On the Log-Brunn-Minkowski conjecture

(based on various joint works with Andrea Colesanti, John Hosle, Alexander Kolesnikov, Arnaud Marsiglietti, Piotr Nayar, Artem Zvavitch.)

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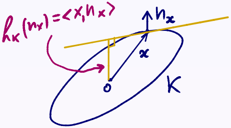
BIRS, Banff, Canada
February 2020.

Notation



- Convex bodies in $\mathbb{R}^n$ denote K, L, M;
- Lebesgue volume in $\mathbb{R}^n$ denote $|\cdot|$ or $|\cdot|_n$;
- Recall Minkowski sum of sets $A, B \subset \mathbb{R}^n$:

$$A + B = \{x + y : x \in A, y \in B\}.$$

- Support function of a convex set K is

$$h_K(y) = \sup_{x \in K} \langle x, y \rangle = \|y\|_{K^\circ};$$

- $h_{K+L} = h_K + h_L$;
- Unit normal to ∂K at $x \in \partial K$ denote n_x;
- $h_K(n_x) = \langle x, n_x \rangle$;
- Second fundamental form of ∂K denote II, mean curvature $H_x = tr(\mathrm{II})$.

Log-concavity of the Lebesgue measure

$$|\lambda K+(1-\lambda)L| \geq |K|^{\lambda}|L|^{1-\lambda}$$

$\frac{1}{n}$-concavity of the Lebesgue measure

$$|\lambda K+(1-\lambda)L|^{\frac{1}{n}} \geq \lambda|K|^{\frac{1}{n}}+(1-\lambda)|L|^{\frac{1}{n}}$$

The Brunn-Minkowski inequality

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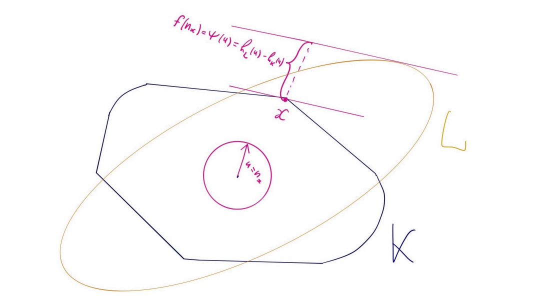
The isoperimetric inequality

For all Borel-measurable sets K with $|K|=|B_2^n|$, one has $|\partial K|\geq|\partial B_2^n|$.

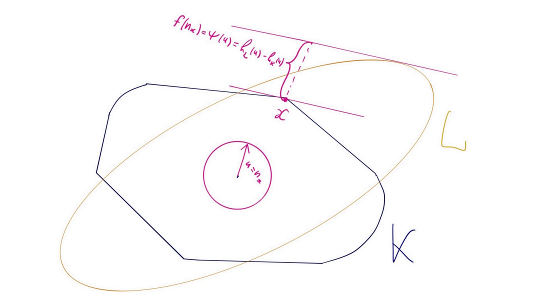
Proof

$$|\partial K|=\liminf_{\epsilon\to 0}\frac{|K+\epsilon B_2^n|-|K|}{\epsilon}\geq\liminf_{\epsilon\to 0}\frac{\left(|K|^{\frac{1}{n}}+\epsilon|B_2^n|^{\frac{1}{n}}\right)^n-|K|}{\epsilon}=n|K|^{\frac{n-1}{n}}|B_2^n|^{\frac{1}{n}}.$$

The local version of the Brunn-Minkowski inequality



- Fix convex sets K and L with support functions h_K and h_L;
- Let $\psi : \mathbb{S}^{n-1} \to \mathbb{R}$ be given by $\psi(u) = h_L(u) - h_K(u)$;
- For $t \in [0,1]$, the body $K_t = (1-t)K + tL$ has support function $h_t = h_K + t\psi$ on $\mathbb{S}^{n-1}$;



