiple reneging rates.

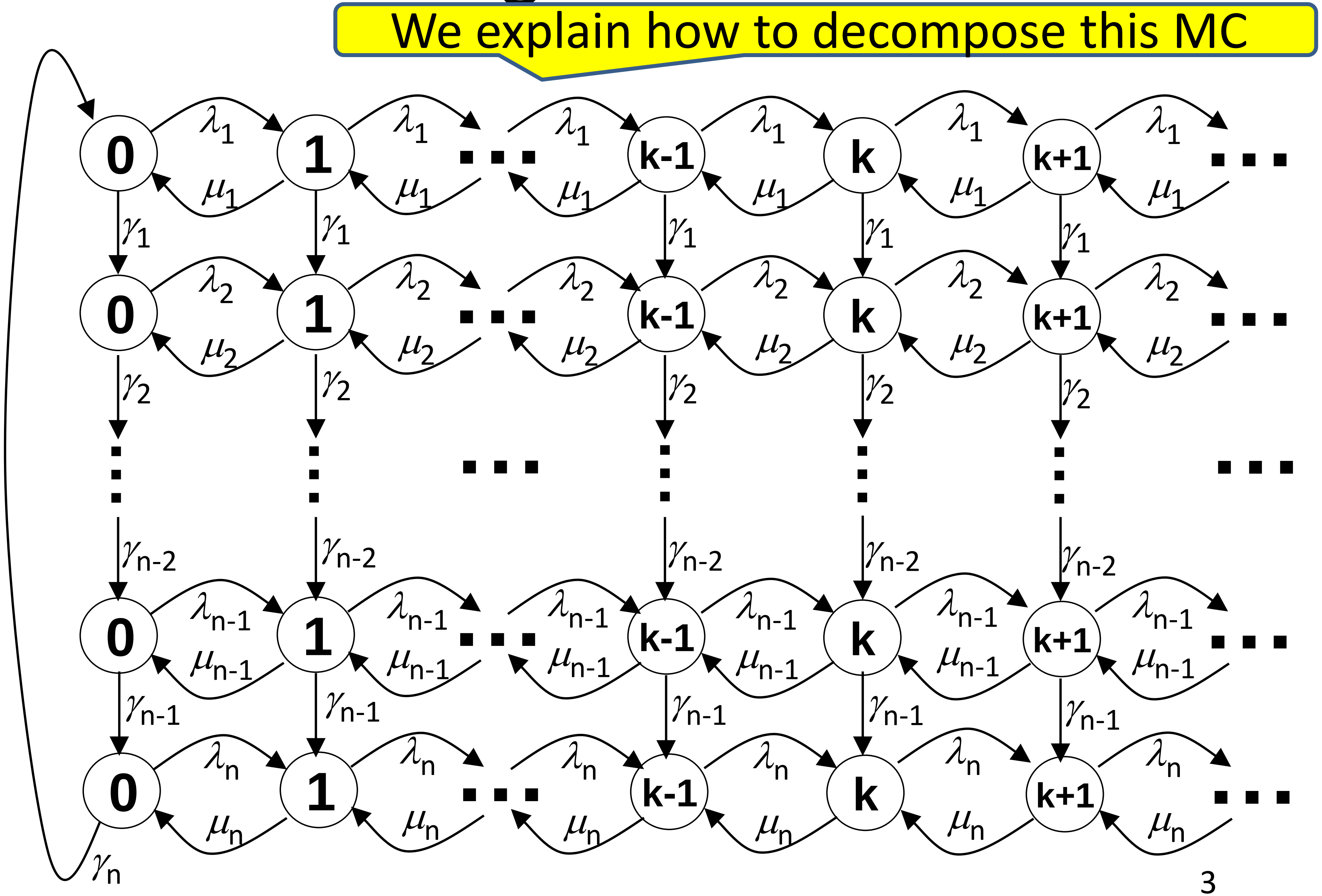


- Last year we presented our decomposition method: $\dfrac{E[f]}{\pi_s} = \sum_{\text{all chain } i} \dfrac{E_i[f]}{\pi_s^i}$
  - We focused on the expectations of the full MC and subchains; the decomposition scheme was a simple truncation.

- This year we focus on another application that requires a non-trivial decomposition scheme.

Markov Modulated Single-Server Queue

- CTMC
- Single-server M/M/1 queue
- Poisson arrivals
- Exponential service times
- Infinite queue capacity
- Transition rate from one M/M/1 queue to the other does not depend on a state
- Process starts over at some point
- Example: Machine deterioration and replacement



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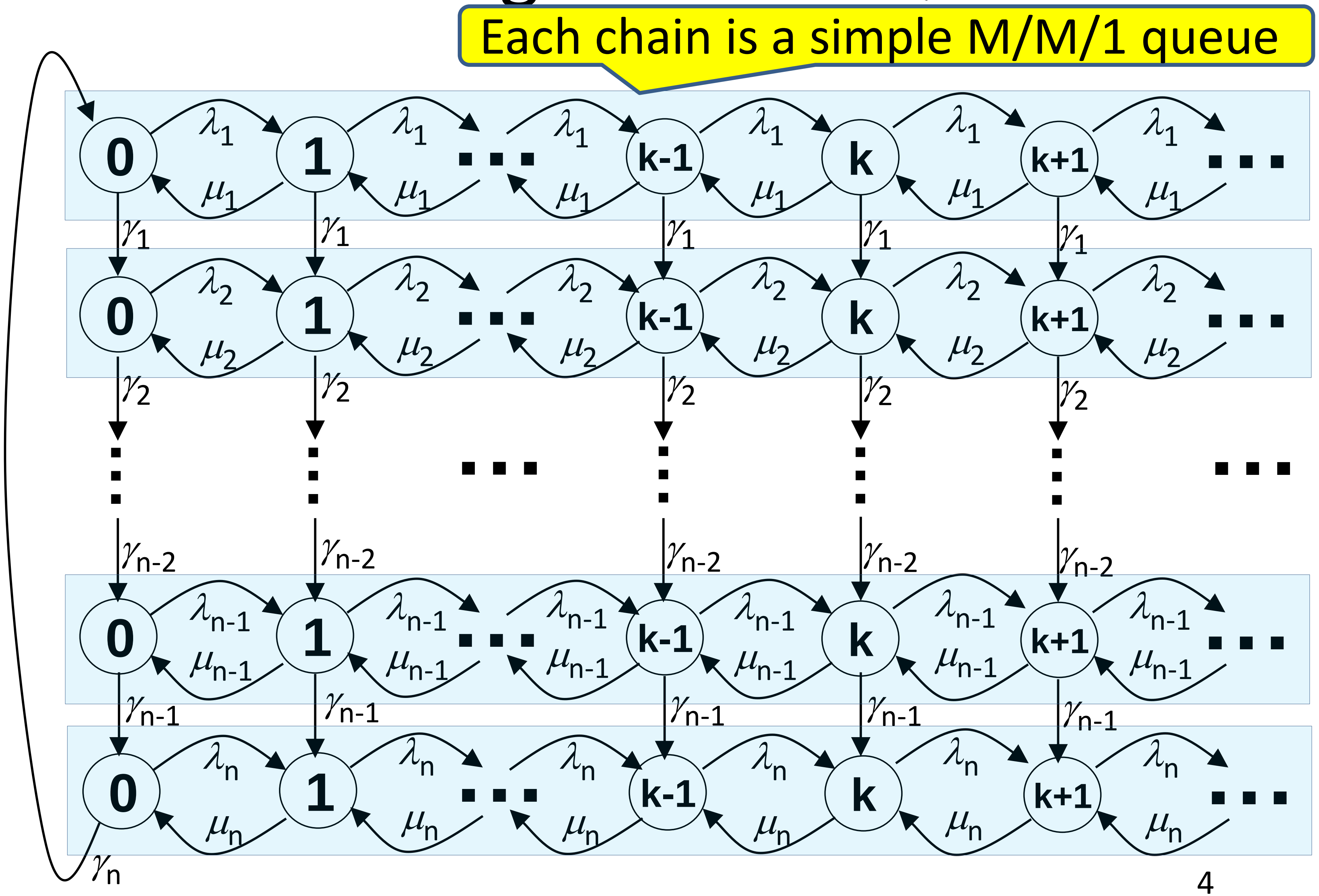
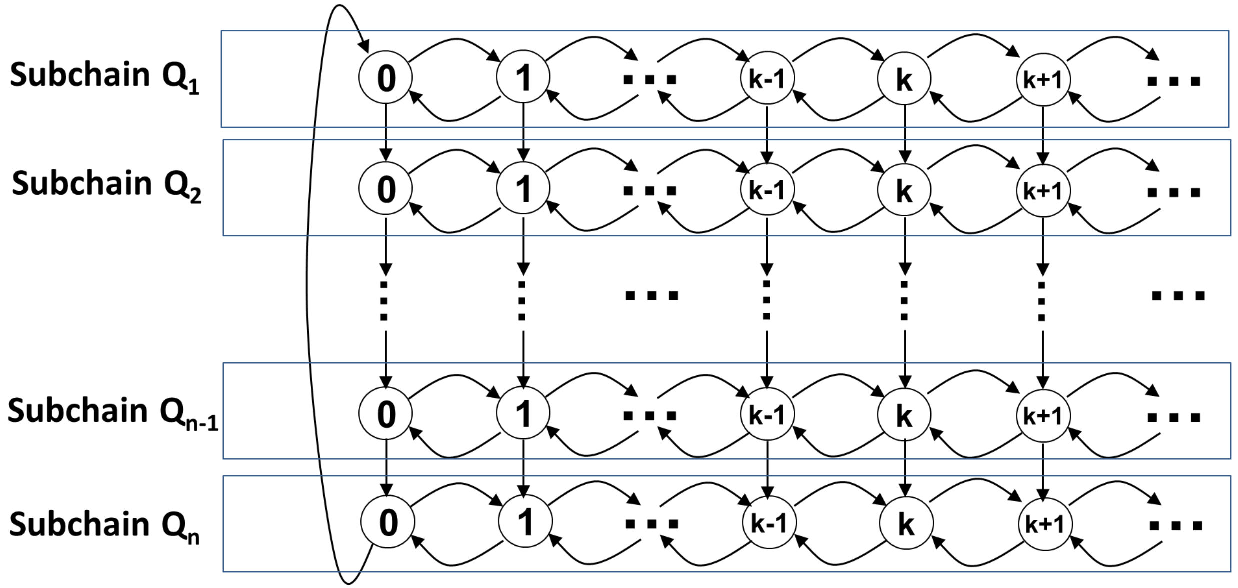


Markov Modulated Single-Server Queue

Why do we use decomposition approach?

1. Efficiency: Decomposition approach decreases a computational cost to solve a large MC.

2. Understanding: A large MC has many different subchains inside. How are these subchains related to each other?

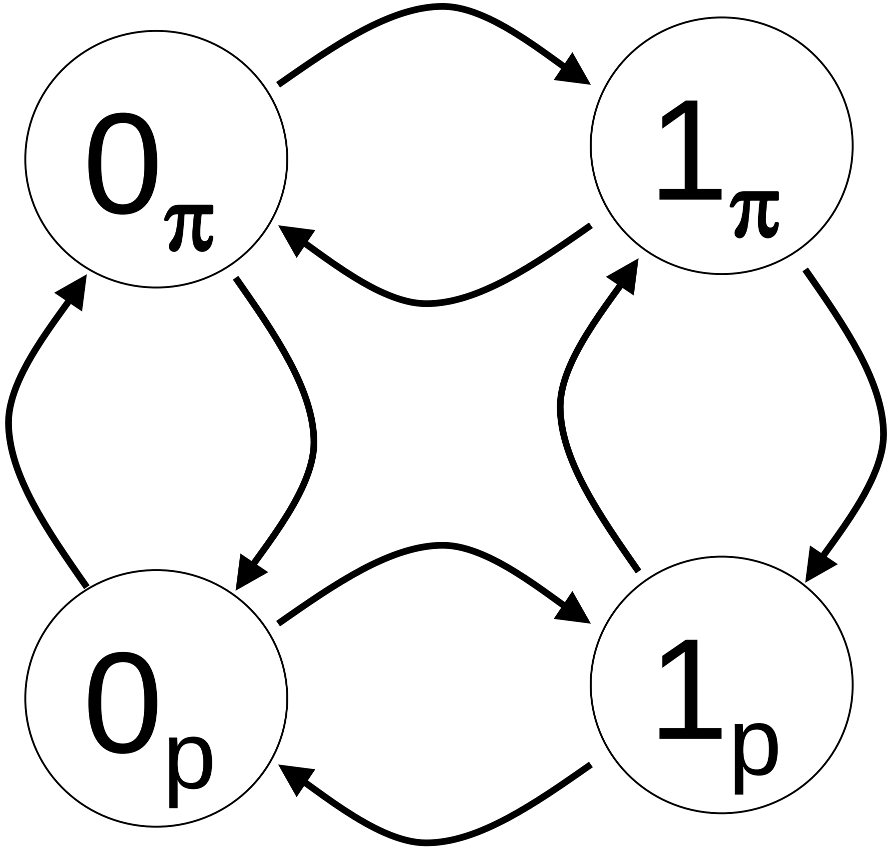


Advantage of analytical approach

Outline

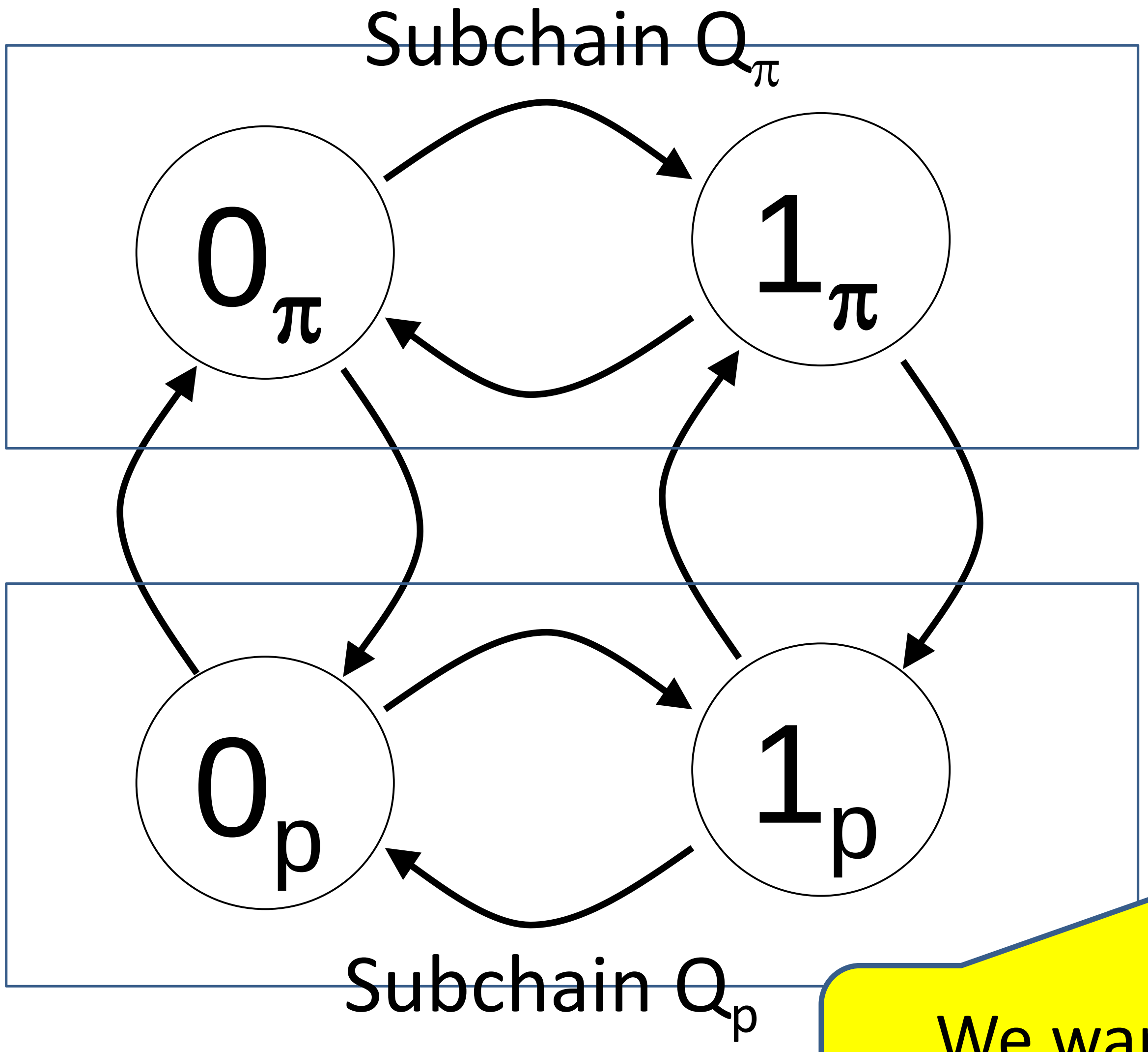
1. Motivational Example: Four-State CTMC
2. Partial Flow Conservation
3. Model: Markov Modulated Single-Server Queueing System
4. Analysis of the Model
5. Summary
6. References

Motivational Example: Four-State CTMC



Let's solve this simple MC by decomposition method

Motivational Example: Four-State CTMC



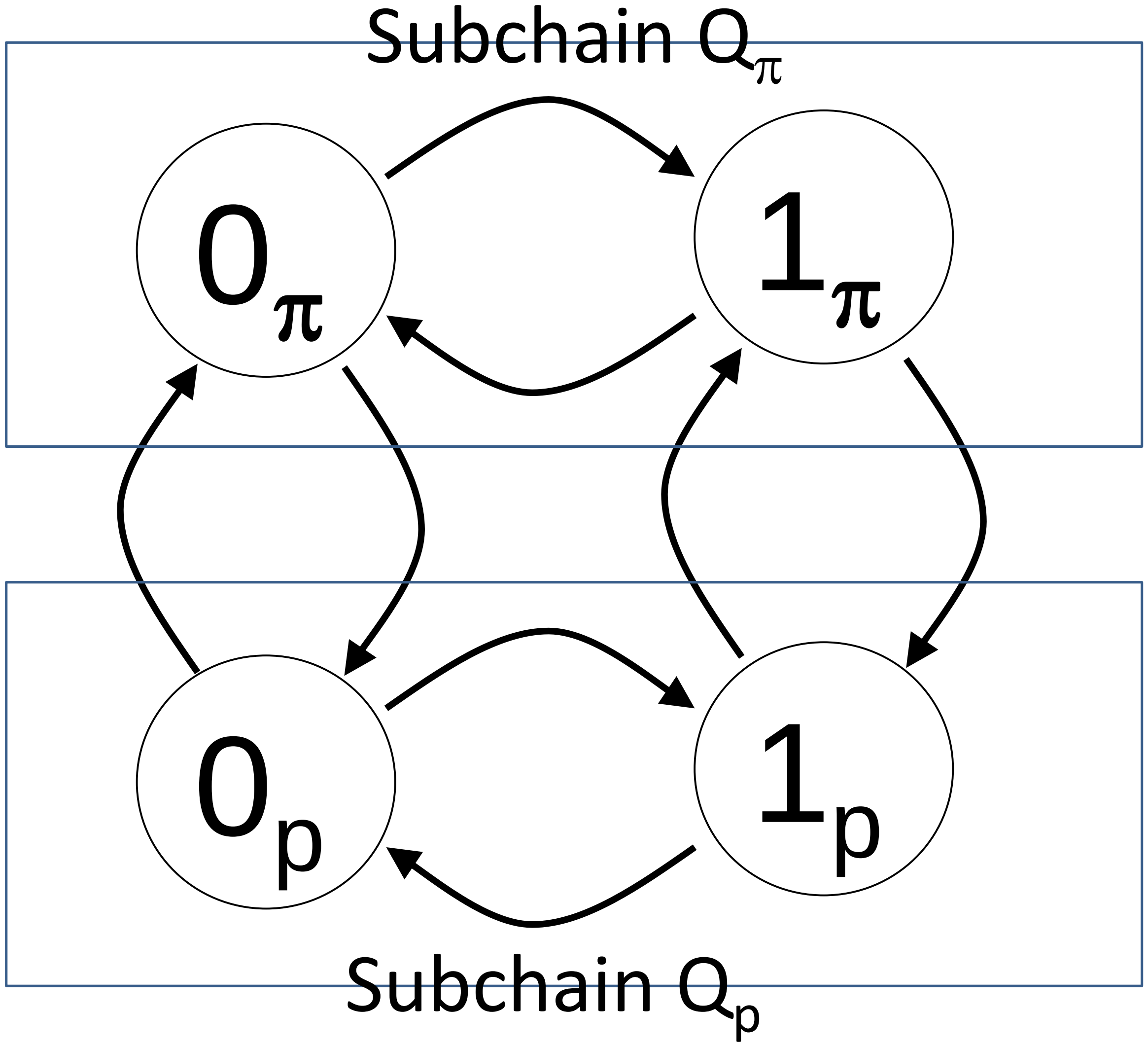
Subchain Q_π with the correct stationary distribution $\{\pi_0, \pi_1\}$

"Correct" means subchain's distribution is proportional to the original full MC.

