On the Computational Complexity of Linear Discrepancy

Lily Li and Aleksandar Nikolov
University of Toronto
September 2020
An array $A$ of $n$ real numbers.
[IN] An array $A$ of $n$ real numbers.

[OUT] Let $S$ be the array of subset sums of $A$ arranged in increasing order.
An array $A$ of $n$ real numbers.

Let $S$ be the array of subset sums of $A$ arranged in increasing order. Find the largest gap $[S_i, S_{i+1}]$ between consecutives terms in $S$. 

\[ A \begin{array}{c}
5 \\
2 \\
1 \\
7 \\
\end{array} \]

\[ S \begin{array}{cccccccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
\end{array} \]
Point Distributions
Point Distributions

\[ \textbf{IN} \] All axis-aligned boxes in \([0,1]^d\), denoted \(\mathcal{R}_d\), and Lebesgue measure \(\lambda_d\) on \(\mathbb{R}^d\).
Point Distributions

[IN] All axis-aligned boxes in $[0,1]^d$, denoted $\mathcal{R}_d$, and Lebesgue measure $\lambda_d$ on $\mathbb{R}^d$.

[OUT] Point set $P \in [0,1]^d$ which minimizes

$$\sup_{R \in \mathcal{R}_d} \left| |R \cap P| - n\lambda_d(R) \right|.$$
Point Distributions

Discretize $[0,1]^d$ to finite set $X$ of size $N$. 

$X$
Point Distributions

Discretize $[0,1]^d$ to finite set $X$ of size $N$.

$A :$ incidence matrix, $R \in \mathcal{R}_d$
Point Distributions

Discretize $[0,1]^d$ to finite set $X$ of size $N$.

$A :$ incidence matrix, $\mathbf{R} \in \mathcal{R}_d$

$$\mathbf{w} = \frac{n}{N} \mathbb{1},$$
Point Distributions

Discretize $[0,1]^d$ to finite set $X$ of size $N$.

$A : \text{incidence matrix}, \ R \in \mathcal{R}_d$

$$w = \frac{n}{N} \mathbb{1},$$

$$\min_{x \in \{0,1\}^N} \|A w - A x\|_\infty$$
Point Distributions

Discretize $[0,1]^d$ to finite set $X$ of size $N$.

$A : \text{incidence matrix}, \ R \in \mathcal{R}_d$

$$w = \frac{n}{N} I,$$

$$\min_{x \in \{0,1\}^N} \|Aw - Ax\|_\infty$$

$$\text{disc}(P) \leq 2\|Aw - Ax\|_\infty$$
ILP

\[ \begin{align*}
\text{max } & \ c^\top w \\
\text{s.t. } & \ Mw \leq b \\
& \ w \in \{0,1\}^n
\end{align*} \]
ILP (Relax-\textbf{-}\textbf{-} Solve\textbf{-}\textbf{-} Round\textbf{-}\textbf{-})

max $c^\top w$

$Mw \leq b$

$w \in \{0,1\}^n$

$w \in [0,1]^n$
ILP (Relax-Solve-Round)

\[
\begin{align*}
\text{max } c^\top w \\
Mw &\leq b \\
w &\in \{0,1\}^n \\
w &\in [0,1]^n
\end{align*}
\]
ILP (Relax-Solve-Round)

\[
\begin{align*}
\max & \quad c^\top w' \\
M w' & \leq b \\
w' & \in \{0,1\}^n \\
w & \in [0,1]^n
\end{align*}
\]
ILP (Relax-Solve-Round)

\[
\begin{align*}
\text{max } & \quad c^\top w \\
\text{s.t. } & \quad Mw \leq b \\
& \quad w' \in \{0,1\}^n \\
& \quad w \in [0,1]^n
\end{align*}
\]

Round \( w \) (approx. satisfy constraints)

\[
\begin{align*}
\min_{x \in \{0,1\}^n} \max_{i \in [m]} |(Mx - Mw)_i| &= \min_{x \in \{0,1\}^n} \|M(x - w)\|_\infty \\
\text{rounded int solution} - \text{LP solution}
\end{align*}
\]
ILP (Relax-Solve-Round)

\[
\begin{align*}
&\max c^\top w \\
&Mw \leq b \\
&w' \in \{0,1\}^n \\
&w \in [0,1]^n
\end{align*}
\]