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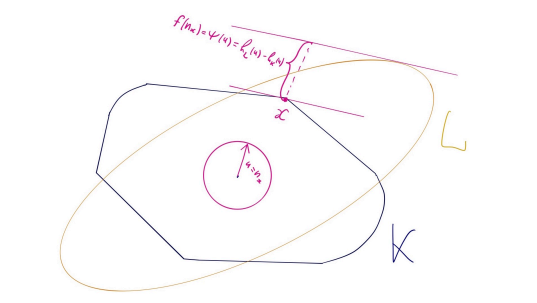
- The Brunn-Minkowski inequality
$$|\lambda K + (1-\lambda)L| \geq |K|^{\lambda}|L|^{1-\lambda}$$
implies that $\log|K_t|$ is concave;
- Let $F(t) = |K_t|$. We deduce $(\log F)''_{t=0} \leq 0$, or
$$F''(0)F(0) - F'(0)^2 \leq 0.$$



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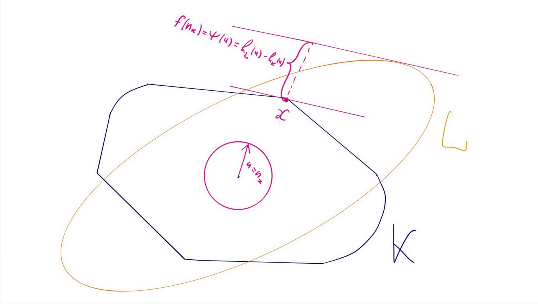

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- Let $f : \partial K \to \mathbb{R}$ be given by $f(x) = \psi(n_x) = h_L(n_x) - h_K(n_x)$;
- $F(0) = |K|$;
- $F'(0) = \int_{\partial K} f$;
- $F''(0) = \int_{\partial K} H_x f^2 - \langle \mathrm{II}^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle$;

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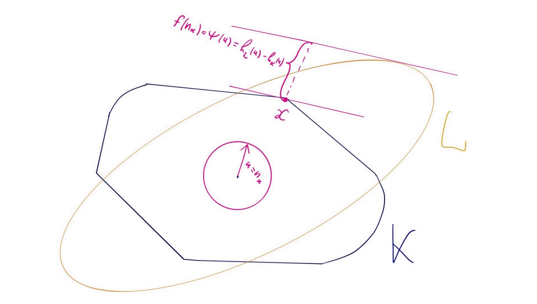

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- $F''(0) = \int_{\partial K} H_x f^2 - \langle \mathrm{II}^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle$;
- Brunn-Minkowski inequality implies, and follows from

$$\int_{\partial K} H_x f^2 - \langle \mathrm{II}^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle - \frac{\left(\int_{\partial K} f\right)^2}{|K|} \leq 0.$$

(Colesanti 2008; Kolesnikov-Milman 2015-2018)

Abstract observation

- Take any algebra A which is a vector space over $\mathbb{R}$;
- Let $Q : A \times A \to \mathbb{R}$ be any symmetric bilinear form;
- Suppose for every $a \in A$,

$$Q(a, a) \leq 0. \tag{1}$$

Abstract observation

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- Fix any element $z \in A$;
- For all $t \in \mathbb{R}$ we have $Q(a+tz, a+tz) \leq 0$, or equivalently

$$Q(a,a) + 2tQ(a,z) + t^2 Q(z,z) \leq 0;$$

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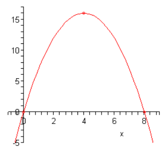

- Optimize in t, plug optimal $t = -\frac{Q(a,z)}{Q(z,z)}$, get the Schwartz inequality

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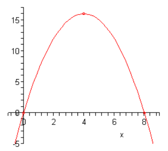

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- (2) is sharper than (1);
- (2) is invariant under $a \to a + tz$.