Subchain Q_p with the correct stationary distribution $\{p_0, p_1\}$

We want to find the correct $\{p_0, p_1\}$ and $\{\pi_0, \pi_1\}$

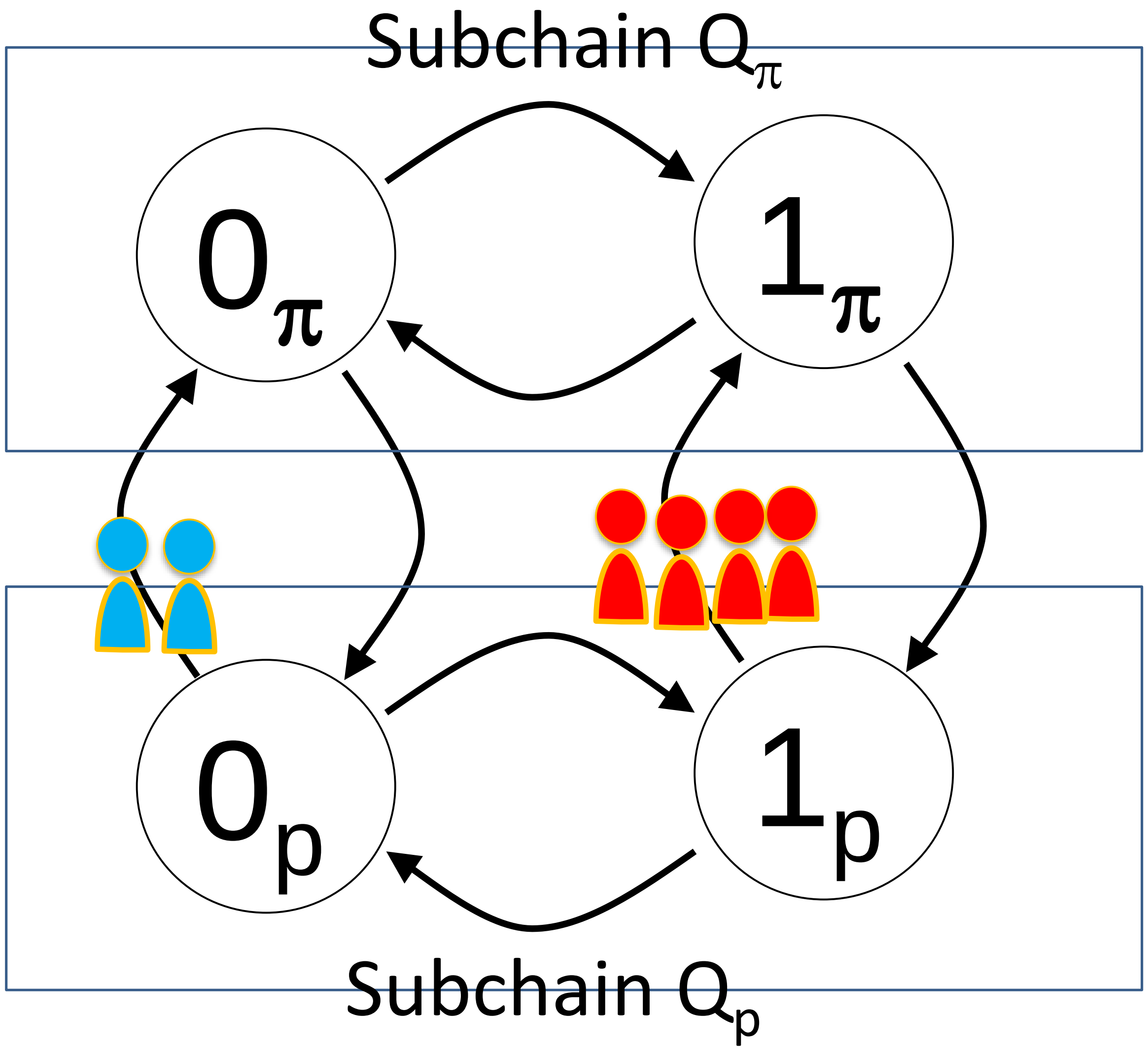
Motivational Example: Four-State CTMC



Standard Decomposition Approach

1. Check how a visit to the other chain returns to the starting chain given it lands at a certain state of the other chain.
2. Redirect flows based on where each visit returns to.

Motivational Example: Four-State CTMC



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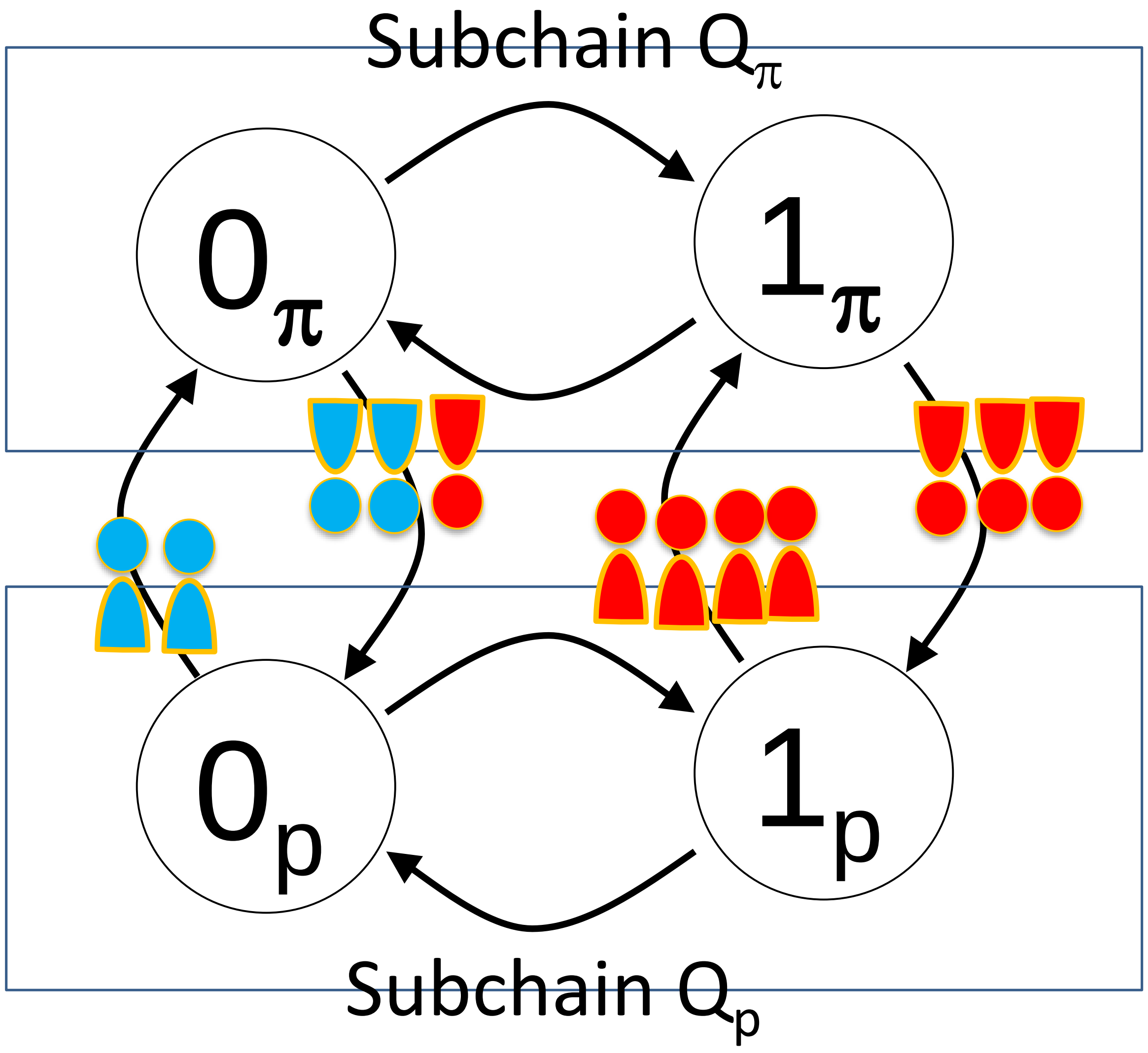

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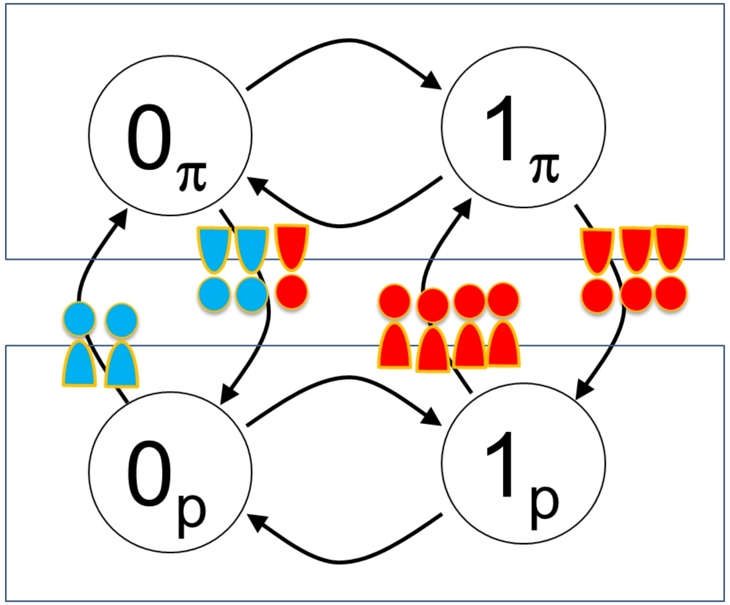

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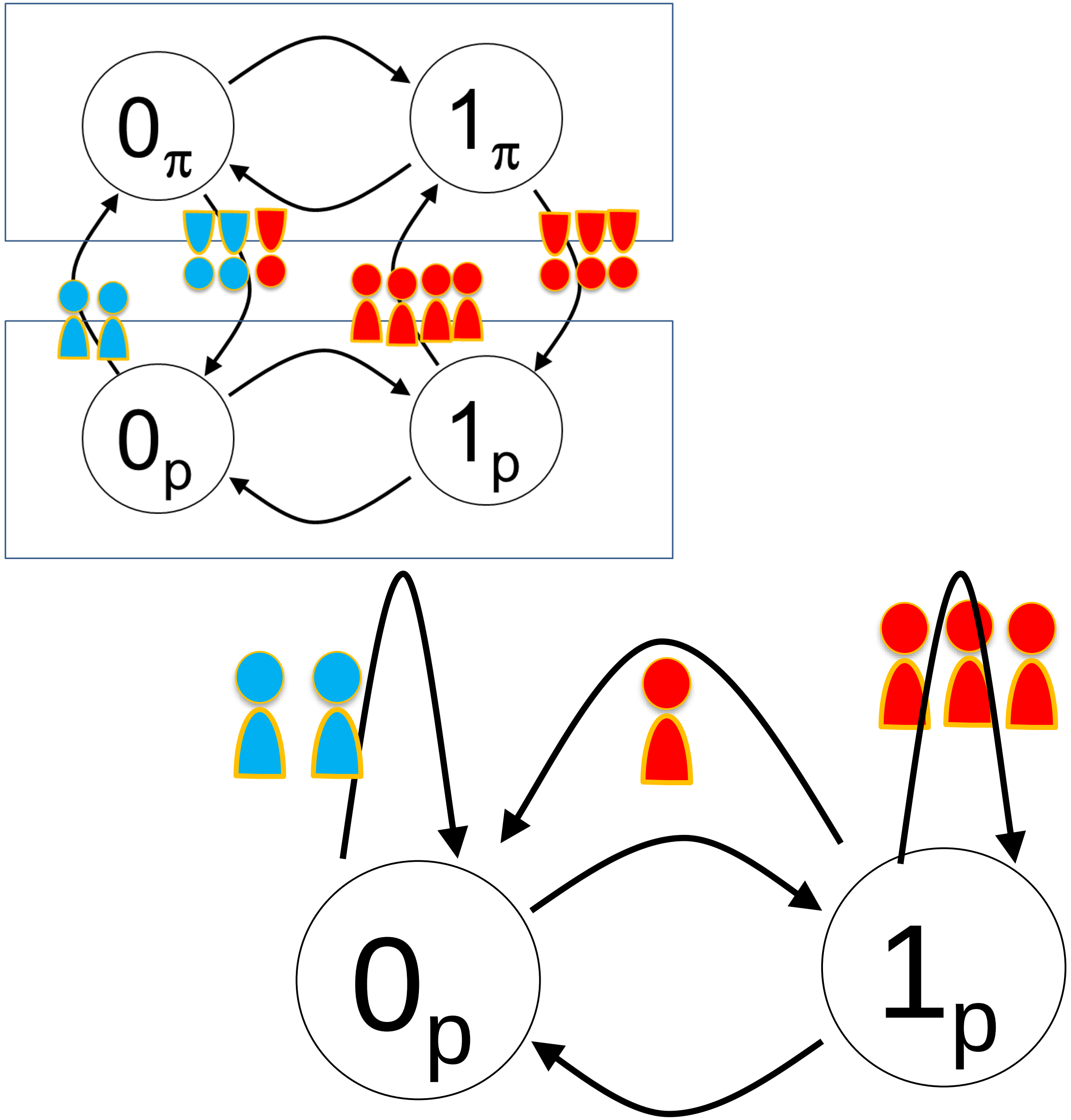
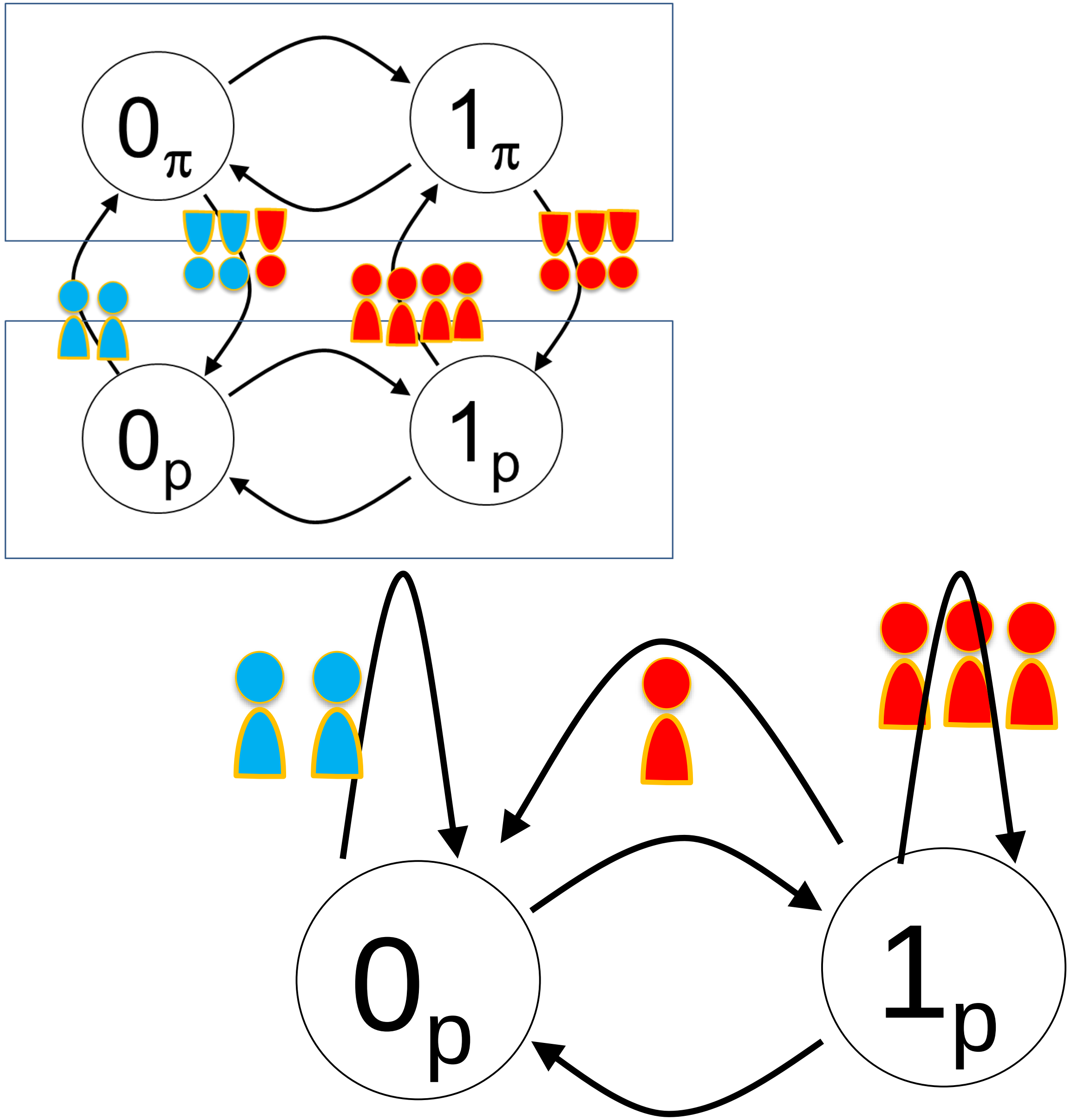


Standard Decomposition Approach

1. Check how a visit to the other chain returns to the starting chain given it lands at a certain state of the other chain.
2. Redirect flows based on where each visit returns to.

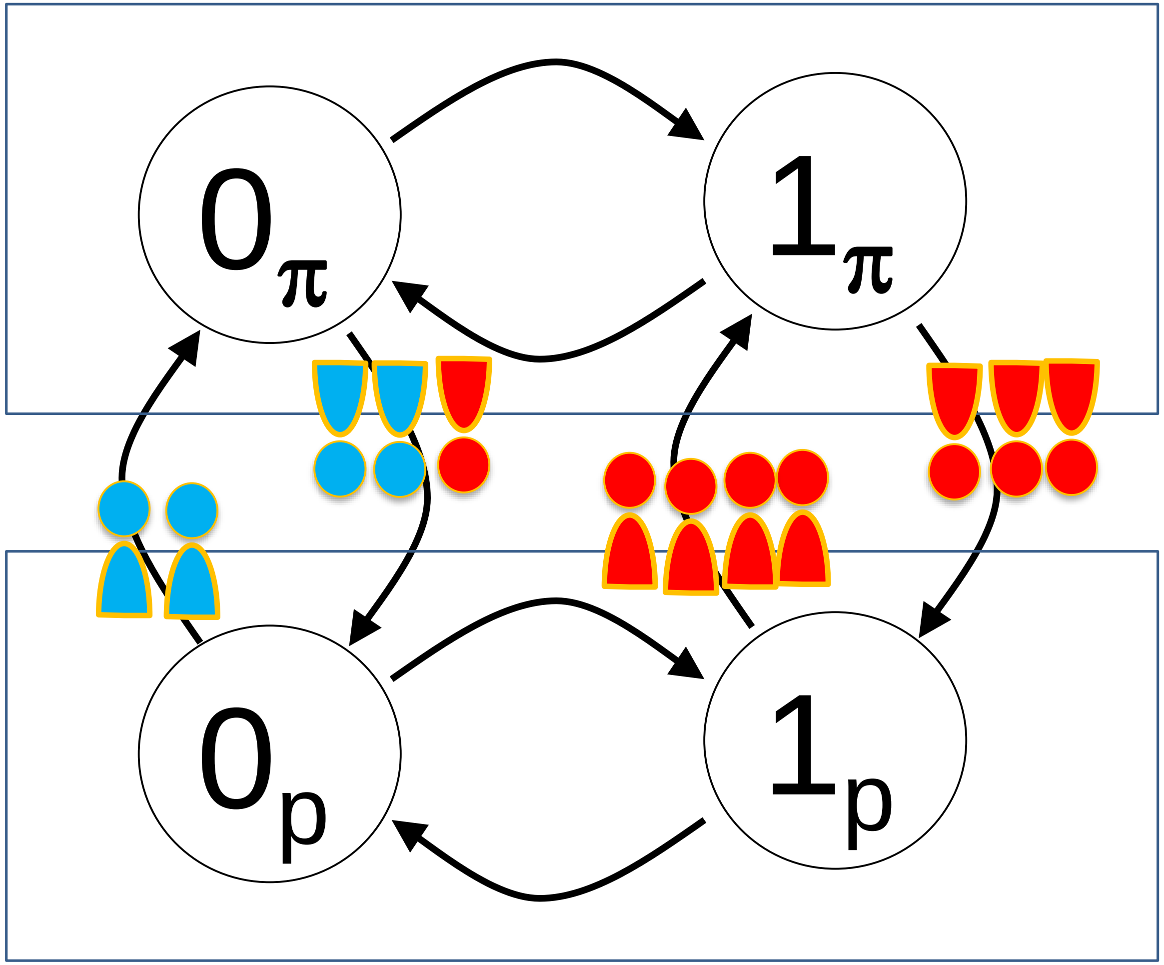
Is there an alternative redirection procedure?