Round \(w\) (approx. satisfy constraints)

\[
\begin{align*}
&\min \max_{x \in \{0,1\}^n, i \in [m]} |(Mx - Mw)_i| = \min_{x \in \{0,1\}^n} \|M(x - w)\|_\infty
\end{align*}
\]

Linear Discrepancy of matrix \(M \in \mathbb{R}^{m \times n}\):

\[
lindisc(M) = \max_{w \in [0,1]^n} \min_{x \in \{0,1\}^n} \|M(x - w)\|_\infty
\]
ILP (Relax-Solve-Round)

\[
\begin{align*}
\text{max } & \quad c^\top w \\
\text{subject to } & \quad Mw \leq b \\
& \quad w' \in \{0,1\}^n \\
& \quad w \in [0,1]^n \\
\end{align*}
\]

Round \( w \) (approx. satisfy constraints)

\[
\min_{x\in\{0,1\}^n} \max_{i\in[m]} |(Mx - Mw)_i| = \min_{x\in\{0,1\}^n} \|M(x - w)\|_\infty
\]

Linear Discrepancy of matrix \( M \in \mathbb{R}^{m \times n} \):

\[
\text{lindisc}(M) = \max_{w\in[0,1]^n} \min_{x\in\{0,1\}^n} \|M(x - w)\|_\infty
\]
ILP (Relax-Solve-Round)

\[
\begin{align*}
&\max c^\top w \\
&Mw \leq b \\
&w' \in \{0,1\}^n \\
&w \in [0,1]^n
\end{align*}
\]

Round \( w \) (approx. satisfy constraints)

\[
\min_{x \in \{0,1\}^n} \max_{i \in [m]} |(Mx - Mw)_i| = \min_{x \in \{0,1\}^n} \|M(x - w)\|_\infty
\]

Linear Discrepancy of matrix \( M \in \mathbb{R}^{m \times n} \):

\[
2\text{lindisc}(M) = \max_{w \in [0,1]^n} \min_{x \in \{0,1\}^n} \|M(x - w)\|_\infty
\]
Summary of results:

$M \in \mathbb{Z}^{d \times n}$ with entries of magnitude bounded by $\delta$;
exact linear discrepancy in time $O(dn) + O(d^2)$.

When $d = 1$, exact linear discrepancy can be computed in $O(n \log n)$.

Linear discrepancy is NP-Hard

$M \in \mathbb{Z}^{m \times n}; 2n + 1$ approximation in polynomial time $\frac{6}{7}$.
Summary of results:

- $M \in \mathbb{Z}^{d \times n}$ with entries of magnitude bounded by $\delta$; exact linear discrepancy in time $O\left( d(n\delta)^{d^2 + d} \right)$.

- When $d = 1$, exact linear discrepancy can be computed in $O(n \log n)$.

- Linear discrepancy is **NP-Hard**.

- $M \in \mathbb{Z}^{m \times n}$; $2^{n+1}$ approximation in polynomial time.
Summary of results:

• $M \in \mathbb{Z}^{d \times n}$ with entries of magnitude bounded by $\delta$; exact linear discrepancy in time $O\left(d(n\delta)^{d^2+d}\right)$

• When $d = 1$, exact linear discrepancy can be computed in $O(n \log n)$.

• Linear discrepancy is NP-Hard

• $M \in \mathbb{Z}^{m \times n}$; $2^{n+1}$ approximation in polynomial time
Claim: Given an array $A$ of $n$ positive real numbers sorted in decreasing order, compute $\text{lindisc}(A)$ in time $O(n)$. 
**Claim:** Given an array $A$ of $n$ positive real numbers sorted in decreasing order, compute $\text{lindisc}(A)$ in time $O(n)$.

Linear Discrepancy of matrix $M \in \mathbb{R}^{1 \times n}$:

$$\text{lindisc}(M) = \max_{w \in [0,1]^n} \min_{x \in \{0,1\}^n} \|M(x - w)\|_\infty$$
Claim: Given an array $A$ of $n$ positive real numbers sorted in decreasing order, compute $\text{lindisc}(A)$ in time $O(n)$.

\[
\text{lindisc}(w, M) = \min_{x \in \{0,1\}^n} \|M(x - w)\|_\infty \geq p \quad \text{Subset Sum}
\]