- The local version of the (multiplicative) Brunn-Minkowski inequality:

$$\int_{\partial K} H_x f^2 - \langle \mathrm{II}^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle - \frac{\left(\int_{\partial K} f\right)^2}{|K|} \leq 0.$$

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- Pick the special function $z(x) = \langle x, n_x \rangle (= h_K(n_x))$;

Optimizing the local version of the Brunn-Minkowski inequality

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- Pick the special function $z(x) = \langle x, n_x \rangle (= h_K(n_x))$;
- Optimize with respect to $f(x) + tz(x)$, using Schwartz inequality get a strengthening

$$\int_{\partial K} H_x f^2 - \langle \mathrm{II}^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle - \frac{n-1}{n} \frac{\left(\int_{\partial K} f\right)^2}{|K|} \leq 0.$$

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- When $K = B_2^n$, we get the sharp Poincare inequality on $\mathbb{S}^{n-1}$:

$$\int_{\mathbb{S}^{n-1}} f^2 - \left(\int_{\mathbb{S}^{n-1}} f\right)^2 \leq \frac{1}{n-1} \int_{\mathbb{S}^{n-1}} |\nabla_\sigma f|^2,$$

where $\int_{\mathbb{S}^{n-1}}$ is normalized.
- The first eigenvalue of Δ on $\mathbb{S}^{n-1}$ is $\frac{1}{n-1}$, and the above is sharp.

- The Local Brunn-Minkowski inequality

$$\int_{\partial K} H_x f^2 - \langle \mathrm{II}^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle - \frac{n-1}{n} \frac{\left(\int_{\partial K} f\right)^2}{|K|} \leq 0$$

  is invariant under $f \to f + t\langle x, n_x \rangle$ ("times change");
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- WLOG suppose that $f(x) = h_M(n_x)$ for some convex body M (or else add a large multiple of $h_K(n_x)$). Get Minkowski's quadratic inequality:

$$V_2(K, M) \leq \frac{V_1(K, M)^2}{|K|}.$$

Invariances of the local version of the Brunn-Minkowski inequality

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- Upshot: the Minkowski quadratic inequality is equivalent to the Brunn-Minkowski inequality.

- Goal: $\int_{\partial K} H_x f^2 - \langle \mathrm{II}^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle - \frac{n-1}{n} \frac{\left(\int_{\partial K} f\right)^2}{|K|} \leq 0.$
- Let $u : K \to \mathbb{R}$ be any function such that $\langle \nabla u, n_x \rangle = f(x)$ for $x \in \partial K$.
- By divergence theorem, $\int_{\partial K} f = \int_K \Delta u$.

Lemma (Kolesnikov, Milman 2015)

$\int_{\partial K} H_x f^2 - \langle \mathrm{II}^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle \leq \int_K (\Delta u)^2 - \|\nabla^2 u\|^2.$

- Goal follows from finding for every $f : \partial K \to \mathbb{R}$ such $u : K \to \mathbb{R}$ with $\langle \nabla u, n_x \rangle = f(x)$ and

$$\mathbb{E}\|\nabla^2 u\|^2 \geq Var(\Delta u) + \frac{1}{n}(\mathbb{E}\Delta u)^2.$$

Solvability of the Neumann system

Let $\Delta u = const$, with the Neumann boundary condition $\langle \nabla u, n_x \rangle = f(x)$.

- For any symmetric matrix A, $\|A\|_{HS}^2 \geq \frac{tr(A)^2}{n}$; thus $\|\nabla^2 u\|^2 \geq \frac{1}{n}(\Delta u)^2$.

Logarithmic sum (Definition)

$$\lambda K +_0 (1-\lambda)L = \bigcap_{u\in\mathbb{S}^{n-1}} \{x\in\mathbb{R}^n : |\langle u,x\rangle| \le h_K(u)^\lambda h_L(u)^{1-\lambda}\}.$$

Note, by AMGM, $\lambda K +_0 (1-\lambda)L \subset \lambda K + (1-\lambda)L$.