This procedure is perfectly fine.
What is the issue?

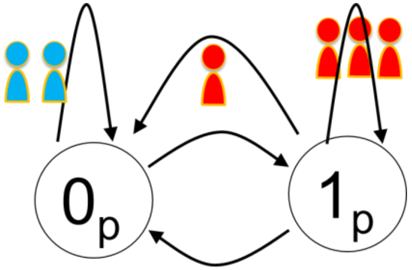
1. It is often hard to trace everybody's move (sample path analysis is complicated).
2. Return probability is dependent on the structure of the other chain.

Is there an alternative redirection procedure?



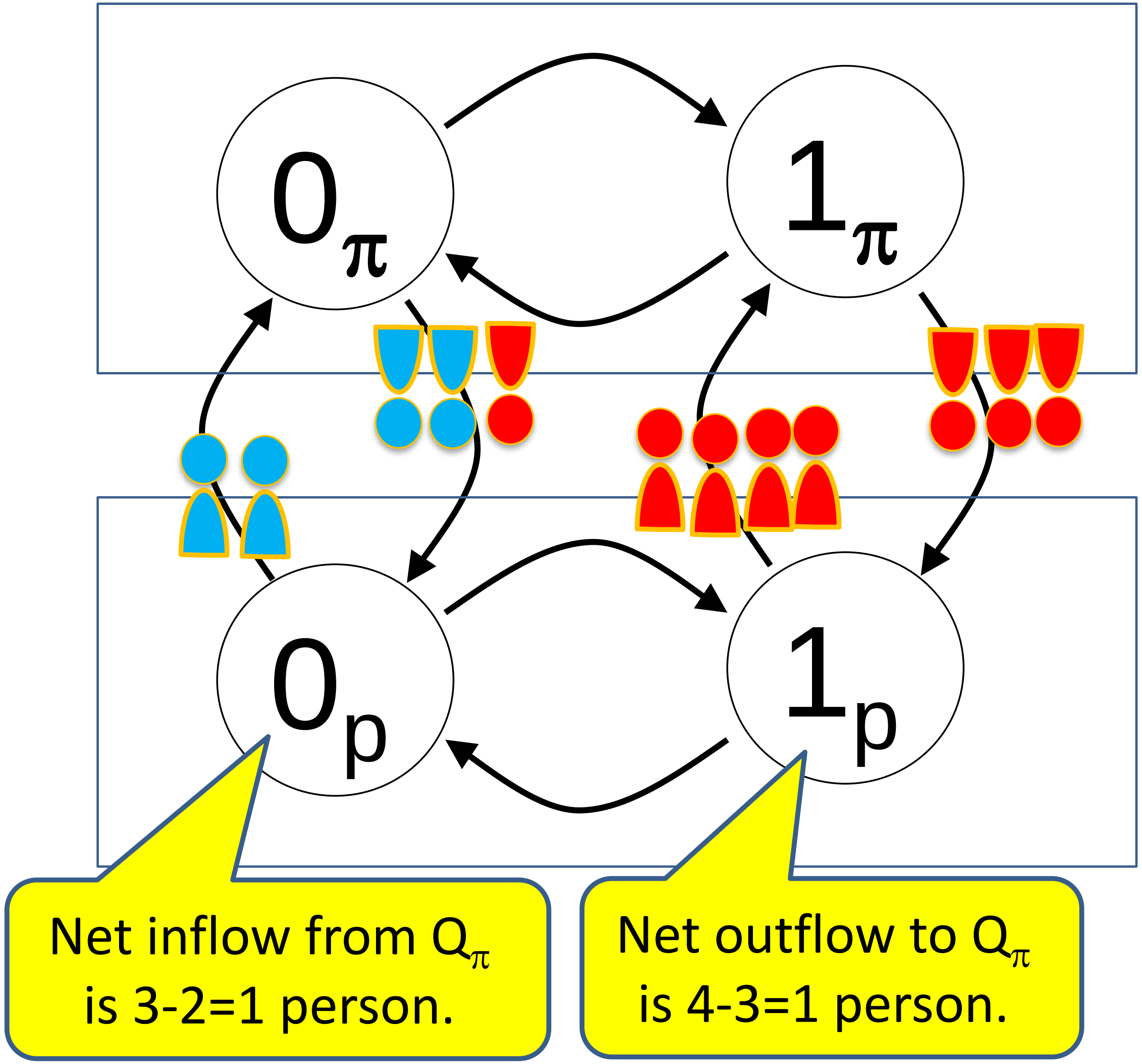
Question: What is a necessary and sufficient condition to maintain the correct distribution after decomposition?

Answer: Conserve the partial flow at each cut.



Hint: This redirection satisfies the condition.

Is there an alternative redirection procedure?



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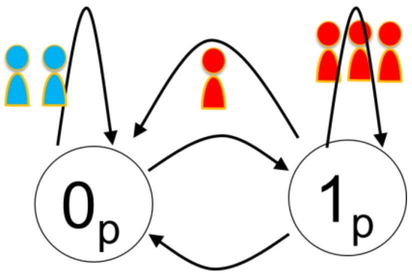
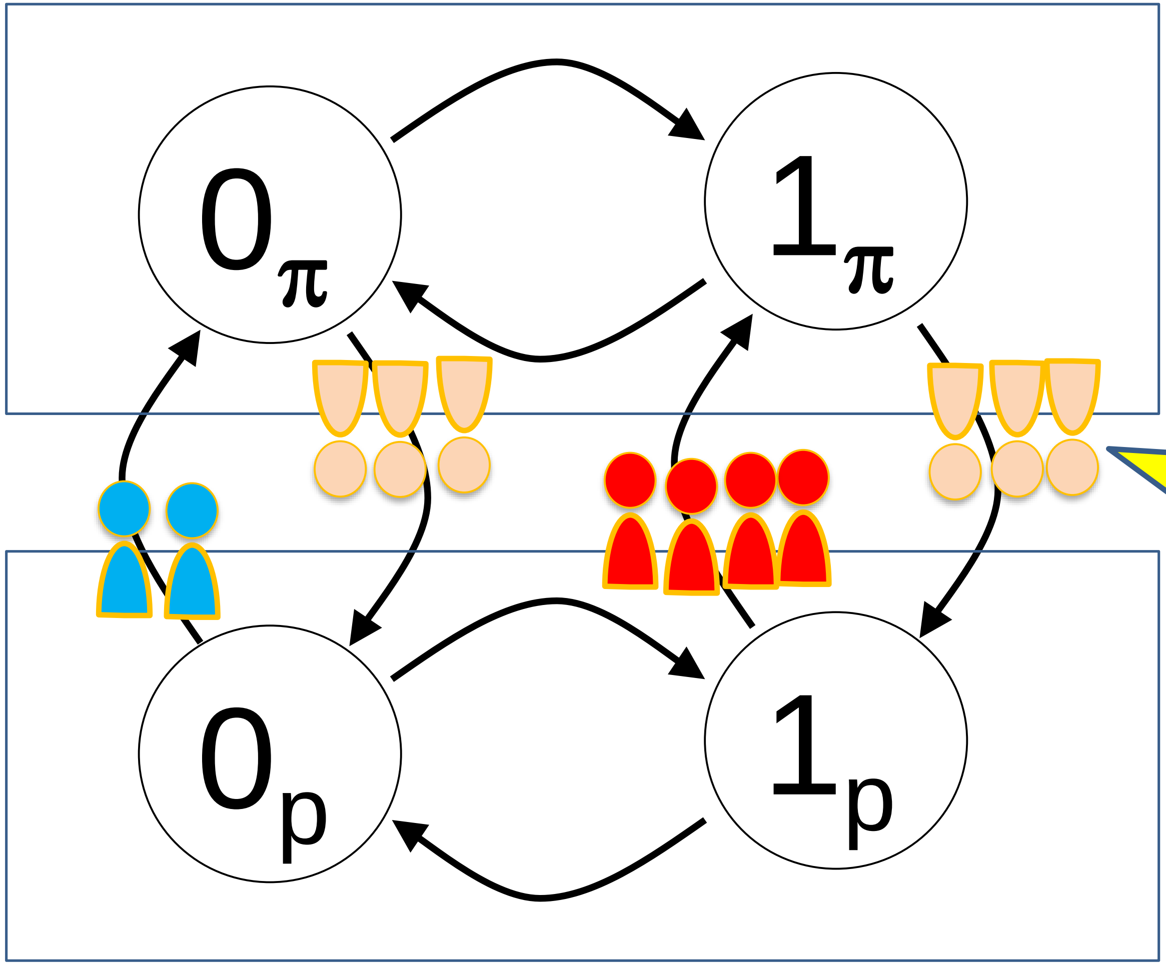


This redirection satisfies the partial flow conservation condition.

Is there an alternative redirection procedure?



Question: If all we want is to conserve partial flows at every cuts, what information do we need to know?

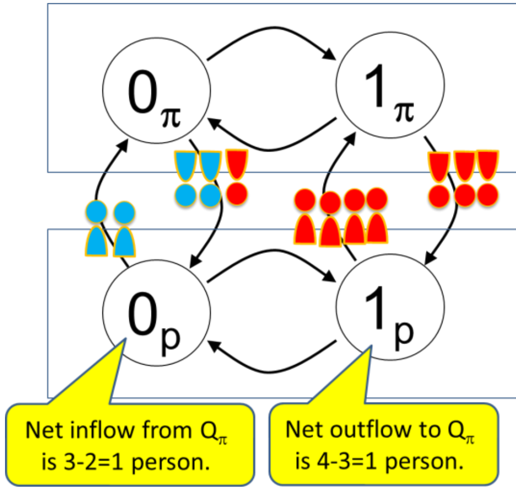
No need to know who is blue and who is red; no need to know who goes to which state.

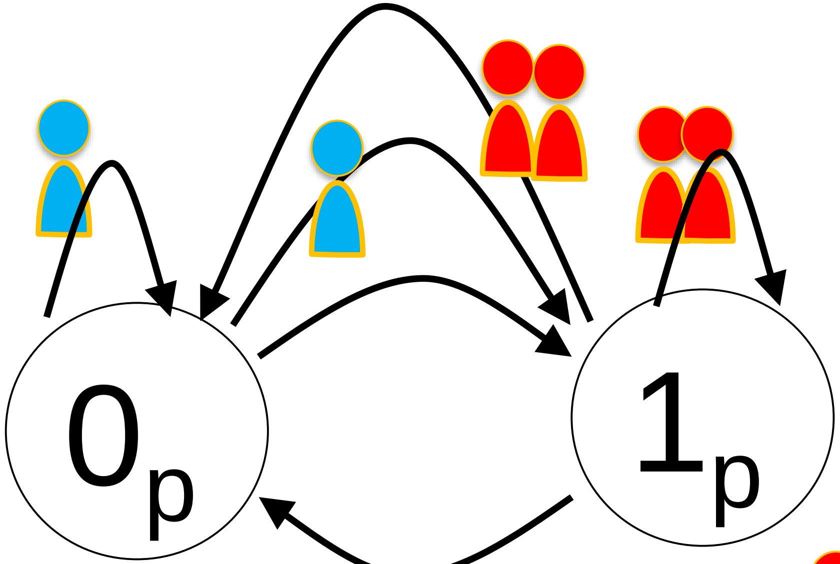
No need to know the total number of returning people (it must be 2+4=6 people).

All we need to know is the proportion of return flows to $\{0_p, 1_p\}$, which is {3/6, 3/6}={50%,50%}

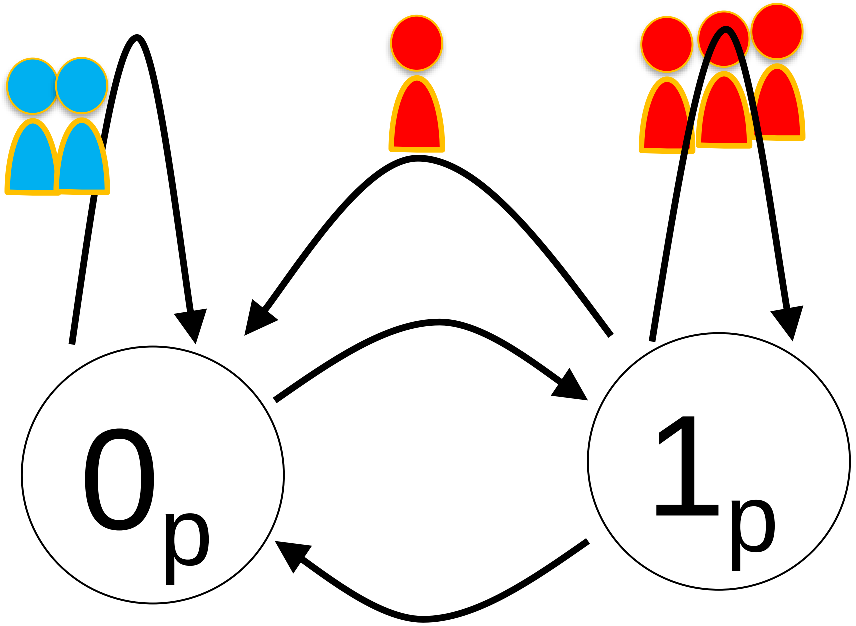
There exist infinitely many redirection methods.

Question: Which redirection method is easy to implement?

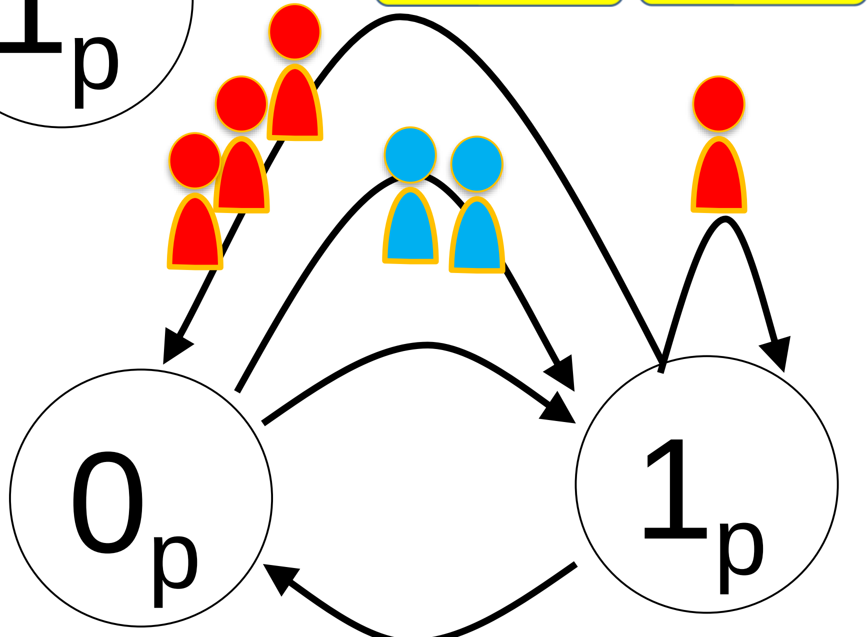




(b)



(a) Standard Method



(c)

There exist infinitely many redirection methods.

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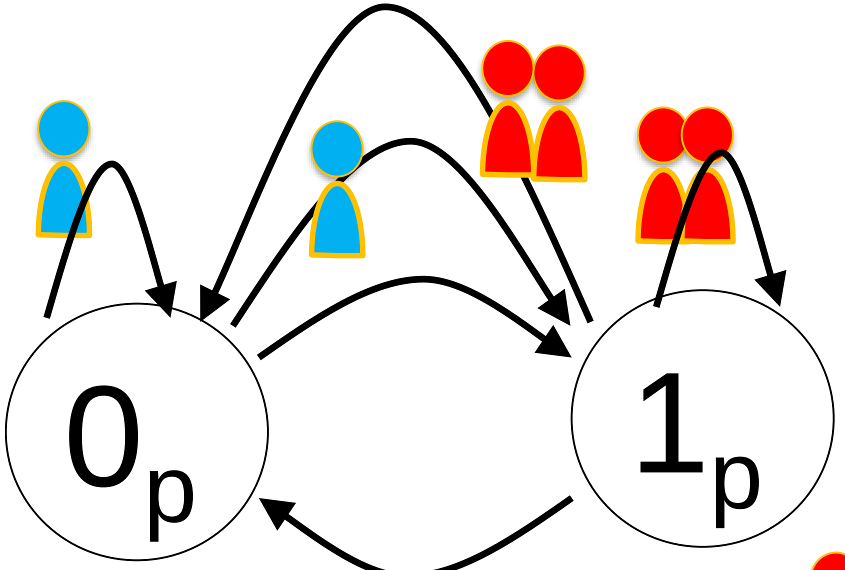
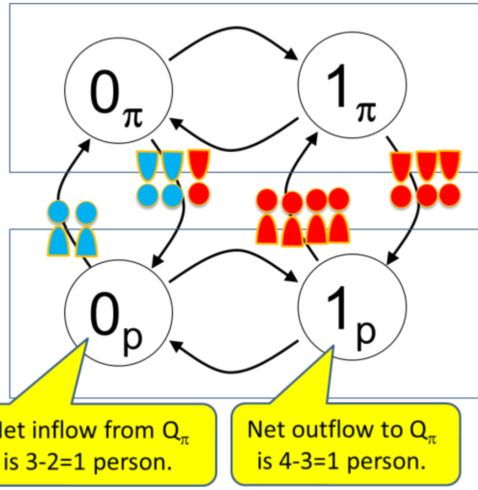
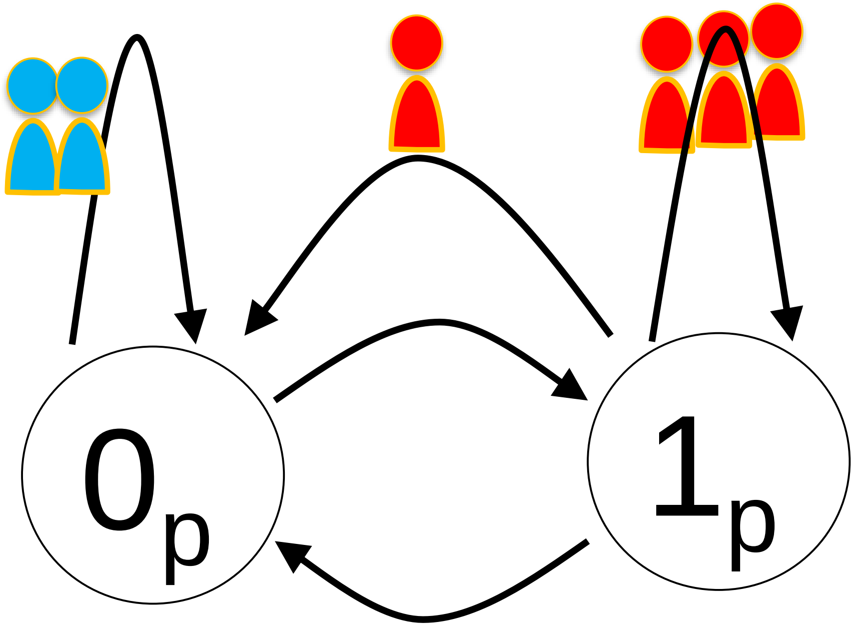


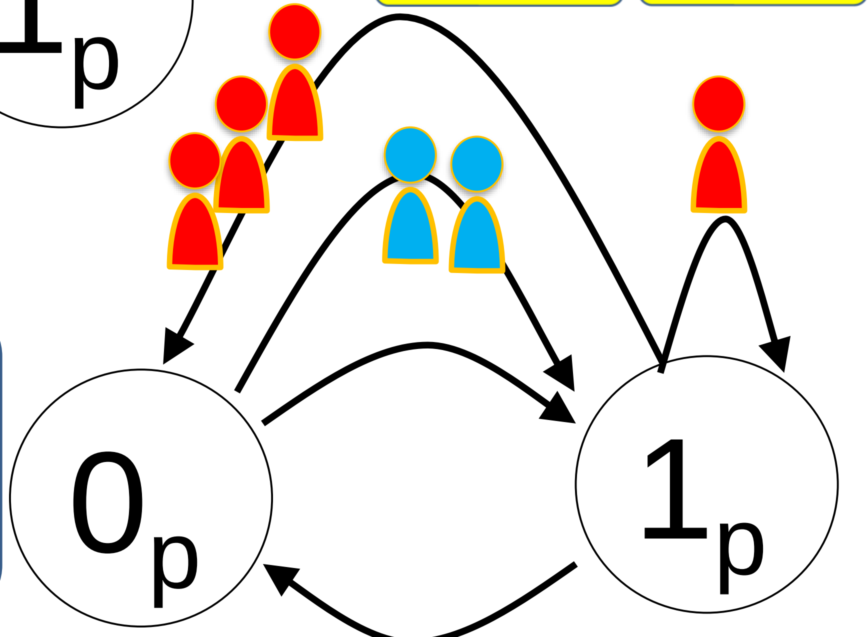
(b)

We choose (b) because we just multiply the outflow by {50%,50%} return flow proportion.