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Log-Brunn-Minkowski conjecture (Böröczky, Lutwak, Yang, Zhang 2011)

For **origin-symmetric convex** sets K and L in $\mathbb{R}^n$,

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- Equivalent to uniqueness of solution of certain Monge-Ampere equations, questions go back to Firey;
- True for $n=2$ (Böröczky, Lutwak, Yang, Zhang 2011), (Stancu for polytopes);
- True for unconditional sets (Saraglou 2013, Cordero-Fradelizi-Maurey);
- True for complex convex bodies (Rotem 2017).

- Fix convex sets K and L with support functions h_K and h_L;
- Let $\psi : \mathbb{S}^{n-1} \to \mathbb{R}$ be given by $\psi(u) = \frac{h_L(u)}{h_K(u)}$;
- Locally, $K_t := tK +_0 (1-t)L$ has support function

$$h_t = h_K \psi^t = h_K + t\varphi + O(t^2),$$

where $\varphi = h_K \log \frac{h_L}{h_K}$;

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- Let $f : \partial K \to \mathbb{R}$ be given by $f(x) = \varphi(n_x) = h_K(n_x) \log \frac{h_L(n_x)}{h_K(n_x)}$;
- $F(0) = |K|$;
- $F'(0) = \int_{\partial K} f$;
- $F''(0) = \int_{\partial K} H_x f^2 - \langle \mathrm{II}^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle + \int_{\partial K} \frac{f^2}{\langle x, n_x \rangle}$.

Theorem (Colesanti, L, Marsiglietti 2016)

The Log-Brunn-Minkowski inequality would imply, for every symmetric convex K and every even function $f : \partial K \to \mathbb{R}$,

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Colesanti-L-Marsiglietti

The local version of the Log-Brunn-Minkowski inequality is true when $K = B_2^n$.

- Indeed, the Local Log-Brunn-Minkowski inequality with $K = B_2^n$ is equivalent to the following Poincare inequality:

$$Var_{\mathbb{S}^{n-1}}(f) \leq \frac{1}{n} \mathbb{E}_{\mathbb{S}^{n-1}} |\nabla_\sigma f|^2,$$

for all even functions f, which is known to be true, moreover, with constant $\frac{1}{2n} < \frac{1}{n}$.

Kolesnikov-Milman

The local version of the Log-Brunn-Minkowski inequality is true when $K = B_p^n$, for all $p \in [2, \infty]$.

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The local version of the Log-Brunn-Minkowski inequality implies the global version of the Log-Brunn-Minkowski inequality!

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- However, when K is fixed, no global result follows. The global conjecture

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with arbitrary symmetric L is not known for any K.

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  with arbitrary symmetric L is not known for any K.
- Could one prove the Local Log BM for some nice "speed function" f, for all K? (one such answer will come after two slides...)

- (Kolesnikov-Milman) The Local Log BM inequality

$$\int_{\partial K} H_x f^2 - \langle \mathrm{II}^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle + \int_{\partial K} \frac{f^2}{\langle x, n_x \rangle} \leq \frac{\left(\int_{\partial K} f\right)^2}{|K|}$$

is invariant under $f \to f + t\langle x, n_x \rangle$.

- (Putterman) Therefore, it is equivalent to the strengthening of Minkowski's quadratic inequality

$$n(n-1)V_2(K,M) + \int_{\partial K} \frac{h_M^2}{\langle x, n_x \rangle} dH_{n-1}(x) \leq \frac{n^2 V_1(K,M)^2}{|K|}.$$

- Furthermore, the Local (and global) Log BM is invariant under linear transformations.
- In the case of Log-Brunn-Minkowski conjecture, the invariance under $f \to f + t\langle x, n_x \rangle$ corresponds to the invariance of the global version under $L \to tL$, while the invariance under $f \to sf$ corresponds to "time change".

Example: $K = B_\infty^n$

- The inequality

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becomes (using symmetry!)

$$n(n-1)V_2(B_\infty^n, M) + 2\cdot 2^{n-1}\sum_{i=1}^n h_M^2(e_i) \leq 2^{-4}\cdot 4\cdot 2^{2n-2}\left(\sum_{i=1}^n h_M(e_i)\right)^2.$$

The local version of the Log-Brunn-Minkowski inequality for $K = B_\infty^n$

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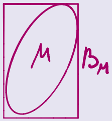
- Mixed volumes are monotone, thus $V_2(B_\infty^n, M) \leq V_2(B_\infty^n, B_M)$, where B_M is the parallelepiped with sides $2h_M(e_1), ..., 2h_M(e_n)$.
- $n(n-1)V_2(B_\infty^n, B_M) = 4 \cdot 2^{n-2} \sum_{i \neq j} h_M(e_i) h_M(e_j)$.



The local version of the Log-Brunn-Minkowski inequality for $K = B^n_\infty$

Example: $K = B^n_\infty$

- The inequality

$$n(n-1)V_2(K,M) + \int_{\partial K} \frac{h_M^2}{\langle x, n_x\rangle} dH_{n-1}(x) \leq \frac{n^2 V_1(K,M)^2}{|K|}$$

  becomes (using symmetry!)

$$n(n-1)V_2(B^n_\infty, M) + 2\cdot 2^{n-1}\sum_{i=1}^n h_M^2(e_i) \leq 2^{-4}\cdot 4\cdot 2^{2n-2}\left(\sum_{i=1}^n h_M(e_i)\right)^2.$$

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- Thus the inequality boils down to an equality

$$\left(\sum_{i=1}^n h_M(e_i)\right)^2 = \left(\sum_{i=1}^n h_M(e_i)\right)^2.$$

The local Log-Brunn-Minkowski inequality for interval M

Theorem (Kolesnikov, L, 2020+)

The local version of the Log-Brunn-Minkowski conjecture

$$n(n-1)V_2(K,M)+\int_{\partial K}\frac{h_M^2}{\langle x,n_x\rangle}dH_{n-1}(x)\leq\frac{n^2V_1(K,M)^2}{|K|}$$

is true when $M=[-v,v]$, for any vector $v\in\mathbb{R}^n$.

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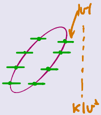
Proof

- The function $|K+t[-v,v]|=|K|+2t|v|\cdot|K|v^{\perp}|$ is linear in t
- Thus $n(n-1)V_2(K,M)=|K+t[-v,v]|''_{t=0}=0$



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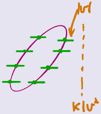

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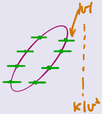

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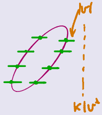
- Thus the statement rewrites

$$\int_{\mathbb{S}^{n-1}}\frac{|\langle u,v\rangle|^2}{h_K(u)}dS_K(u)\le\frac{4|K|v^{\perp}|^2}{|K|}.$$



Proof (continued)

- Goal: $\int_{\mathbb{S}^{n-1}} \left(\frac{|\langle u,v \rangle|}{h_K(u)} \right) |\langle u,v \rangle| dS_K(u) \leq \frac{4|K|v^{\perp}|^2}{|K|}$

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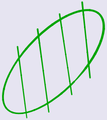
- By Fubbini's theorem, for every $u \in \mathbb{S}^{n-1}$, $|K| = \int_{-h_K(u)}^{h_K(u)} |K \cap (u^{\perp} + tu)| dt$, and thus

$$\frac{1}{h_K(u)} \leq \frac{2}{|K|} |K \cap u^{\perp}|.$$



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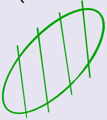

- Since the projection of a subset is smaller than the projection of a set,

$$\frac{|\langle u,v\rangle|}{h_K(u)} \leq \frac{2}{|K|} |K \cap u^{\perp}| \cdot |\langle u,v\rangle| = \frac{2}{|K|} |K \cap u^{\perp}|v^{\perp}| \leq \frac{2}{|K|} |K|v^{\perp}|.$$

The local Log-Brunn-Minkowski inequality for interval M

Proof (continued)

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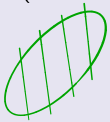

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- We conclude

$$\int_{\mathbb{S}^{n-1}} \left(\frac{|\langle u,v\rangle|}{h_K(u)}\right)|\langle u,v\rangle| dS_K(u) \leq \frac{2|K|v^\perp|}{|K|}\int_{\mathbb{S}^{n-1}} |\langle u,v\rangle| dS_K = \frac{4|K|v^\perp|^2}{|K|}.$$

(the last passage is the Cauchy's projection formula again.) □

Question: what if M is a 2-dimensional square?