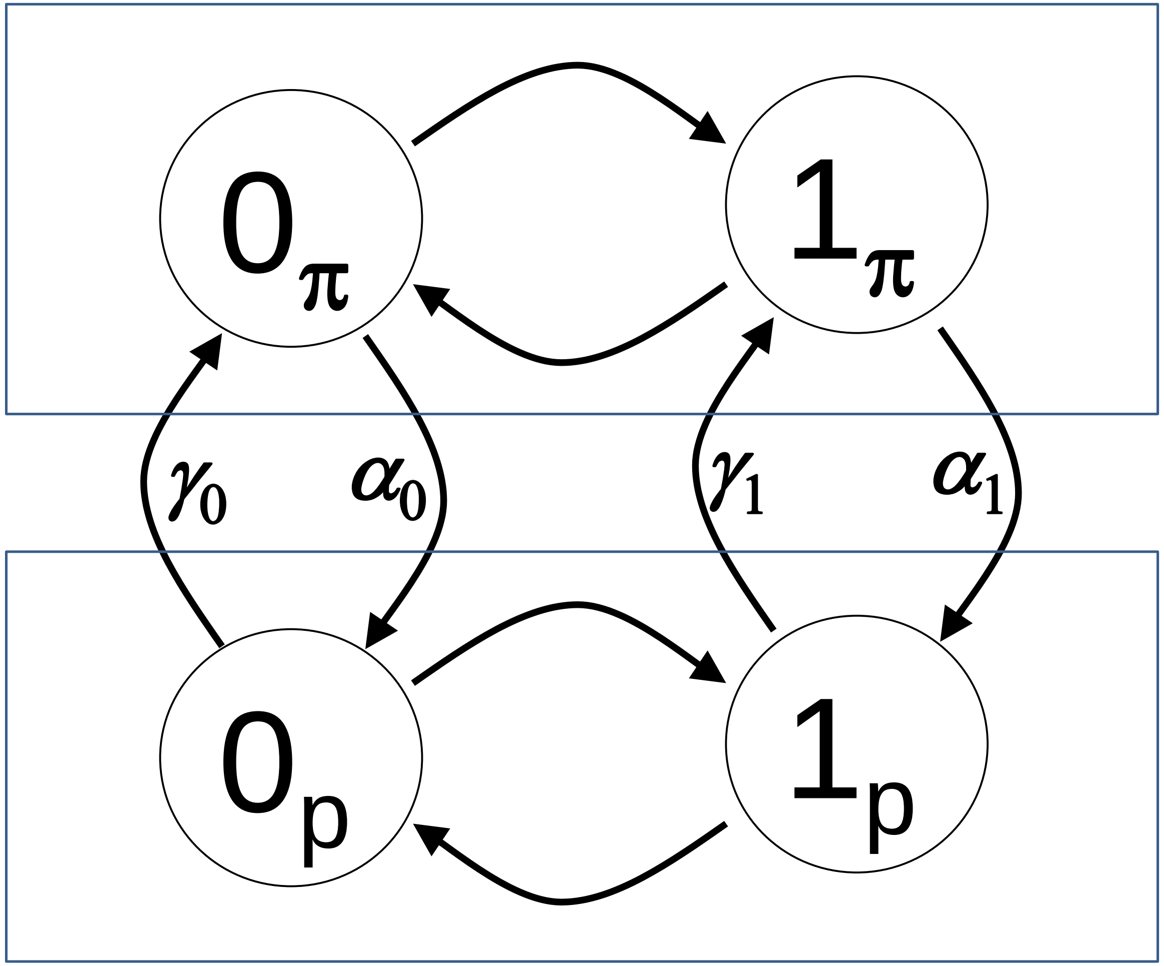


(a) Standard Method



(c)

Now, let's decompose MC and add terminations.



Subchain Q_π with the correct stationary distribution $\{\pi_0, \pi_1\}$

Subchain Q_p with the correct stationary distribution $\{p_0, p_1\}$

Termination refers to the added transitions at the boundary states of the decomposed subchain. Termination **conserves the partial flow** at each cut.

Now, let's decompose MC and add terminations.

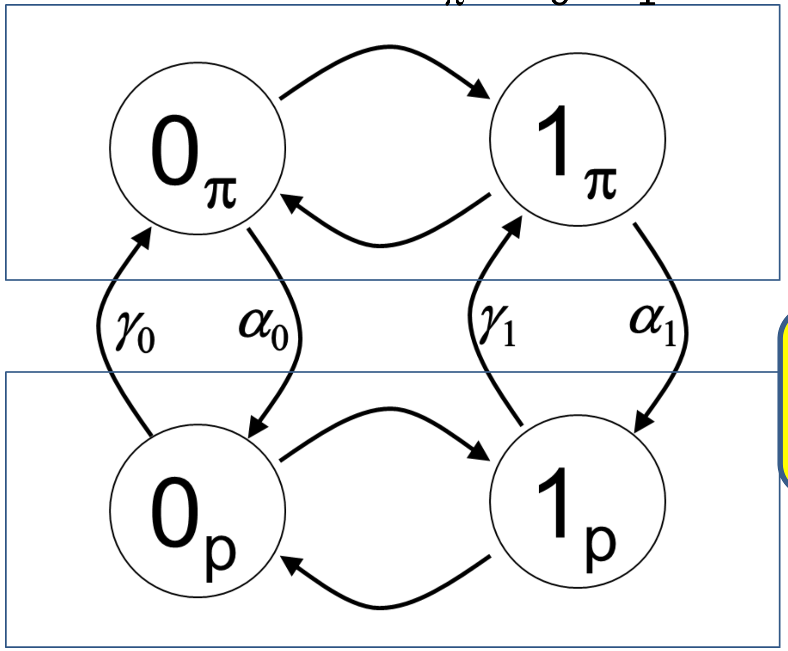
Do we need self-transitions?

$$\gamma_0 \cdot \frac{\alpha_0 \pi_0}{\alpha_0 \pi_0 + \alpha_1 \pi_1}$$

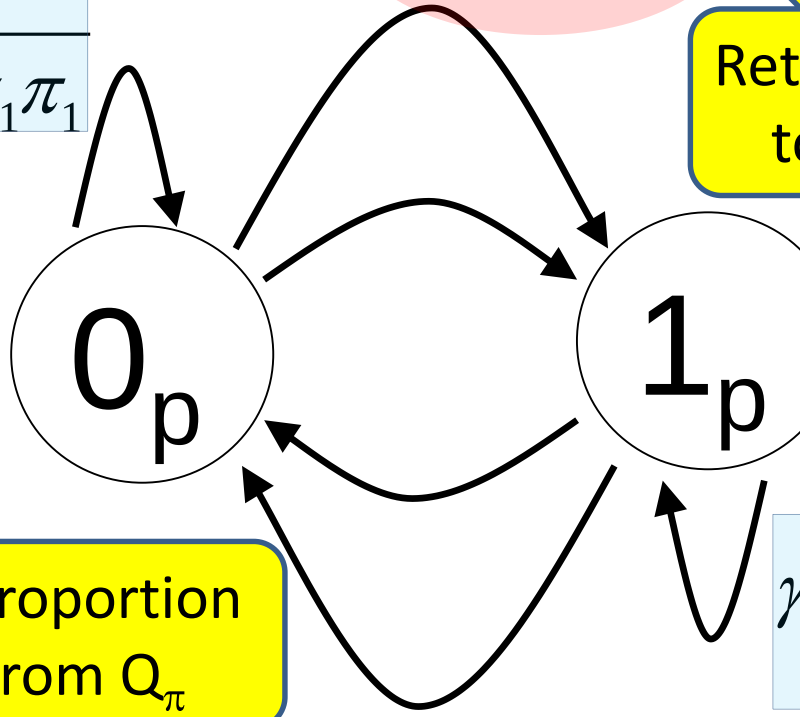
$$\gamma_0 \cdot \frac{\alpha_1 \pi_1}{\alpha_0 \pi_0 + \alpha_1 \pi_1}$$

Return flow proportion to state 1_p from Q_π

Subchain Q_π $\{\pi_0, \pi_1\}$



Subchain Q_p $\{p_0, p_1\}$



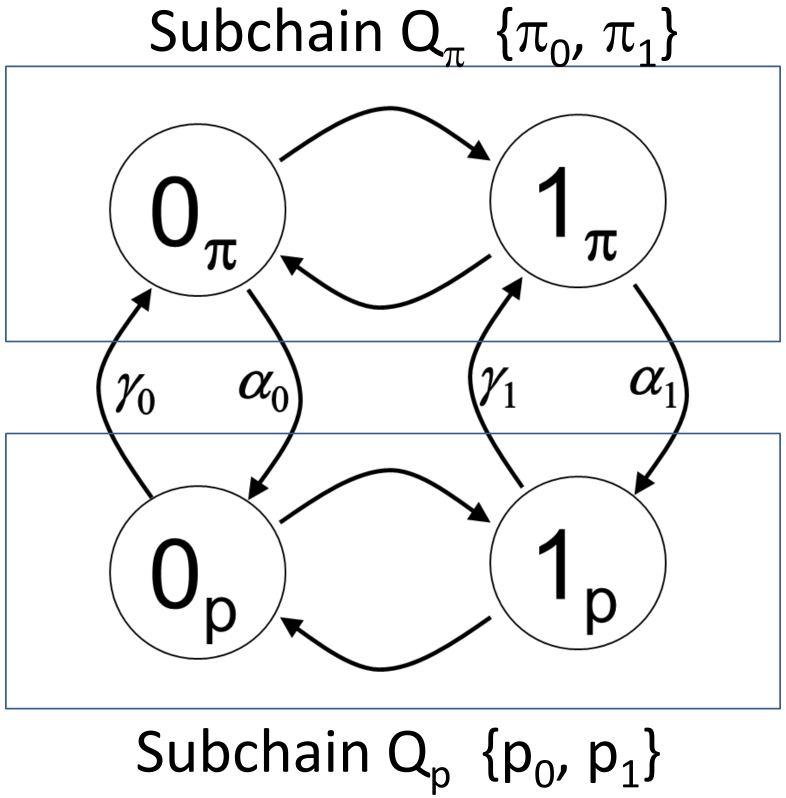
Do we need self-transitions?

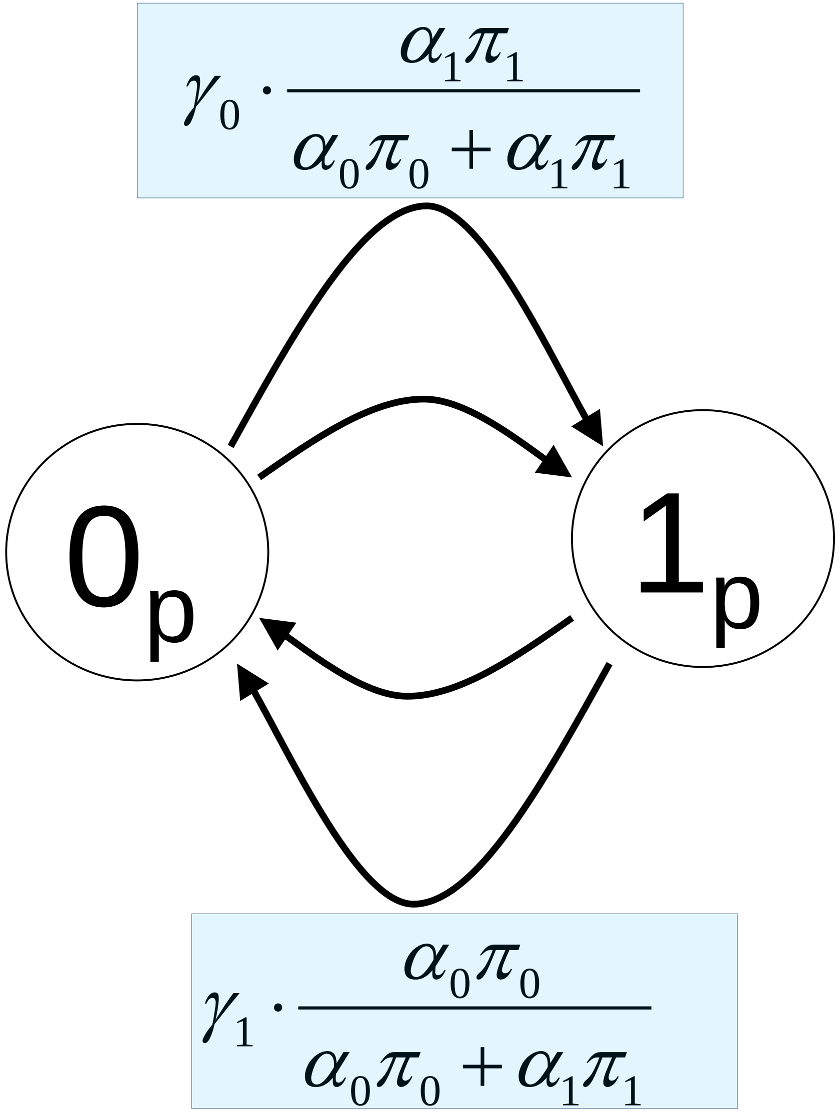
$$\gamma_1 \cdot \frac{\alpha_1 \pi_1}{\alpha_0 \pi_0 + \alpha_1 \pi_1}$$

Return flow proportion to state 0_p from Q_π

$$\gamma_1 \cdot \frac{\alpha_0 \pi_0}{\alpha_0 \pi_0 + \alpha_1 \pi_1}$$

Now, let's decompose MC and add terminations.

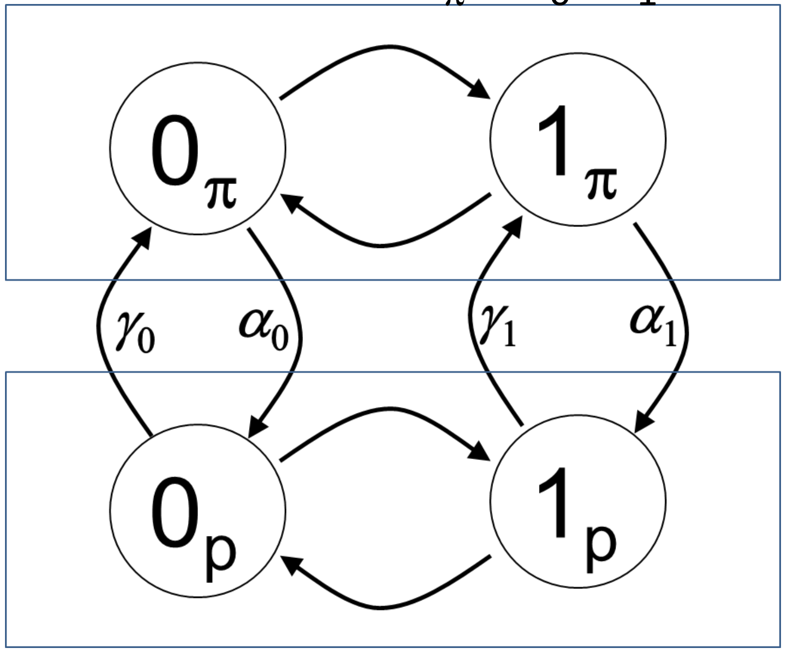




When does subchain become independent?

Question: When is Q_p's stationary distribution independent from Q_π?

Subchain Q_π $\{\pi_0, \pi_1\}$



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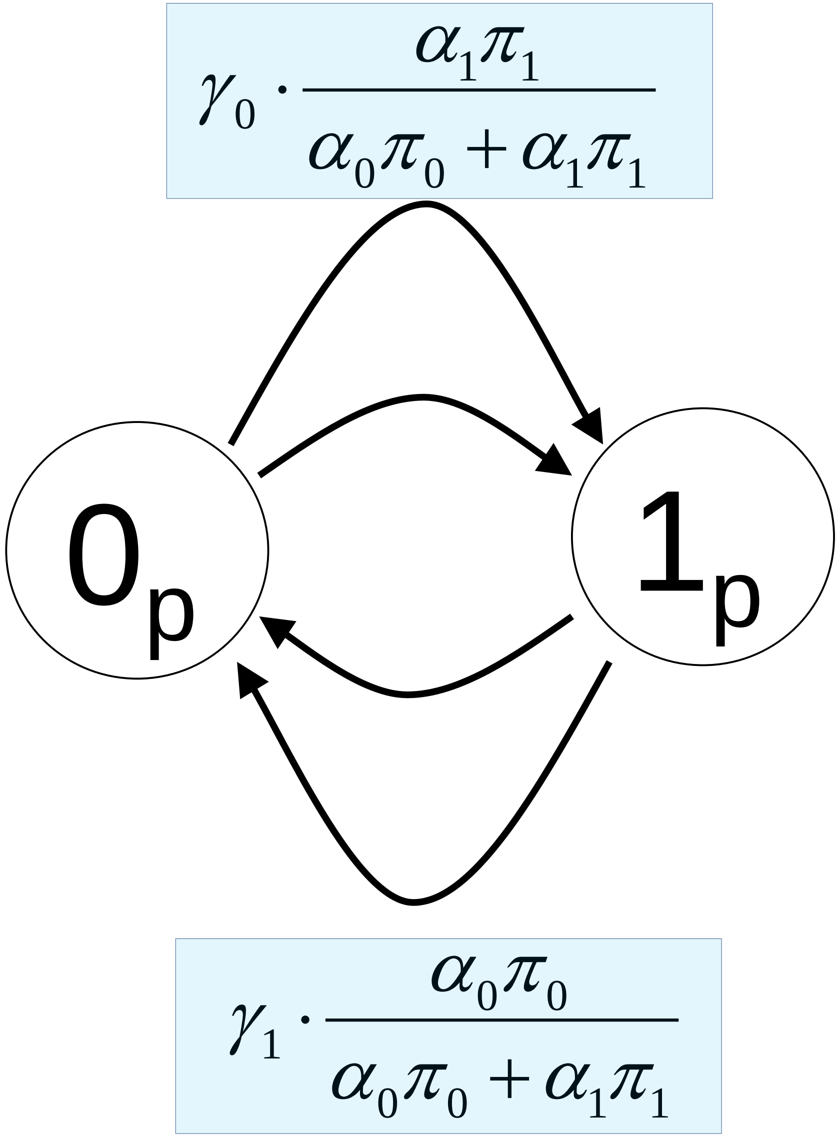

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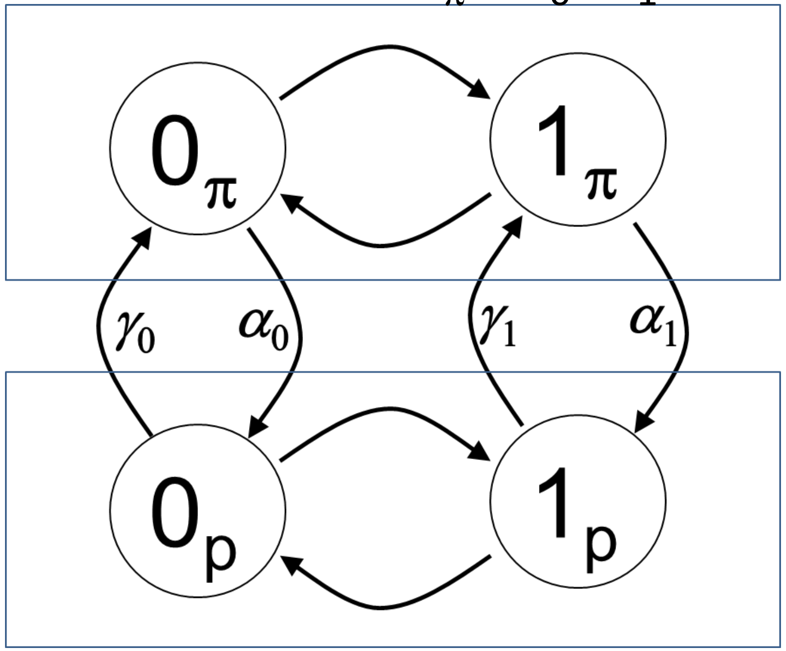
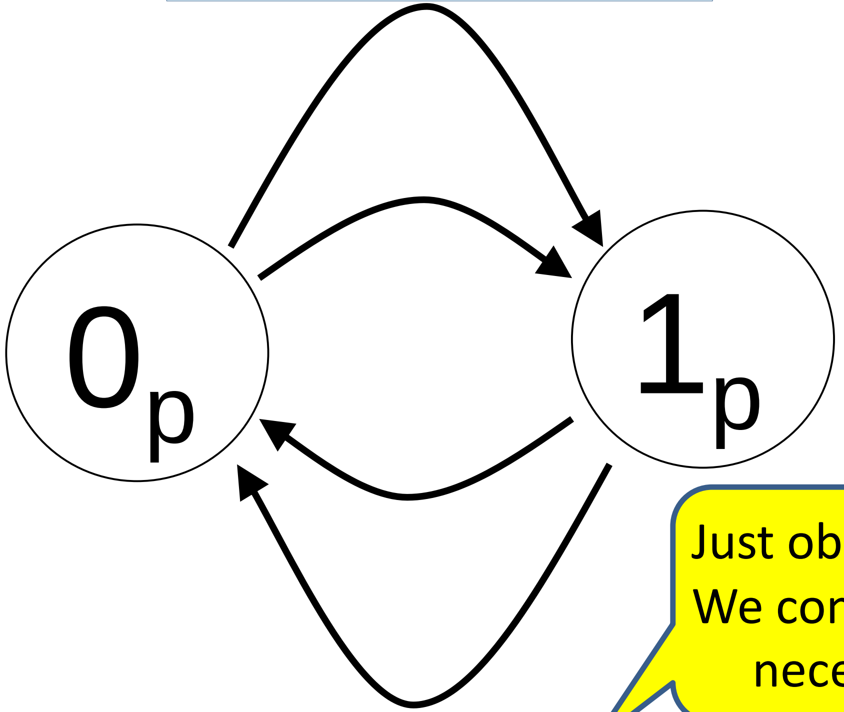