When $M=[-e_1,e_1]\times[-e_2,e_2]$, the Local Log BM inequality

$$n(n-1)V_2(K,M)+\int_{\partial K}\frac{h_M^2}{\langle x,n_x\rangle}dH_{n-1}(x)\leq\frac{n^2V_1(K,M)^2}{|K|}$$

becomes

Conjecture: Local Log BM when M is a square

$$8|K|span(e_1,e_2)^{\perp}|+\int_{\mathbb{S}^{n-1}}\frac{(|u_1|+|u_2|)^2}{h_K(u)}dS_K(u)\leq\frac{4\left(|K|e_1^{\perp}|+|K|e_2^{\perp}|\right)^2}{|K|}.$$

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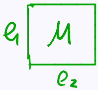

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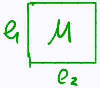

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Observation

If the above inequality is true for all K, then the Local Log-Brunn-Minkowski inequality holds whenever

- K is any symmetric convex body and M is a zonoid (limit of a sum of intervals), or
- K is a zonoid and M is any symmetric convex body.

The L_p-Brunn-Minkowski conjecture

L_p-Minkowski sum (Definition)

$$\lambda K +_p (1-\lambda)L = \bigcap_{u \in \mathbb{S}^{n-1}} \{x \in \mathbb{R}^n : |\langle u, x\rangle|^p \le \lambda h_K(u)^p + (1-\lambda) h_L(u)^p\}.$$

L_p-Brunn-Minkowski conjecture (Böröczky, Lutwak, Yang, Zhang 2011)

For **origin-symmetric convex** sets K and L in $\mathbb{R}^n$, for $p \in [0,1]$

$$|\lambda K +_p (1-\lambda)L| \ge |K|^\lambda |L|^{1-\lambda}.$$

Equivalently, by homogeneity (and/or the earlier story)

$$|\lambda K +_p (1-\lambda)L|^{\frac{p}{n}} \ge \lambda |K|^{\frac{p}{n}} + (1-\lambda)|L|^{\frac{p}{n}}.$$

- For $p \in [0,1]$,

$$\lambda K +_0 (1-\lambda)L \subset \lambda K +_p (1-\lambda)L \subset \lambda K + (1-\lambda)L.$$

- The conjecture interpolates between the Log-Brunn-Minkowski conjecture ($p = 0$) and the Brunn-Minkowski inequality ($p = 1$).

- Kolesnikov-Milman developed the local version of the L_p-Brunn-Minkowski inequality

$$n(n-1)V_2(K,M)+(1-p)\int_{\partial K}\frac{h_M^2}{\langle x,n_x\rangle}dH_{n-1}(x)\le\frac{n-p}{n}\frac{n^2V_1(K,M)^2}{|K|}$$

- Kolesnikov-Milman: true for $p\in[1-cn^{-1.5},1]$!
- Chen-Huang-Li-Liu: local implies global (with equality cases)
- Putterman: local implies global (simple and useful proof)
- Conclusion: the L_p-Brunn-Minkowski conjecture

$$|\lambda K+_p(1-\lambda)L|^{\frac{p}{n}}\ge\lambda|K|^{\frac{p}{n}}+(1-\lambda)|L|^{\frac{p}{n}}.$$

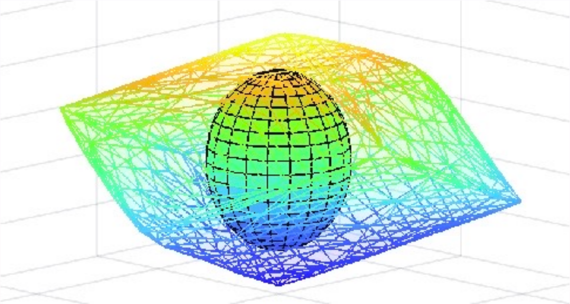
is true when $p\in[1-cn^{-1.5},1]$!

The L_p-Brunn-Minkowski conjecture

Theorem (Hosle, Kolesnikov, L 2020+)

For origin-symmetric convex sets K and L in $\mathbb{R}^n$ such that $K \subset L$, for $p \in [1 - cn^{-0.75}, 1]$

$$|\lambda K +_p (1-\lambda)L| \geq |K|^{\lambda}|L|^{1-\lambda}.$$



Remark: note that this is not the dilation-invariant version.

The L_p-Brunn-Minkowski conjecture for measures

Theorem (Saraglou 2014)

If the L_p-Brunn-Minkowski conjecture holds for some $p \in [0,1]$, then for any even log-concave measure μ and any pair of origin-symmetric convex sets K and L in $\mathbb{R}^n$,

$$\mu(\lambda K +_p (1-\lambda)L) \geq \mu(K)^{\lambda} \mu(L)^{1-\lambda}.$$

- Considering the case $p = 0$ and $K = aL$, note that the above would imply the B-conjecture of Banazchyk-Latala, posed in the 1990s.
- Cordero-Fradelizi-Maurey 2008: true when $K = aL$ and μ is Gaussian.

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L, Marsiglietti, Nayar, Zvavitch 2017

If the Log-Brunn-Minkowski conjecture holds, then for every even log-concave measure μ and any pair of origin-symmetric convex sets K and L in $\mathbb{R}^n$,

$$\mu(\lambda K + (1-\lambda)L)^{\frac{1}{n}} \geq \lambda\mu(K)^{\frac{1}{n}} + (1-\lambda)\mu(L)^{\frac{1}{n}}.$$

The Gardner-Zvavitch conjecture

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Moreover, (3) strengthens when p decreases.

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Conjecture (Gardner, Zvavitch 2007)

For the Gaussian measure μ, and symmetric convex sets K and L,

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Theorem (Kolesnikov, L 2018)

For the Gaussian measure μ, and convex sets K and L containing the origin,

$$\mu(\lambda K + (1-\lambda)L)^{\frac{1}{2n}} \geq \lambda\mu(K)^{\frac{1}{2n}} + (1-\lambda)\mu(L)^{\frac{1}{2n}}.$$

Theorem (Hosle, Kolesnikov, L 2020+)

Let γ be the Gaussian measure, and let K and L be symmetric convex sets containing the ball rB_2^n. Then for any $\lambda > 0$,

1. $\gamma(\lambda K +_p (1-\lambda)L) \geq \gamma(K)^{\lambda}\gamma(L)^{1-\lambda}$, whenever $p \geq 0$ and
$$p \geq 1 - \frac{2r^2}{n+1}.$$
2. In particular, the Gaussian Log-Brunn-Minkowski inequality holds for all convex sets K and L containing $\sqrt{0.5(n+1)}B_2^n$.
3. More generally, $\gamma(\lambda K +_p (1-\lambda)L)^{\frac{q}{n}} \geq \lambda\gamma(K)^{\frac{q}{n}} + (1-\lambda)\gamma(L)^{\frac{q}{n}}$, provided that
$$4q + \frac{n+1}{r^2}(1-p) \leq 2.$$
4. Assuming further that $K \subset L$, we show that $\gamma(\lambda K +_p (1-\lambda)L) \geq \gamma(K)^{\lambda}\gamma(L)^{1-\lambda}$, whenever $p \geq 0$ and $p \geq 1 - \frac{r}{\sqrt{n}+0.25}$.

- In one of the steps of the proof, we deduced the "local to global" result for general measures, following the approach of Putterman.

Thanks for your attention!