Subchain Q_p $\{p_0, p_1\}$

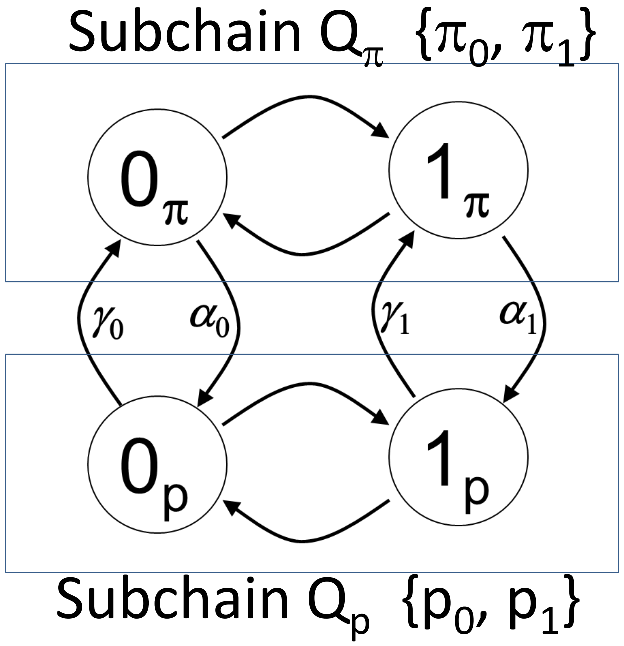
$$\gamma_0 \cdot \frac{\alpha_1 \pi_1}{\alpha_0 \pi_0 + \alpha_1 \pi_1}$$



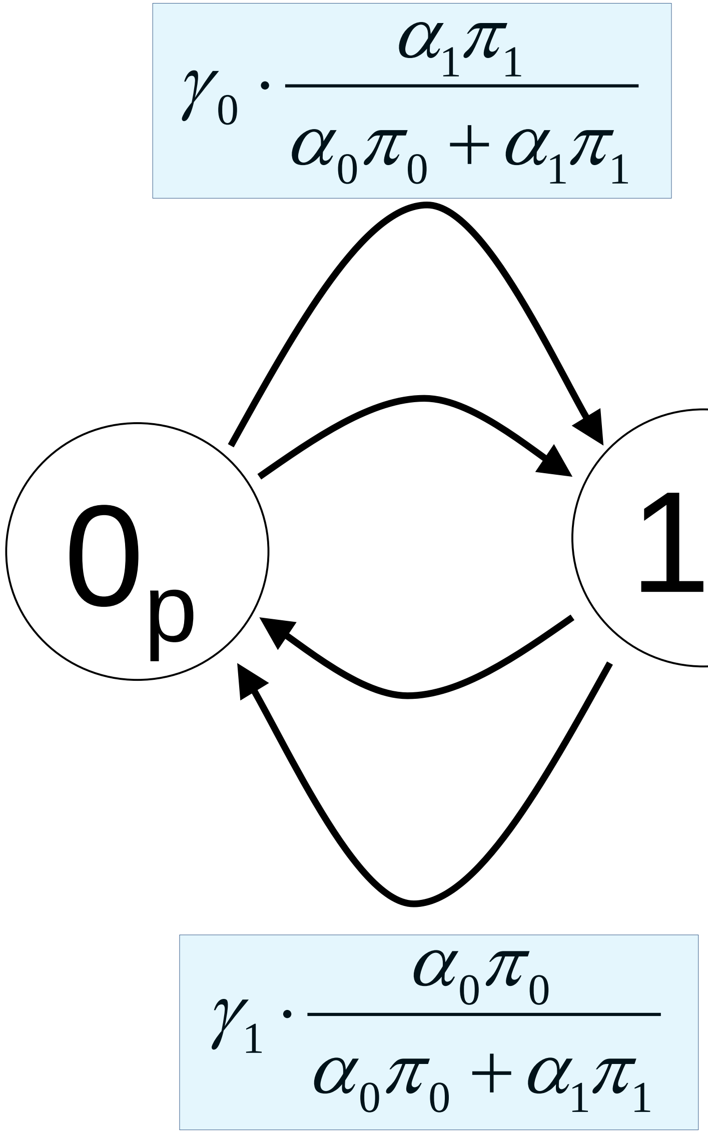
$$\gamma_1 \cdot \frac{\alpha_0 \pi_0}{\alpha_0 \pi_0 + \alpha_1 \pi_1}$$

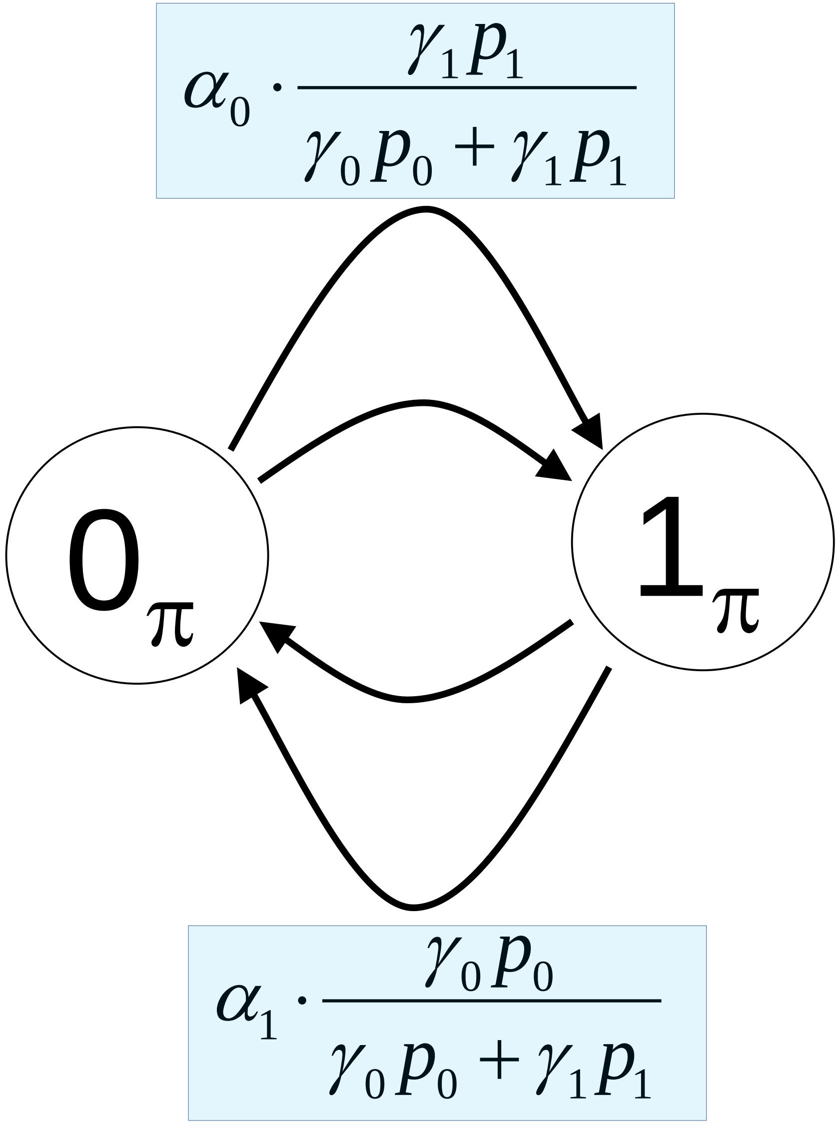
Just observe the terminations. We conclude α_0=0 or α_1=0 are necessary and sufficient.

Trichotomy of Decomposition Analysis: Case I

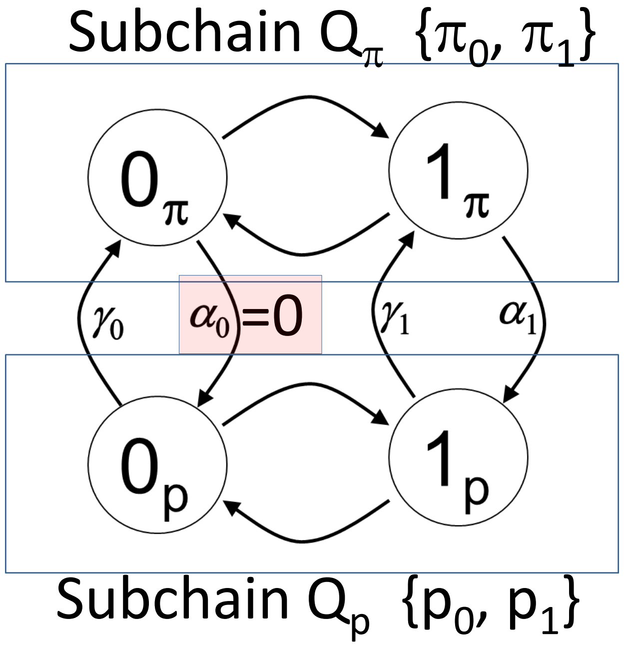


If α_0, α_1 , γ_0, and γ_1 are all non zero, then both subchain's distributions must be obtained simultaneously (or recursively numerically).

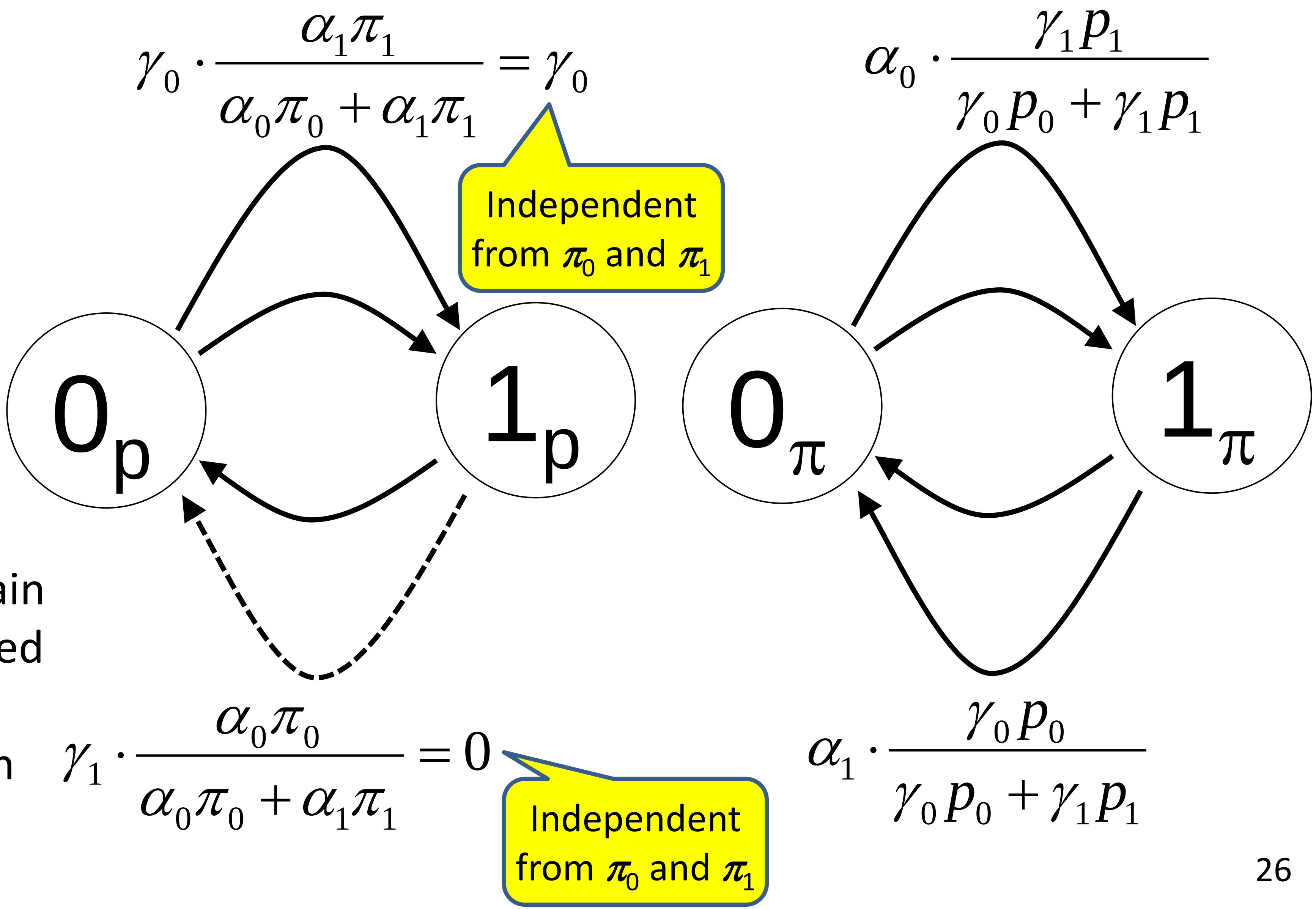




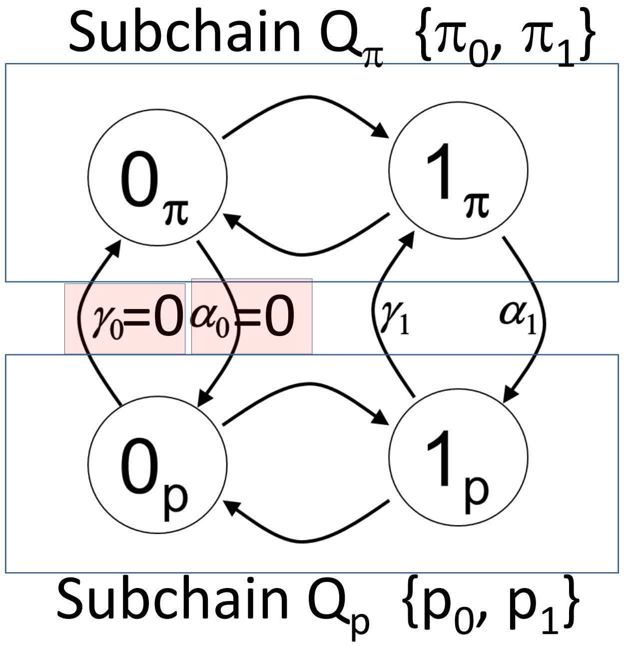
Trichotomy of Decomposition Analysis: Case II



If α_0=0, but α_1 , γ_0, and γ_1 are non zero, then subchain Q_p's distribution is obtained independently, and given Q_p's distribution, subchain Q_π's distribution is obtained.



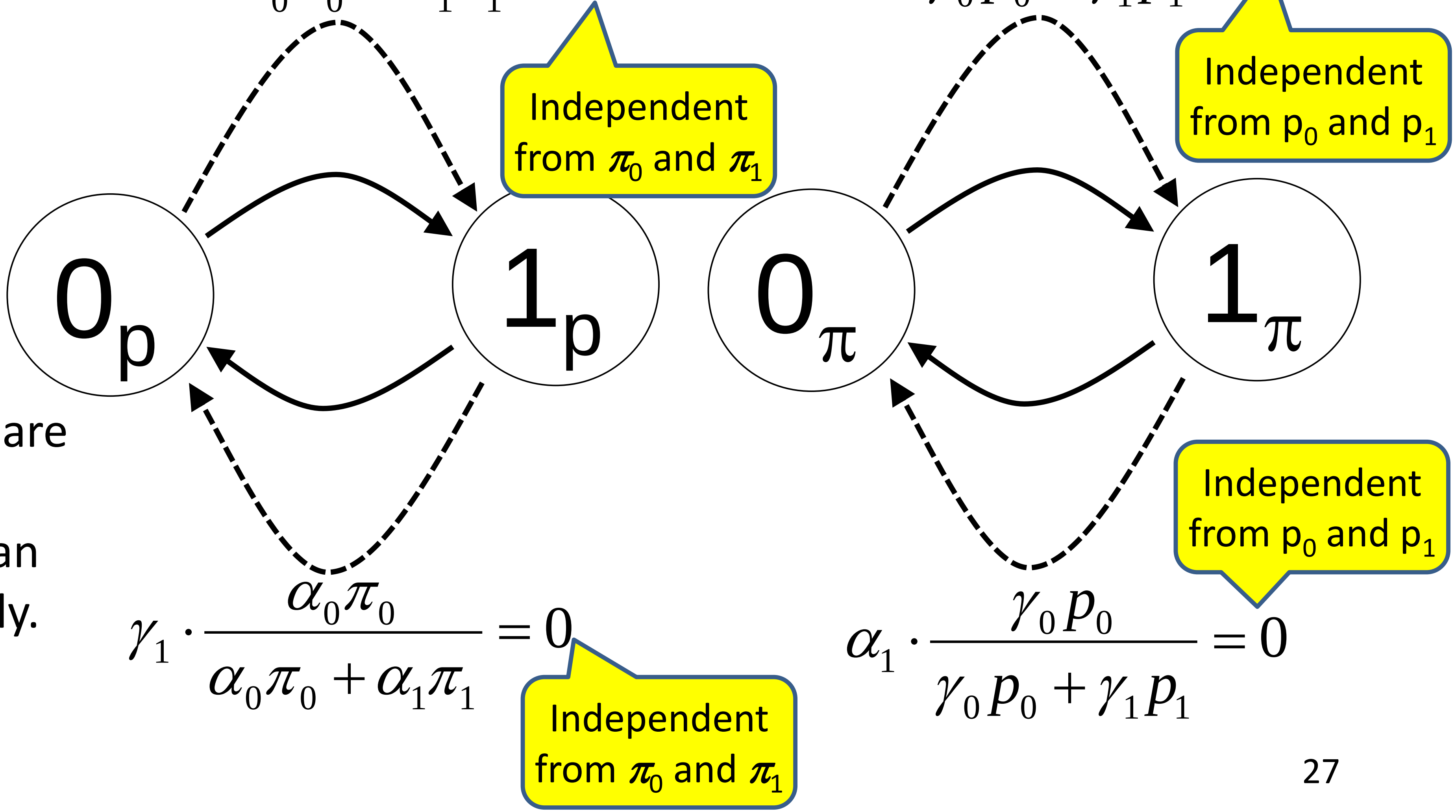
Trichotomy of Decomposition Analysis: Case III(a)



If $\alpha_0 = \gamma_0 = 0$, but α_1 and γ_1 are non zero, then both subchain's distributions can be obtained independently. (Termination becomes Truncation.)

$$\gamma_0 \cdot \frac{\alpha_1 \pi_1}{\alpha_0 \pi_0 + \alpha_1 \pi_1} = 0$$

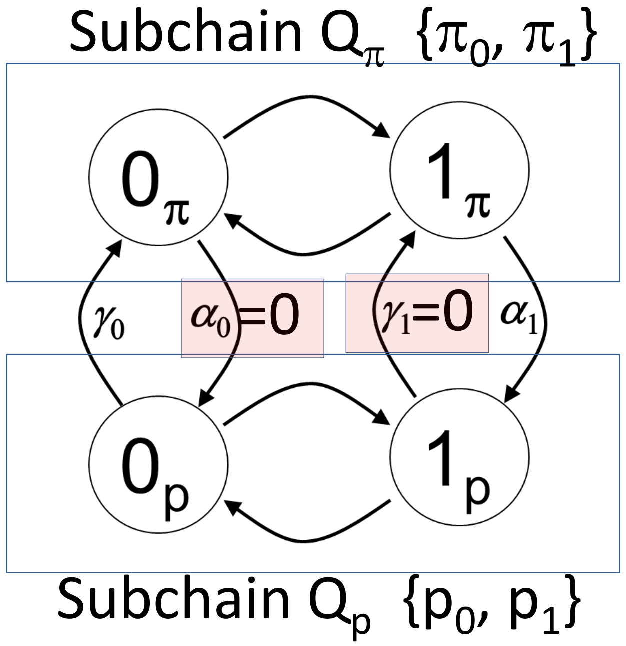
$$\alpha_0 \cdot \frac{\gamma_1 p_1}{\gamma_0 p_0 + \gamma_1 p_1} = 0$$



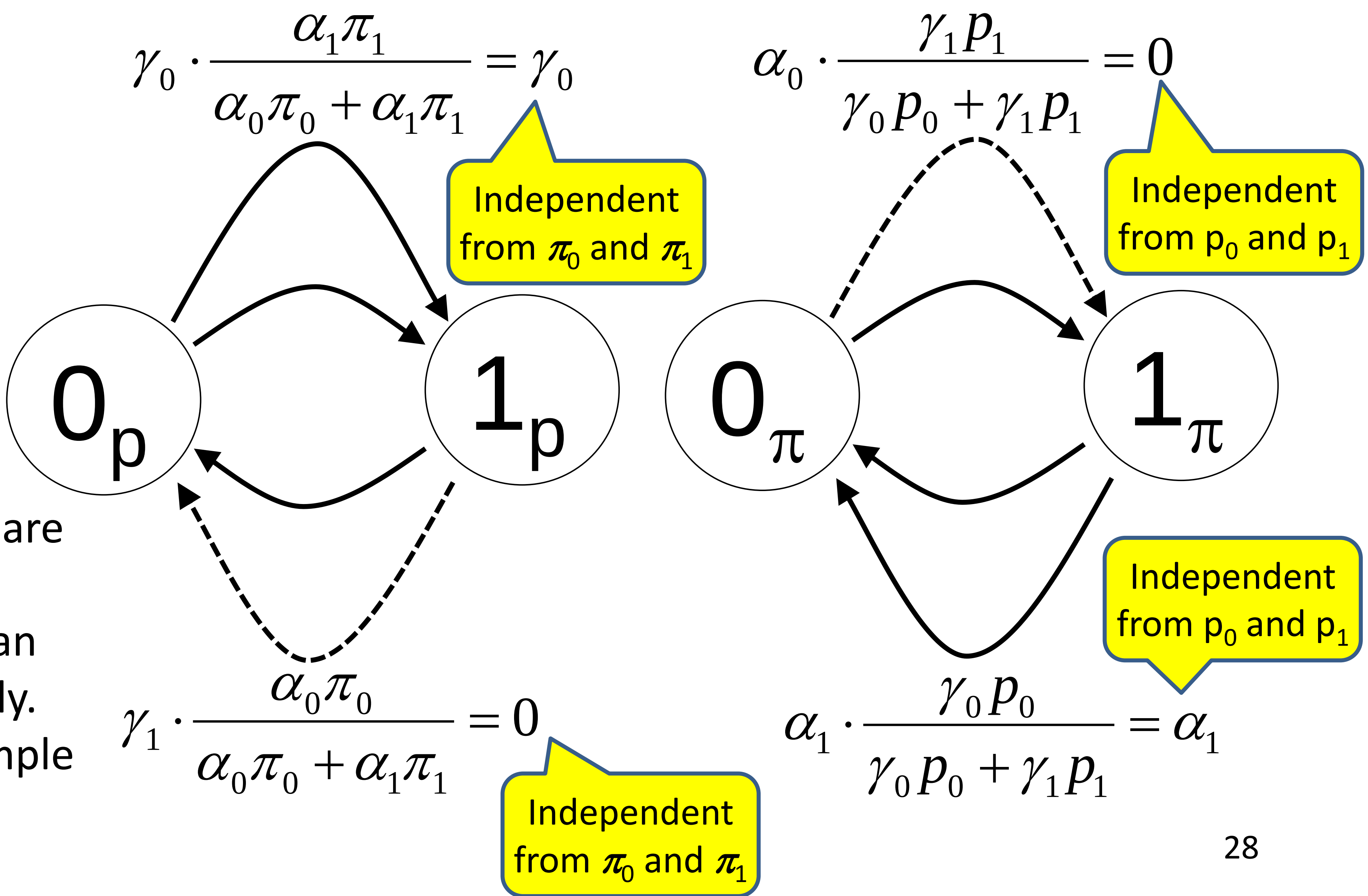
$$\gamma_1 \cdot \frac{\alpha_0 \pi_0}{\alpha_0 \pi_0 + \alpha_1 \pi_1} = 0$$

$$\alpha_1 \cdot \frac{\gamma_0 p_0}{\gamma_0 p_0 + \gamma_1 p_1} = 0$$

Trichotomy of Decomposition Analysis: Case III(b)



If $\alpha_0 = \gamma_1 = 0$, but α_1 and γ_0 are non zero, then both subchain's distributions can be obtained independently. (Termination becomes simple redirection of links.)



What are the benefits of our decomposition method?

- We can always decompose a MC: Given other subchains' distributions, termination is always possible.

- We can always find dependencies among subchains (i.e., how subchains impact other chains) based on how subchains are connected with each other.

Whether the result is simple or not depends on the model we solve.
If we observe a structural pattern in a MC, the result may become simpler.

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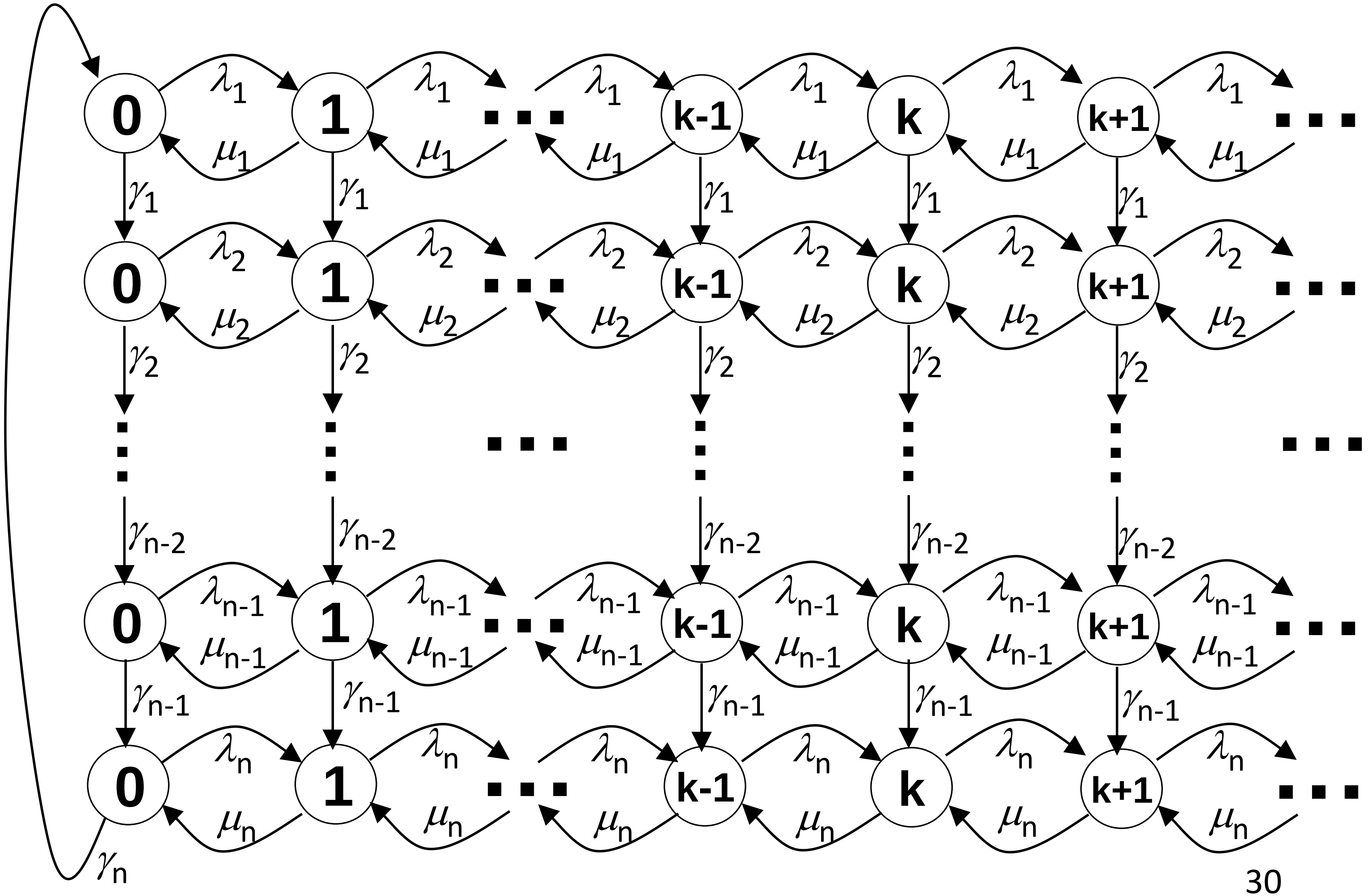
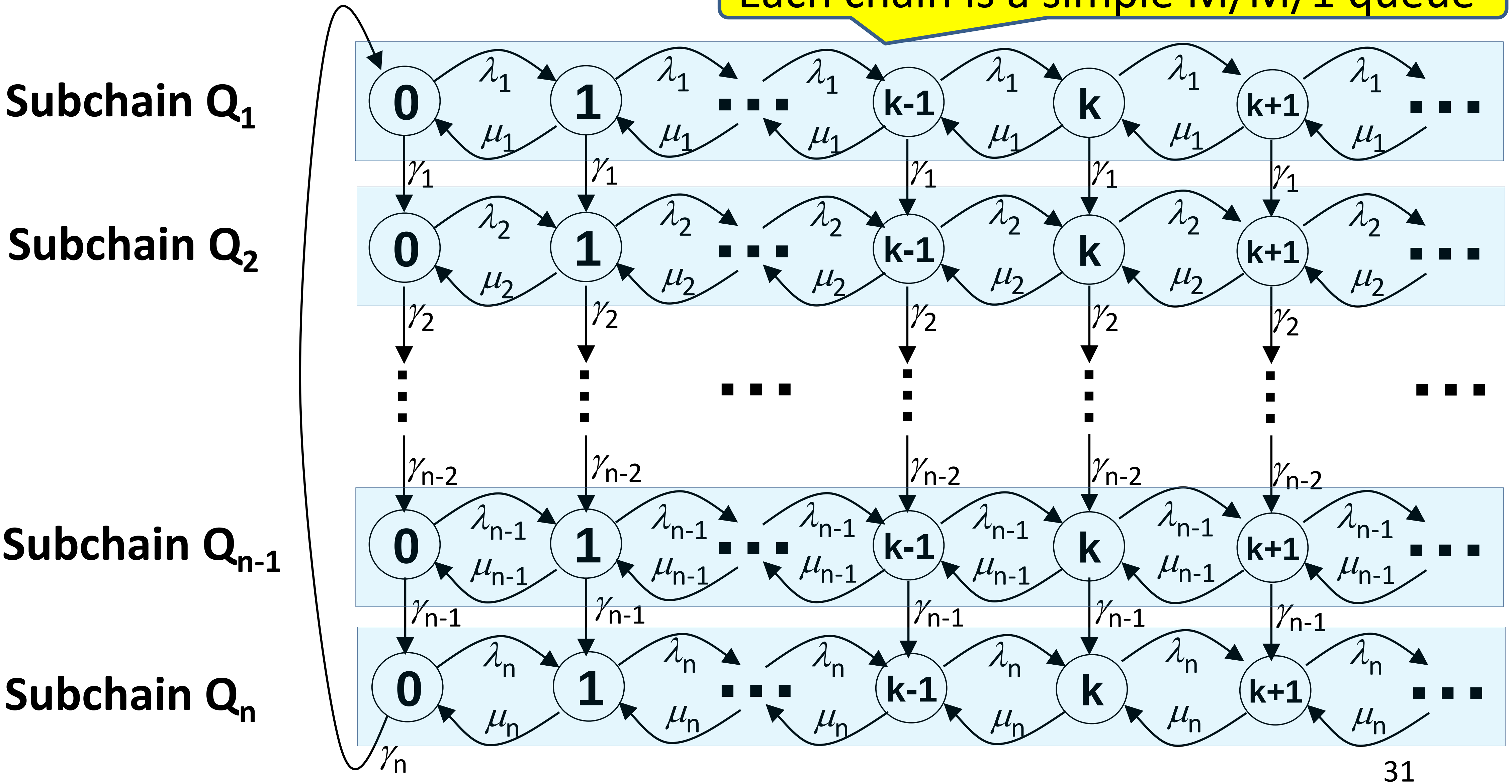


Markov Modulated Single-Server Queue

Each chain is a simple M/M/1 queue



Check how two M/M/1 queues are connected with each other.

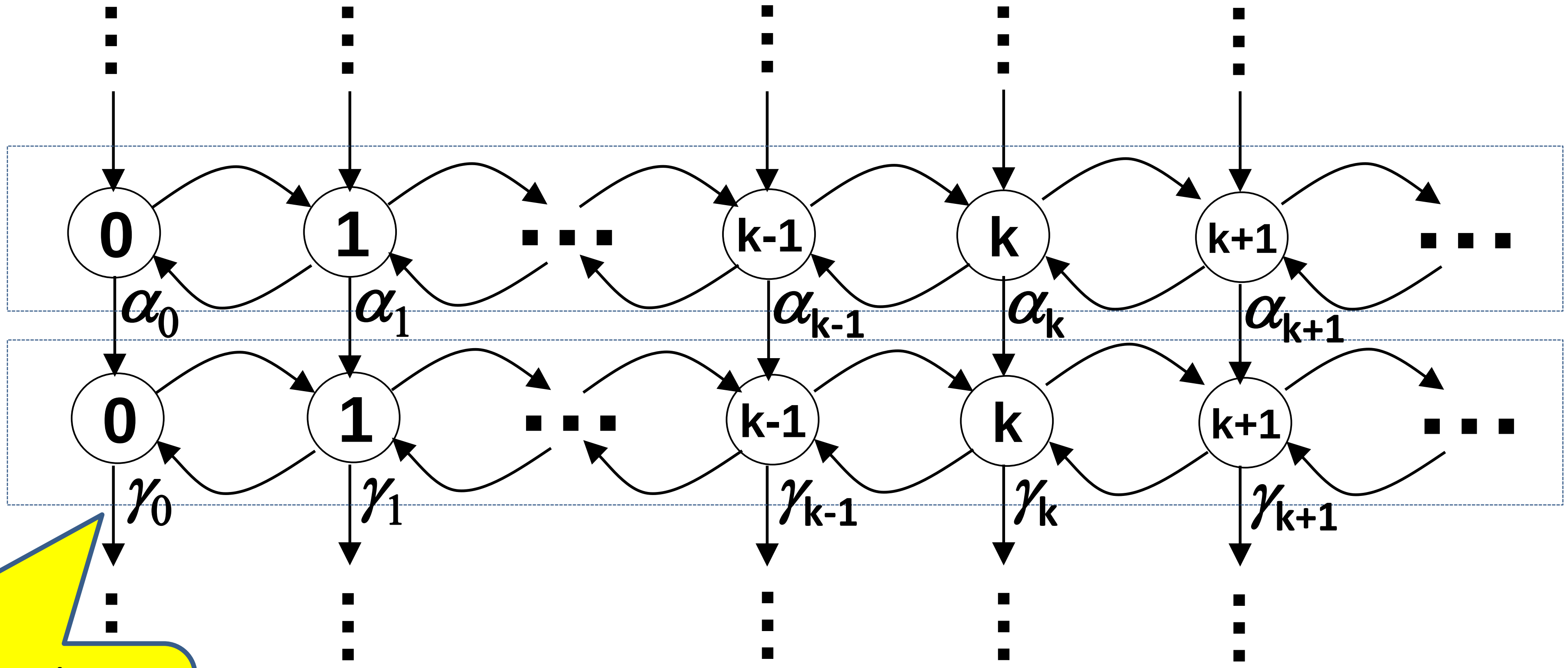
Comes back from subchain Q_p

Subchain Q_π

$\{\pi_0, \pi_1, \pi_2, \pi_3 \ldots\}$

Subchain Q_p

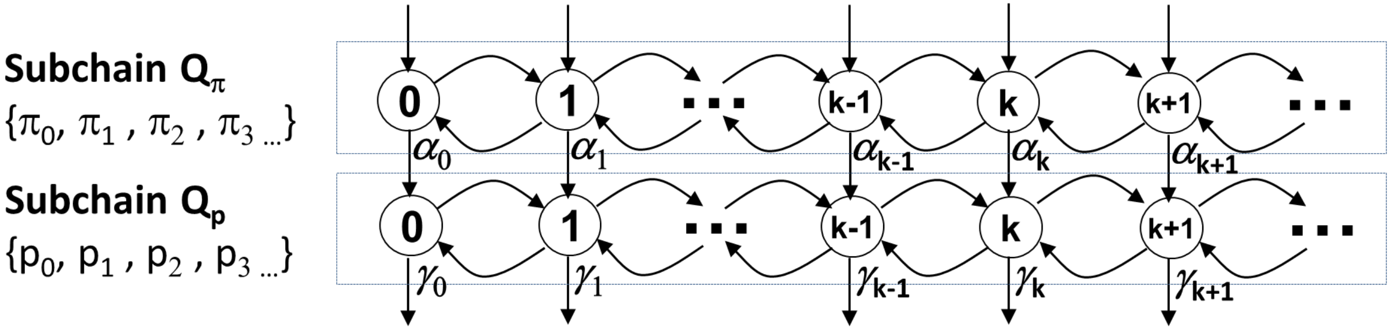
$\{p_0, p_1, p_2, p_3 \ldots\}$



Here, to make a general discussion, transition rates from one M/M/1 queue to the other are state-dependent

Eventually, goes to subchain Q_π

It is convenient to define proportions of flows.



Proportions are normalized (by definition)

z transforms are defined

Proportion of inflow into state k at Q_p

$$\pi_k = \frac{\alpha_k \pi_k}{\overline{\alpha}} = \frac{\alpha_k \pi_k}{\sum_{i=0}^{\infty} \alpha_i \pi_i}$$

$$\overline{\alpha} = \sum_{i=0}^{\infty} \alpha_i \pi_i$$

Proportion of outflow from state k at Q_p

$$p_k = \frac{\gamma_k p_k}{\overline{\gamma}} = \frac{\gamma_k p_k}{\sum_{i=0}^{\infty} \gamma_i p_i}$$

$$\overline{\gamma} = \sum_{i=0}^{\infty} \gamma_i p_i$$

$$\Pi(z) = \sum_{k=0}^{\infty} \pi_k z^k$$

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$$P(z) = \sum_{k=0}^{\infty} p_k z^k$$

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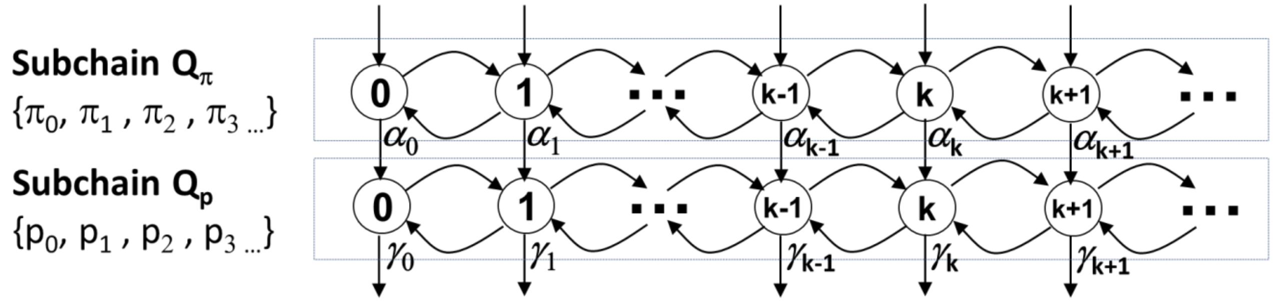
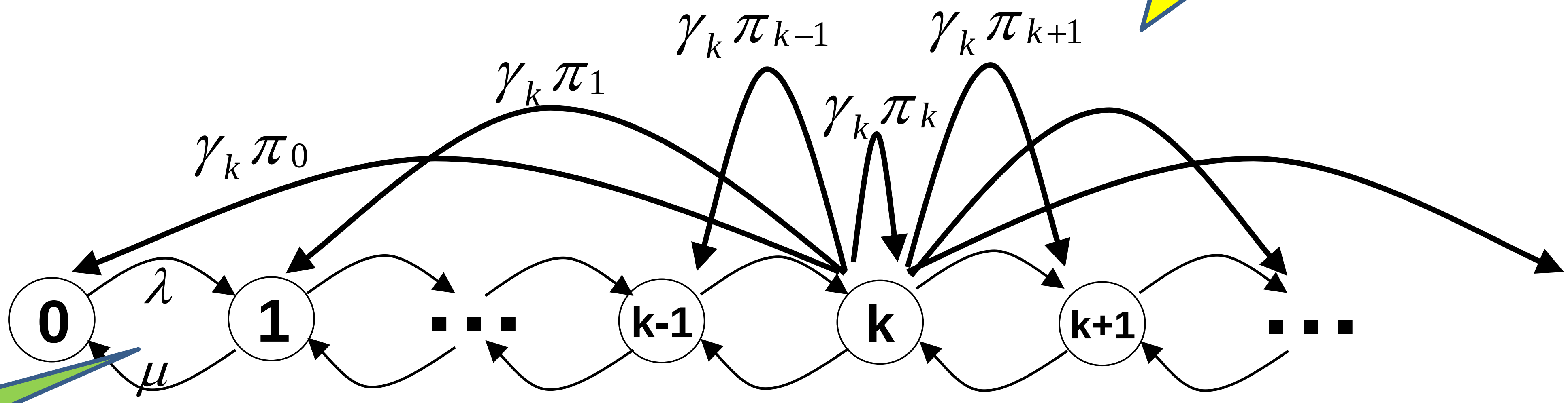
Now, let's decompose MC and add terminations.



Only terminations at state **k** are shown in this diagram

Subchain Q_p

$\{p_0, p_1, p_2, p_3 ...\}$



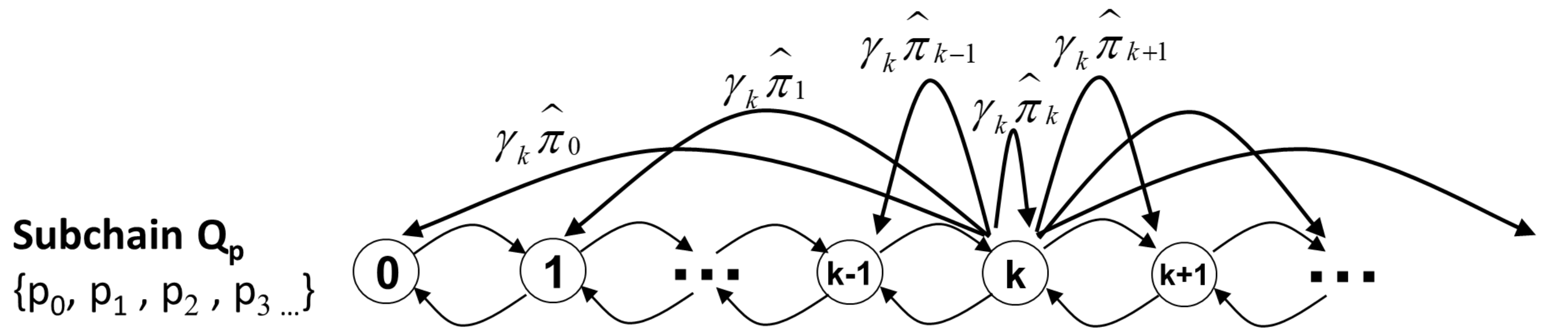
We assume arrival rate λ and service rate μ for this M/M/1 queue

Key information is the proportion of inflow into state **k** at subchain Q_p

$$\pi_k = \frac{\alpha_k \pi_k}{\overline{\alpha}} = \frac{\alpha_k \pi_k}{\sum_{i=0}^{\infty} \alpha_i \pi_i}$$

Quick Question: When can Q_p be solved independently?

What is the (general) solution?



$$P(z) = \frac{(1-z)p_0 + \frac{\bar{\gamma}}{\mu} z \left[P(z) - \Pi(z) \right]}{\left(1 - \frac{\lambda}{\mu} z\right)(1-z)}$$

$$\pi_k = \frac{\alpha_k \pi_k}{\bar{\alpha}} = \frac{\alpha_k \pi_k}{\sum_{i=0}^{\infty} \alpha_i \pi_i}$$

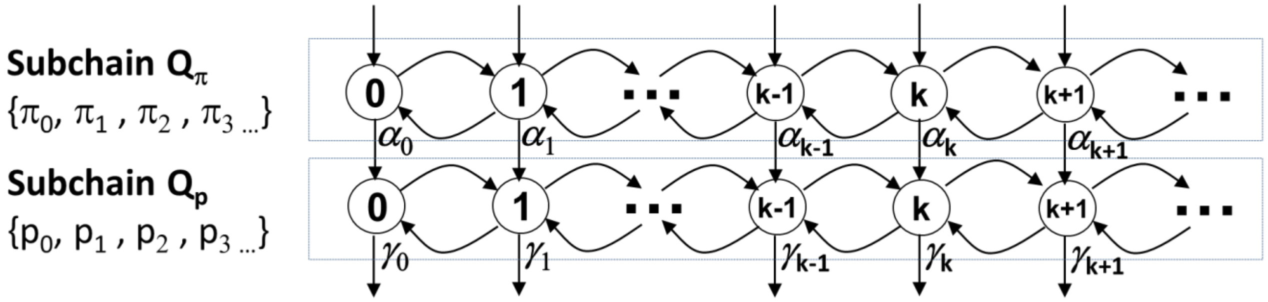
$$\Pi(z) = \sum_{k=0}^{\infty} \pi_k z^k$$

$$p_k = \frac{\gamma_k p_k}{\bar{\gamma}} = \frac{\gamma_k p_k}{\sum_{i=0}^{\infty} \gamma_i p_i}$$

$$P(z) = \sum_{k=0}^{\infty} p_k z^k$$

I omitted the derivation, but it is straightforward. Note that *P*(1)=0/0, so L'Hospital's rule should be used to determine p_0.

Two special cases that show up in the model



$$P(z) = \frac{(1-z)p_0 + \frac{\bar{\gamma}}{\mu} z \left[P(z) - \Pi(z) \right]}{\left(1 - \frac{\lambda}{\mu} z\right)(1-z)}$$

$$\pi_k = \frac{\alpha_k \pi_k}{\bar{\alpha}} = \frac{\alpha_k \pi_k}{\sum_{i=0}^{\infty} \alpha_i \pi_i}$$

$$\Pi(z) = \sum_{k=0}^{\infty} \pi_k z^k$$

$$p_k = \frac{\gamma_k p_k}{\bar{\gamma}} = \frac{\gamma_k p_k}{\sum_{i=0}^{\infty} \gamma_i p_i}$$

$$P(z) = \sum_{k=0}^{\infty} p_k z^k$$

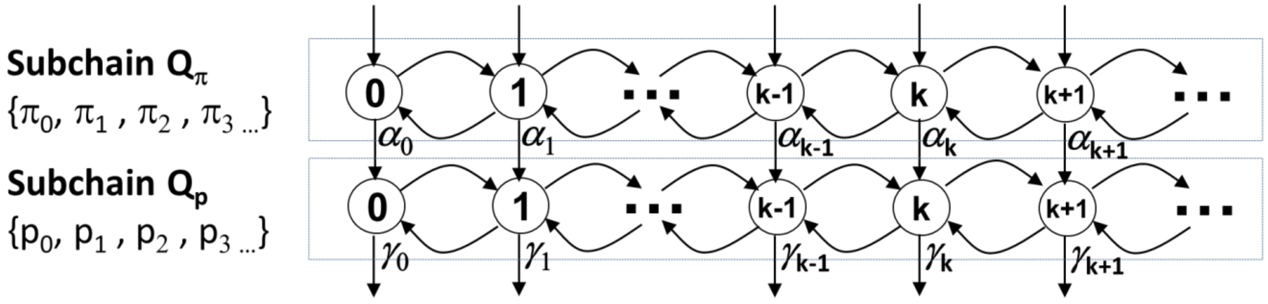
Single-Channel Case) If $\alpha_0 = \alpha > 0$ ($\gamma_0 = \gamma > 0$) and all other α_k (or γ_k, respectively) are zero, then

$$\pi_0 = 1, \pi_k = 0, \forall k \geq 1, \Pi(z) = 1 \text{ (Or } p_0 = 1, p_k = 0, \forall k \geq 1, P(z) = 1)$$

Multi-Channel Case) If $\alpha_k = \alpha > 0$ ($\gamma_k = \gamma > 0$) for all k, then

$$\pi_k = \pi_k, \forall k, \Pi(z) = \Pi(z) \text{ (Or } p_k = p_k, \forall k, P(z) = P(z))$$

Two special cases that show up in the model



$$P(z) = \frac{(1-z)p_0 + \frac{\bar{\gamma}}{\mu} z \left[P(z) - \Pi(z) \right]}{\left(1 - \frac{\lambda}{\mu} z\right)(1-z)}$$

$$\pi_k = \frac{\alpha_k \pi_k}{\bar{\alpha}} = \frac{\alpha_k \pi_k}{\sum_{i=0}^{\infty} \alpha_i \pi_i}$$

$$\Pi(z) = \sum_{k=0}^{\infty} \pi_k z^k$$

$$p_k = \frac{\gamma_k p_k}{\bar{\gamma}} = \frac{\gamma_k p_k}{\sum_{i=0}^{\infty} \gamma_i p_i}$$

$$P(z) = \sum_{k=0}^{\infty} p_k z^k$$

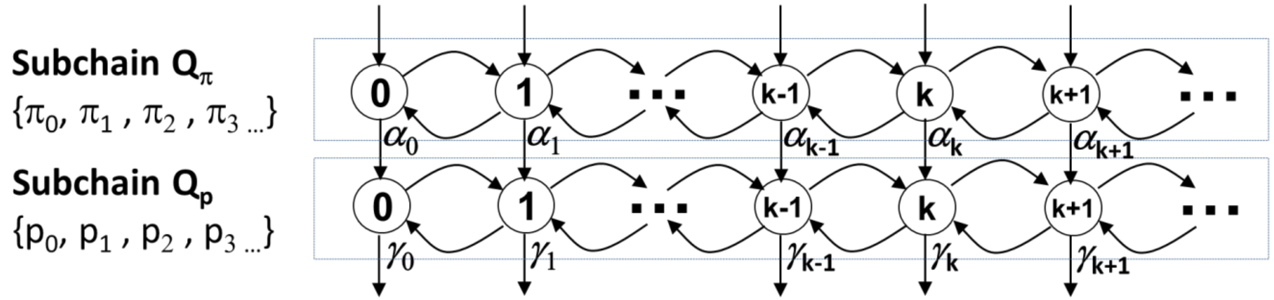
Single-Channel Case) If $\alpha_0 = \alpha > 0$ ($\gamma_0 = \gamma > 0$) and all other α_k (or γ_k, respectively) are zero, then

$$\pi_0 = 1, \pi_k = 0, \forall k \geq 1, \Pi(z) = 1 \text{ (Or } p_0 = 1, p_k = 0, \forall k \geq 1, P(z) = 1)$$

Multi-Channel Case) If $\alpha_k = \alpha > 0$ ($\gamma_k = \gamma > 0$) for all k, then

$$\pi_k = \pi_k, \forall k, \Pi(z) = \Pi(z) \text{ (Or } p_k = p_k, \forall k, P(z) = P(z))$$

Example: Two M/M/1 connected only at states 0



Single-Channel Case) If $\alpha_0=\alpha>0$, $\gamma_0=\gamma>0$ and all other α_k , γ_k are zero, then

$$\pi_0 = p_0 = 1, \pi_k = p_k = 0, \forall k \geq 1, \Pi(z) = P(z) = 1$$

$$P(z) = \frac{(1-z)p_0 + \frac{\bar{\gamma}}{\mu} z \left[P(z) - \Pi(z) \right]}{\left(1 - \frac{\lambda}{\mu} z\right)(1-z)} = \frac{(1-z)p_0}{\left(1 - \frac{\lambda}{\mu} z\right)(1-z)} = \frac{p_0}{\left(1 - \frac{\lambda}{\mu} z\right)}$$

We obtain a familiar M/M/1 result

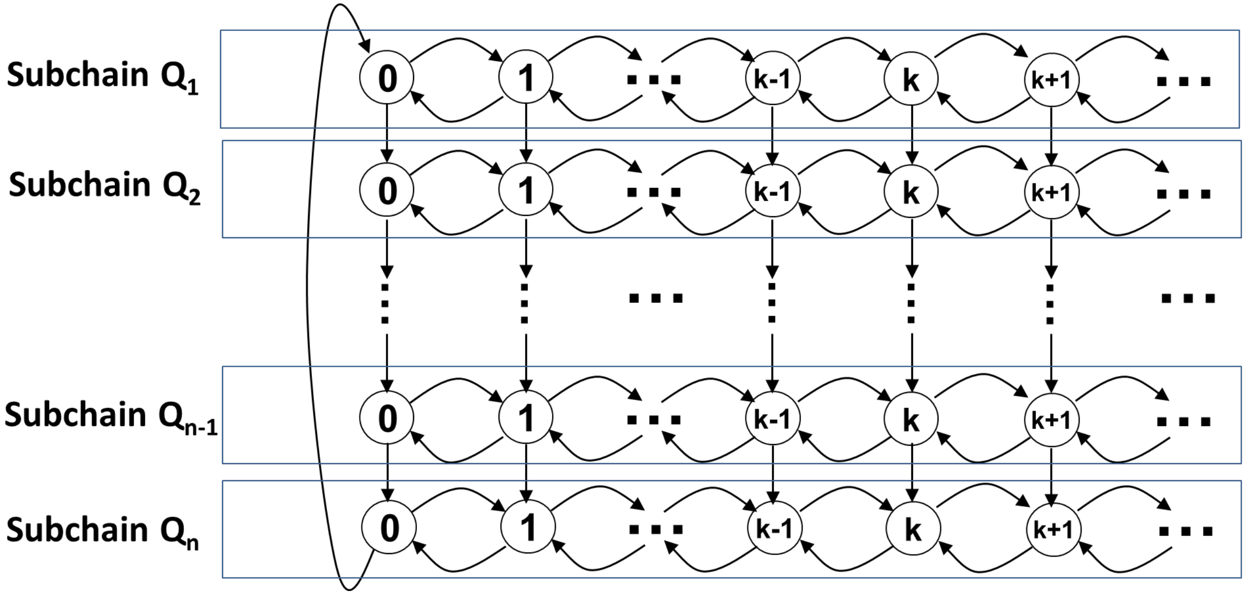
$$\pi_k = \frac{\alpha_k \pi_k}{\bar{\alpha}} = \frac{\alpha_k \pi_k}{\sum_{i=0}^{\infty} \alpha_i \pi_i}$$

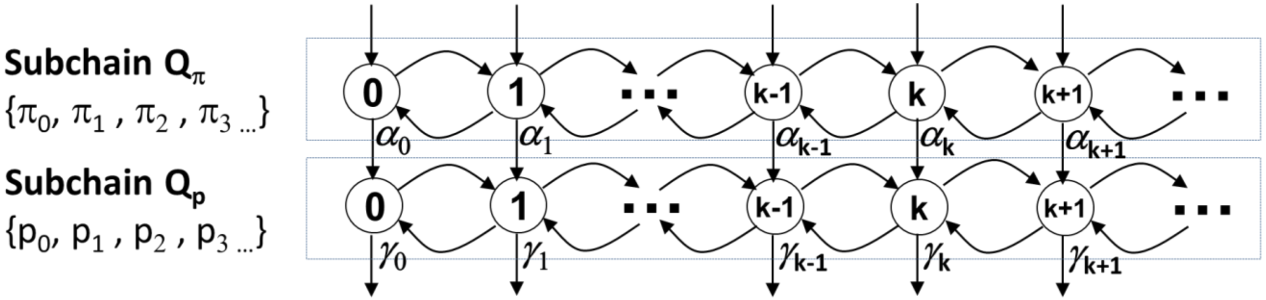
$$\Pi(z) = \sum_{k=0}^{\infty} \pi_k z^k$$

$$p_k = \frac{\gamma_k p_k}{\bar{\gamma}} = \frac{\gamma_k p_k}{\sum_{i=0}^{\infty} \gamma_i p_i}$$

$$P(z) = \sum_{k=0}^{\infty} p_k z^k$$

Solving Q_1: It can be solved independently.





Single-Channel for α) $\alpha_0=\alpha>0$ and all other α_k are zero

$$\pi_0 = 1, \pi_k = 0, \forall k \ge 1, \Pi(z) = 1$$

Multi-Channel for γ) $\gamma_k=\gamma>0$ for all k

$$p_k = p_k, \forall k, P(z) = P(z)$$

$$P(z) = \frac{(1-z)p_0 + \frac{\bar{\gamma}}{\mu} z \left[P(z) - \Pi(z) \right]}{\left(1 - \frac{\lambda}{\mu} z\right)(1-z)}$$

Result does not depend on other chains

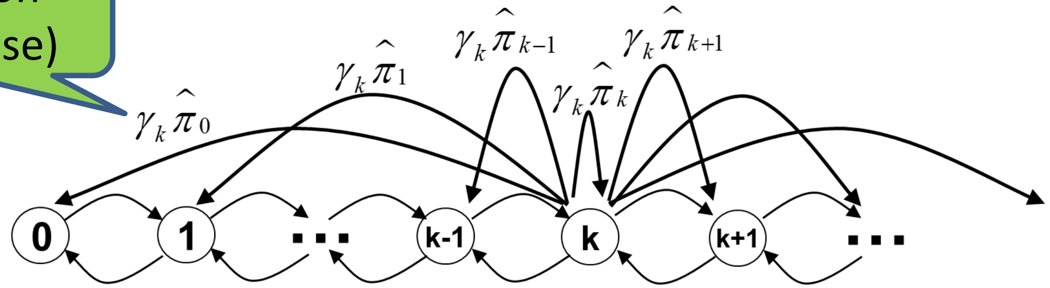
$$P(z) = \frac{(1-z)p_0 - \frac{\gamma}{\mu} z}{\left(1 - \frac{\lambda}{\mu} z\right)(1-z) - \frac{\gamma}{\mu} z}, \quad p_0 = \frac{\gamma}{\mu} \frac{\omega_1}{1-\omega_1}$$

ω_1 is a pole of *P(z)*, satisfying $0 < \omega_1 < 1$.

Solving Q_1: What termination did we use?

Termination (general case)

Subchain Q_p
$\{p_0, p_1, p_2, p_3 \ldots\}$



Single-Channel for α) $\alpha_0 = \alpha > 0$ and all other α_k are zero

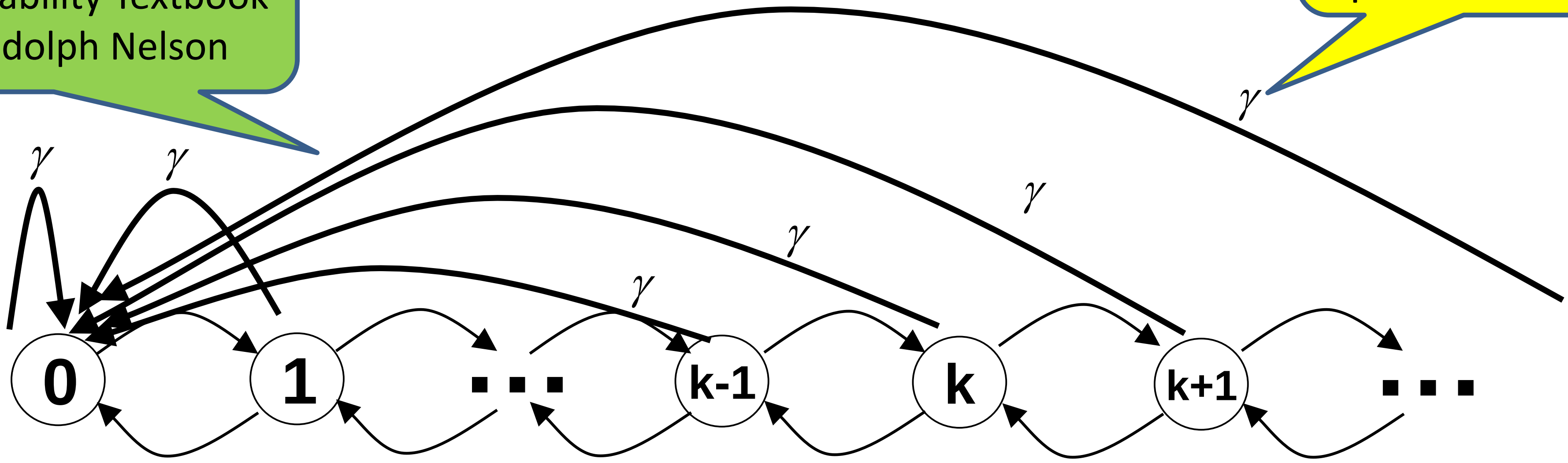
$$\pi_0 = 1, \pi_k = 0, \forall k \geq 1$$

Multi-Channel for γ) $\gamma_k = \gamma > 0$ for all k

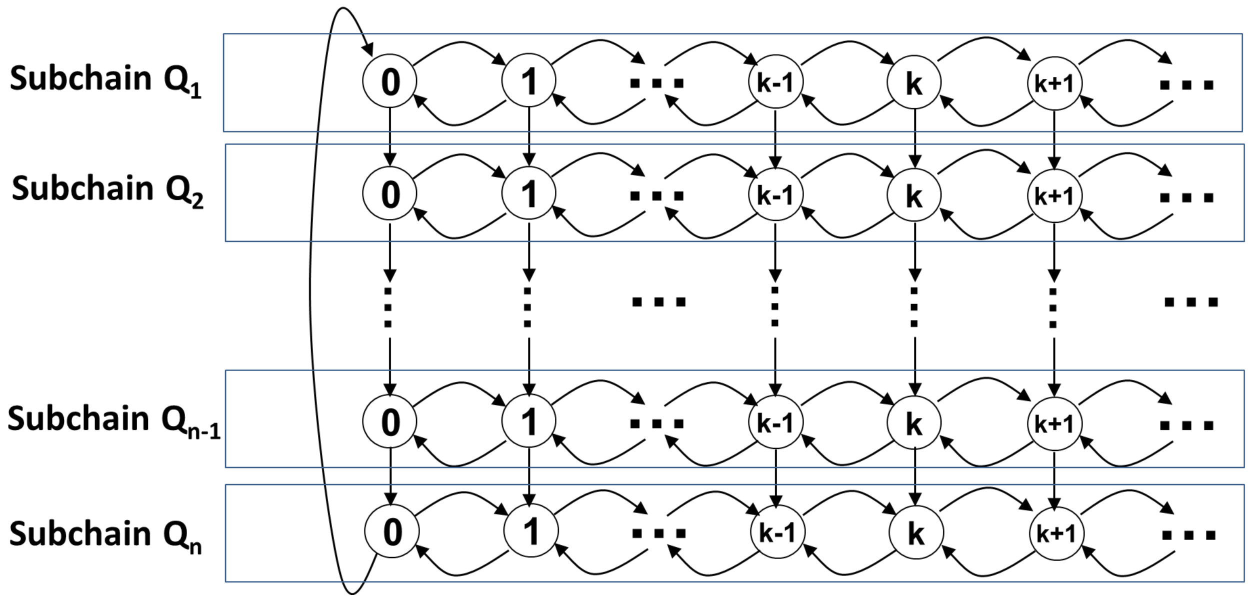
Termination does not depend on the previous chain

This MC shows up as Example 8.13 in the Probability Textbook (1995) by Randolph Nelson

Subchain Q_p
$\{p_0, p_1, p_2, p_3 \ldots\}$



Solving Q_2 … Q_{n-1}: They depend on the previous chain.



Multi-Channel for α) $\alpha_k = \alpha > 0$ for all k

$$\pi_k = \pi_k, \forall k, \Pi(z) = \Pi(z)$$

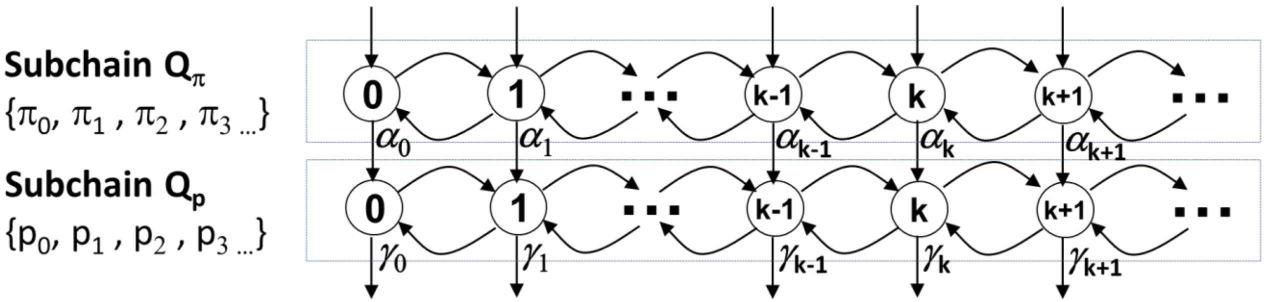
Multi-Channel for γ) $\gamma_k = \gamma > 0$ for all k

$$p_k = p_k, \forall k, P(z) = P(z)$$

$$P(z) = \frac{(1-z)p_0 + \frac{\bar{\gamma}}{\mu} z \left[P(z) - \Pi(z) \right]}{\left(1 - \frac{\lambda}{\mu} z\right)(1-z)}$$

Very similar to the result of Q_1, but it depends on the previous chain

$$P(z) = \frac{(1-z)p_0 - \frac{\gamma}{\mu}\Pi(z)}{\left(1 - \frac{\lambda}{\mu} z\right)(1-z) - \frac{\gamma}{\mu} z}, \quad p_0 = \frac{\gamma}{\mu} \frac{\omega_1}{1-\omega_1} \Pi(\omega_1)$$

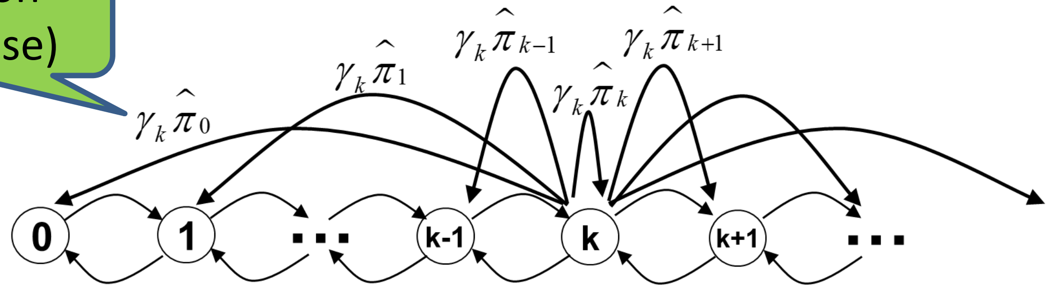


ω_1 is a pole of $P(z)$, satisfying $0 < \omega_1 < 1$.

Solving Q_2 … Q_{n-1}: What termination did we use?

Termination (general case)

Subchain Q_p
$\{p_0, p_1, p_2, p_3 \ldots\}$



Multi-Channel for α) α_k=α >0 for all k

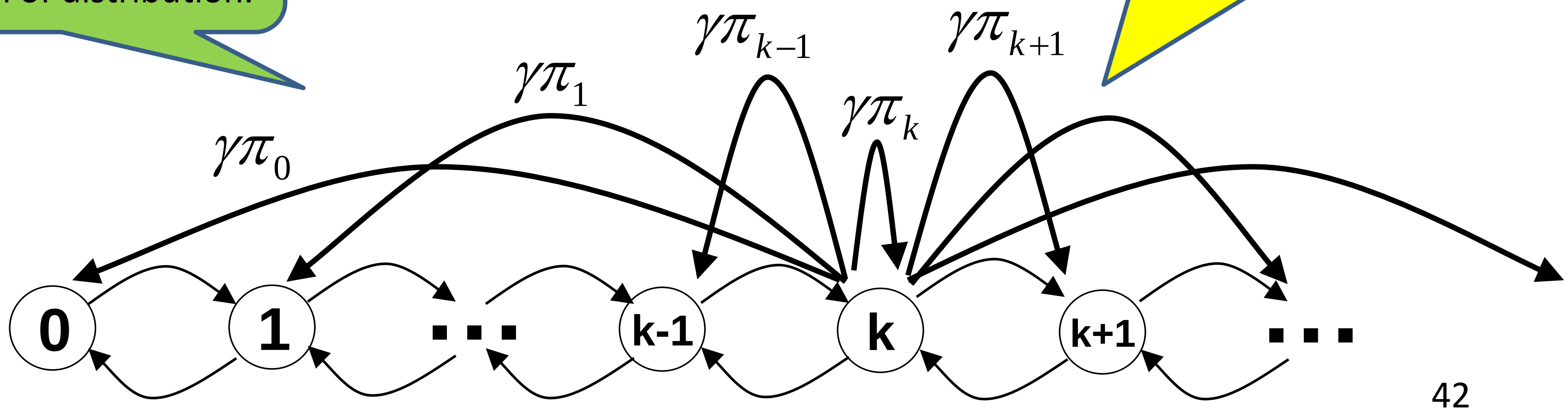
$$\pi_k = \pi_k, \forall k$$

Multi-Channel for γ) γ_k=γ >0 for all k

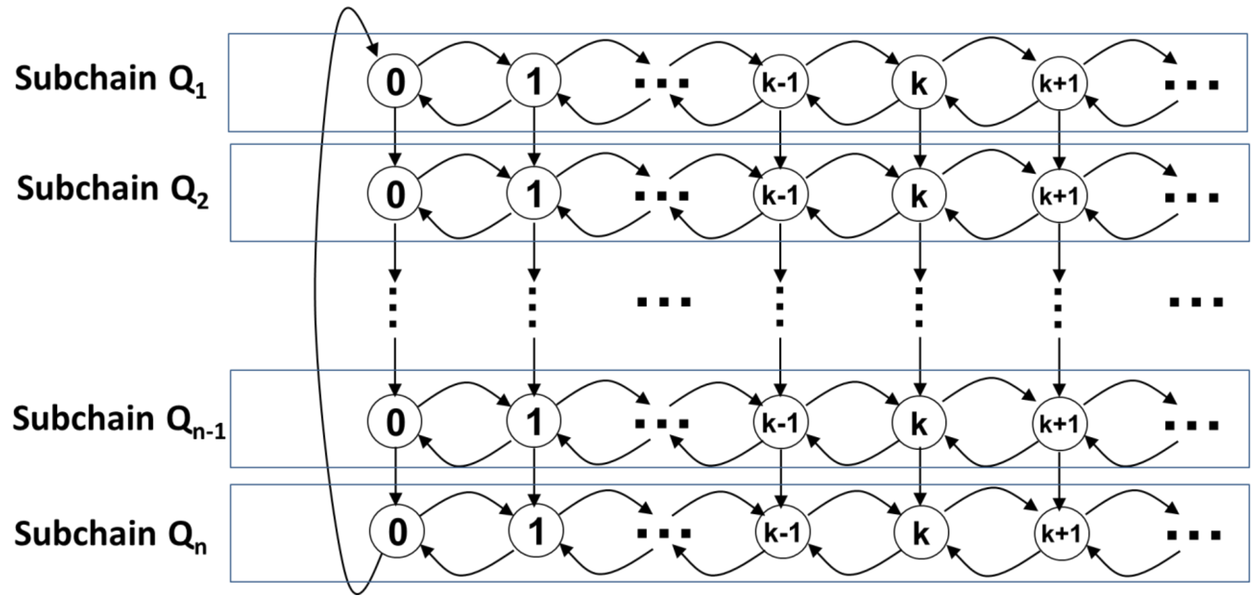
This MC looks complicated, but actually, it is easy to solve if we use z-transform of distribution.

We show termination (redirection of flows) at state k only

Subchain Q_p
$\{p_0, p_1, p_2, p_3 \ldots\}$



Solving Q_n: Given Q_{n-1}, Q_n can be solved.



Multi-Channel for α) $\alpha_k = \alpha > 0$ for all k

$$\pi_k = \pi_k, \forall k, \Pi(z) = \Pi(z)$$

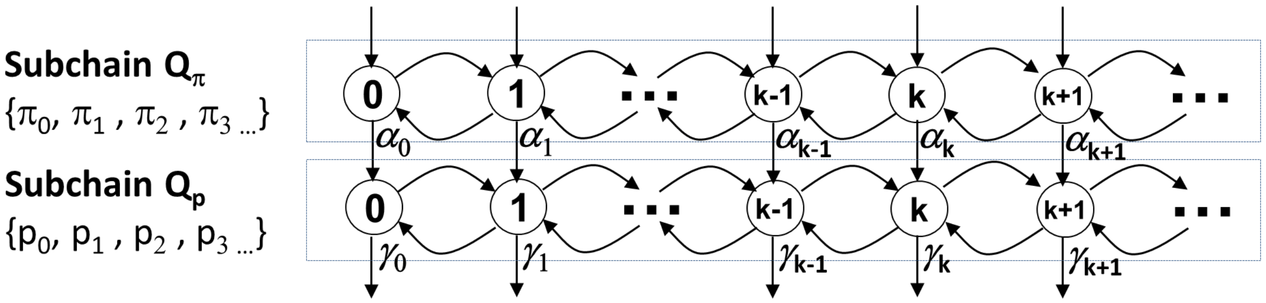
Single-Channel for γ) $\gamma_0 = \gamma > 0$ and all other γ_k are zero

$$p_0 = 1, p_k = 0, \forall k \geq 1, P(z) = 1$$

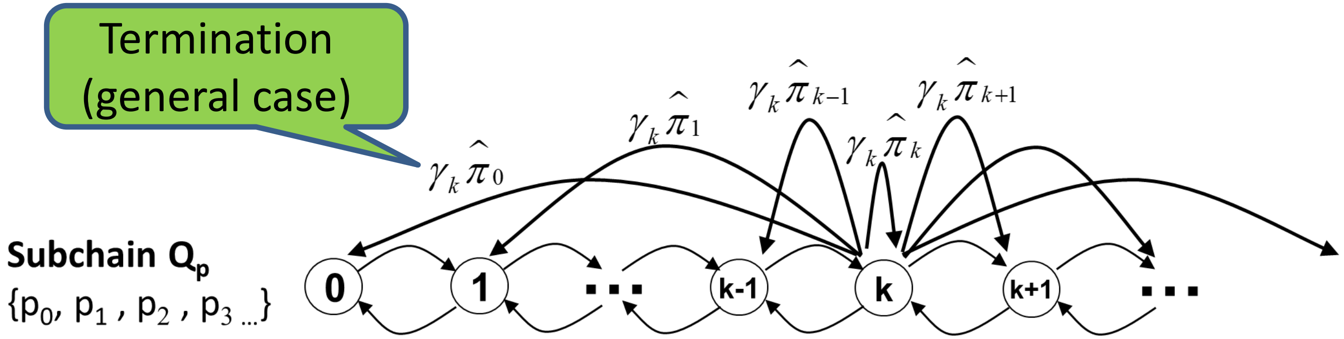
$$P(z) = \frac{(1-z)p_0 + \frac{\bar{\gamma}}{\mu} z \left[P(z) - \Pi(z)\right]}{\left(1 - \frac{\lambda}{\mu} z\right)(1-z)}$$

Result depends on the previous chain

$$P(z) = \frac{(1-z) + \frac{\gamma}{\mu} z \left[1 - \Pi(z)\right]}{\left(1 - \frac{\lambda}{\mu} z\right)(1-z)} p_0, \quad p_0 = \frac{\gamma}{\mu} \frac{1 - \frac{\lambda}{\mu}}{1 + \frac{\gamma}{\mu} \Pi'(1)}$$



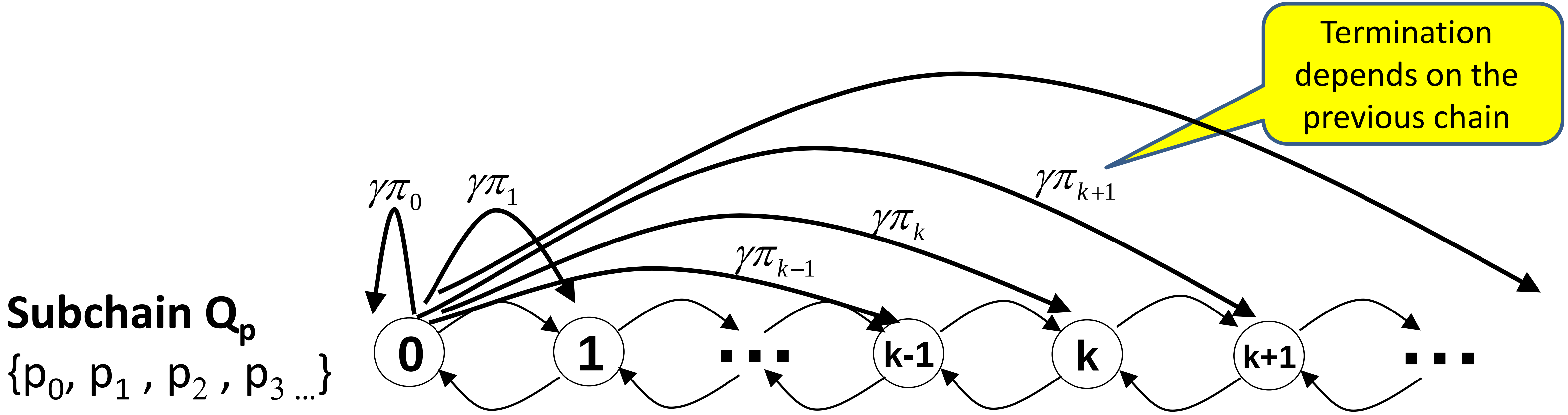
Solving Q_n: What termination did we use?



Multi-Channel for α) $\alpha_k = \alpha > 0$ for all k

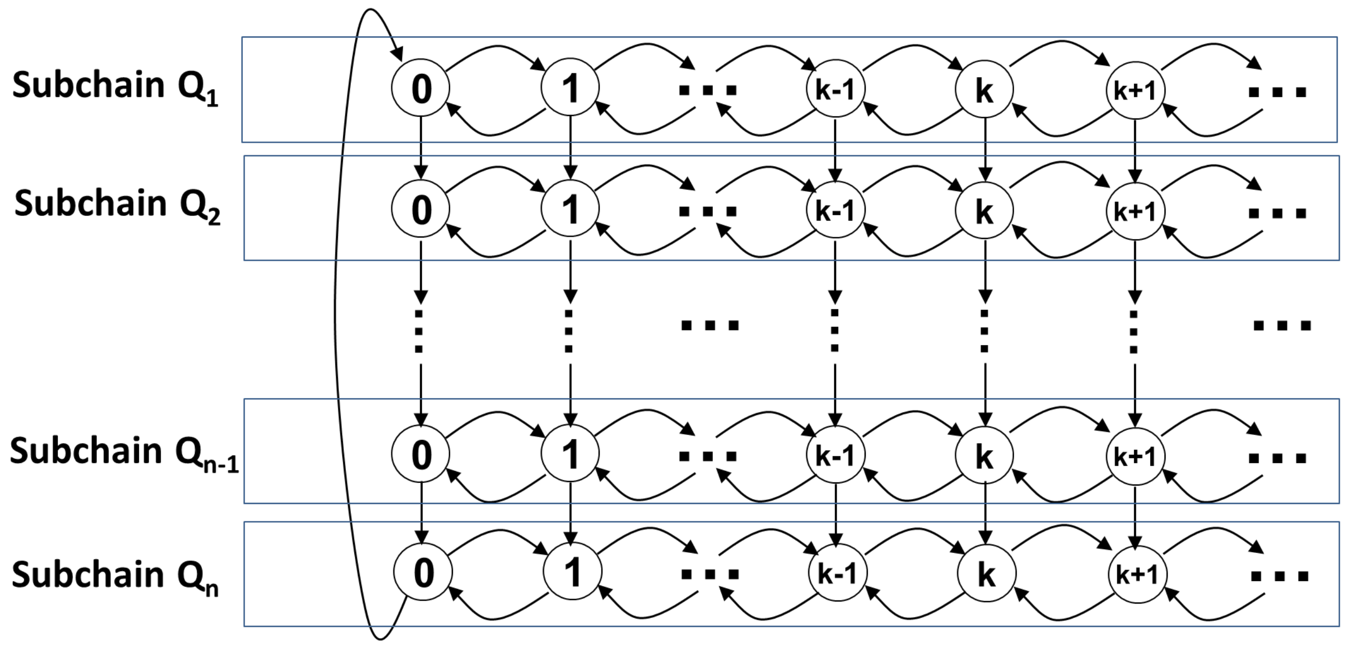
$$\pi_k = \pi_k, \forall k$$

Single-Channel for γ) $\gamma_0 = \gamma > 0$ and all other γ_k are zero



Solving Markov Modulated Single-Server Queue

No iteration is necessary



Step 1: We first solve subchain Q_1.

Step 2: Given the result of Q_1, solve Q_2.

Step 3: Given the result of Q_2, solve Q_3.

…

Step n: Given the result of Q_{n-1}, solve Q_n.

Done. (Ok, we need to normalize them. But it is straightforward after we find distributions of subchains...)

$$P(z)=\frac{(1-z)p_0-\frac{\gamma}{\mu}z}{\left(1-\frac{\lambda}{\mu}z\right)(1-z)-\frac{\gamma}{\mu}z},\ p_0=\frac{\gamma}{\mu}\frac{\omega_1}{1-\omega_1}$$

Q_1 is solved independently

$$P(z)=\frac{(1-z)p_0-\frac{\gamma}{\mu}\Pi(z)}{\left(1-\frac{\lambda}{\mu}z\right)(1-z)-\frac{\gamma}{\mu}z},\ p_0=\frac{\gamma}{\mu}\frac{\omega_1}{1-\omega_1}\Pi(\omega_1)$$

Q_2 to Q_{n-1} are solved given the previous chain’s solution

$$P(z)=\frac{(1-z)+\frac{\gamma}{\mu}z\left[1-\Pi(z)\right]}{\left(1-\frac{\lambda}{\mu}z\right)(1-z)}p_0,\ p_0=\frac{\gamma}{\mu}\frac{1-\frac{\lambda}{\mu}}{1+\frac{\gamma}{\mu}\Pi'(1)}$$

Q_n is solved given the solution of Q_{n-1}

Summary

- Our decomposition method utilizes a **termination** scheme; Termination refers to the added transition rates at the boundary states of the decomposed subchain.
- Our termination scheme is based on **partial flow conservation**; it does not rely on return rates.
- Our method reveals how subchains are dependent on each other based on how subchains are connected with each other.
- It is easy to implement both numerically and analytically.

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