Linear Discrepancy of matrix $M \in \mathbb{R}^{1 \times n}$:

\[
\text{lindisc}(M) = \max_{w \in [0,1]^n} \min_{x \in \{0,1\}^n} \|M(x - w)\|_\infty
\]
Claim: Given an array $A$ of $n$ positive real numbers sorted in decreasing order, compute $\text{lindisc}(A)$ in time $O(n)$. 
Claim: Given an array $A$ of $n$ positive real numbers sorted in decreasing order, compute $\text{lindisc}(A)$ in time $O(n)$. 
Claim: Given an array $A$ of $n$ positive real numbers sorted in decreasing order, compute $\text{lindisc}(A)$ in time $O(n)$. 
Claim: Given an array \( A \) of \( n \) positive real numbers sorted in decreasing order, compute \( lindisc(A) \) in time \( O(n) \).

Lemma: Let \( A = [a_1, \ldots, a_n] \) with \( a_i \geq a_{i+1} \). For any \( k \leq n \), let \( A_k = [a_1, \ldots, a_k] \). Then
\[
2lindisc(A_k) = \max(a_k, 2lindisc(A_{k-1}) - a_k).
\]
Lemma: Let $A = [a_1, ..., a_n]$ with $a_i \geq a_{i+1}$. For any $k \leq n$, let $A_k = [a_1, ..., a_k]$. Then

$$2\text{lindisc}(A_k) = \max(a_k, 2\text{lindisc}(A_{k-1}) - a_k).$$
Lemma: Let $A = [a_1, \ldots, a_n]$ with $a_i \geq a_{i+1}$. For any $k \leq n$, let $A_k = [a_1, \ldots, a_k]$. Then

$$2\text{lindisc}(A_k) = \max(a_k, 2\text{lindisc}(A_{k-1}) - a_k).$$

$k = 0 \quad 2\text{lindisc}(A_0) = 0$

\[
\begin{array}{c}
A \\
\hline
7 & 5 & 2 & 1
\end{array}
\]

\[
\begin{array}{ccccccccccccccc}
S & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15
\end{array}
\]
Lemma: Let $A = [a_1, \ldots, a_n]$ with $a_i \geq a_{i+1}$. For any $k \leq n$, let $A_k = [a_1, \ldots, a_k]$. Then

$$2\text{lindisc}(A_k) = \max(a_k, 2\text{lindisc}(A_{k-1}) - a_k).$$

$k = 0 \quad 2\text{lindisc}(A_0) = 0$

$k = 1 \quad 2\text{lindisc}(A_1) = 7$

\[
\begin{array}{cccccc}
A & 7 & 5 & 2 & 1 \\
S & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15
\end{array}
\]
Lemma: Let $A = [a_1, ..., a_n]$ with $a_i \geq a_{i+1}$. For any $k \leq n$, let $A_k = [a_1, ..., a_k]$. Then

$$2\text{lindisc}(A_k) = \max(a_k, 2\text{lindisc}(A_{k-1}) - a_k).$$

$k = 1 \quad 2\text{lindisc}(A_1) = 7$

$k = 2 \quad 2\text{lindisc}(A_2) = 5$
**Lemma:** Let $A = [a_1, \ldots, a_n]$ with $a_i \geq a_{i+1}$. For any $k \leq n$, let $A_k = [a_1, \ldots, a_k]$. Then

$$2\text{lindisc}(A_k) = \max(a_k, 2\text{lindisc}(A_{k-1}) - a_k).$$

- $k = 2$ \quad $2\text{lindisc}(A_2) = 5$
- $k = 3$ \quad $2\text{lindisc}(A_3) = 3$
Lemma: Let $A = [a_1, ... , a_n]$ with $a_i \geq a_{i+1}$. For any $k \leq n$, let $A_k = [a_1, ... , a_k]$. Then

$$2\text{lindisc}(A_k) = \max(a_k, 2\text{lindisc}(A_{k-1}) - a_k).$$

$k = 3$ \quad $2\text{lindisc}(A_3) = 3$

$k = 4$ \quad $2\text{lindisc}(A_4) = 2$

\[A\]

\[
\begin{array}{ccccc}
  & 7 & 5 & 2 & 1 \\
\end{array}
\]

\[S\]

\[
\begin{array}{cccccccccccccccc}
  0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
\end{array}
\]
Lemma: Let $A = [a_1, \ldots, a_n]$ with $a_i \geq a_{i+1}$. For any $k \leq n$, let $A_k = [a_1, \ldots, a_k]$. Then

$$2\text{lindisc}(A_k) = \max(a_k, 2\text{lindisc}(A_{k-1}) - a_k).$$

Proof. Show:

1. $2\text{lindisc}(A_k) \leq \max(a_k, 2\text{lindisc}(A_{k-1}) - a_k)$
2. $2\text{lindisc}(A_k) \geq \max(a_k, 2\text{lindisc}(A_{k-1}) - a_k)$
Lemma: Let $A = [a_1, ..., a_n]$ with $a_i \geq a_{i+1}$. For any $k \leq n$, let $A_k = [a_1, ..., a_k]$. Then

$$2\text{lindisc}(A_k) = \max(a_k, 2\text{lindisc}(A_{k-1}) - a_k).$$

Proof. Show:

1. $2\text{lindisc}(A_k) \leq \max(a_k, 2\text{lindisc}(A_{k-1}) - a_k)$
Lemma: Let $A = [a_1, ..., a_n]$ with $a_i \geq a_{i+1}$. For any $k \leq n$, let $A_k = [a_1, ..., a_k]$. Then
\[
2lindisc(A_k) = \max(a_k, 2lindisc(A_{k-1}) - a_k).
\]

Proof. Show:

1. $2lindisc(A_k) \leq \max(a_k, 2lindisc(A_{k-1}) - a_k)$
Lemma: Let $A = [a_1, ..., a_n]$ with $a_i \geq a_{i+1}$. For any $k \leq n$, let $A_k = [a_1, ..., a_k]$. Then

$$2\text{lindisc}(A_k) = \max(a_k, 2\text{lindisc}(A_{k-1}) - a_k).$$

Proof. Show:

1. $2\text{lindisc}(A_k) \leq \max(a_k, 2\text{lindisc}(A_{k-1}) - a_k)$
**Lemma:** Let $A = [a_1, ..., a_n]$ with $a_i \geq a_{i+1}$. For any $k \leq n$, let $A_k = [a_1, ..., a_k]$. Then

$$2\text{lindisc}(A_k) = \max(a_k, 2\text{lindisc}(A_{k-1}) - a_k).$$

**Proof.** Show:

1. $2\text{lindisc}(A_k) \leq \max(a_k, 2\text{lindisc}(A_{k-1}) - a_k)$
Lemma: Let $A = [a_1, ..., a_n]$ with $a_i \geq a_{i+1}$. For any $k \leq n$, let $A_k = [a_1, ..., a_k]$. Then

$$2lindisc(A_k) = \max(a_k, 2lindisc(A_{k-1}) - a_k).$$

Proof. Show:

1. $2lindisc(A_k) \leq \max(a_k, 2lindisc(A_{k-1}) - a_k)$
Lemma: Let \( A = [a_1, \ldots, a_n] \) with \( a_i \geq a_{i+1} \). For any \( k \leq n \), let \( A_k = [a_1, \ldots, a_k] \). Then

\[ 2 \text{lindisc}(A_k) = \max(a_k, 2 \text{lindisc}(A_{k-1}) - a_k). \]

Proof. Show:

1. \( 2 \text{lindisc}(A_k) \leq \max(a_k, 2 \text{lindisc}(A_{k-1}) - a_k) \)
2. \( 2 \text{lindisc}(A_k) \geq \max(a_k, 2 \text{lindisc}(A_{k-1}) - a_k) \)

Two cases: (i) \( a_k \geq 2 \text{lindisc}(A_{k-1}) - a_k \), or (ii) \( a_k < 2 \text{lindisc}(A_{k-1}) - a_k \).
Lemma: Let $A = [a_1, ..., a_n]$ with $a_i \geq a_{i+1}$. For any $k \leq n$, let $A_k = [a_1, ..., a_k]$. Then

$$2\text{lindisc}(A_k) = \max(a_k, 2\text{lindisc}(A_{k-1}) - a_k).$$

Proof. (i) $a_k \geq 2\text{lindisc}(A_{k-1}) - a_k$
Lemma: Let $A = [a_1, ..., a_n]$ with $a_i \geq a_{i+1}$. For any $k \leq n$, let $A_k = [a_1, ..., a_k]$. Then
\[
2\text{lindisc}(A_k) = \max(a_k, 2\text{lindisc}(A_{k-1}) - a_k).
\]

Proof. (i) $a_k \geq 2\text{lindisc}(A_{k-1}) - a_k$
Lemma: Let $A = [a_1, \ldots, a_n]$ with $a_i \geq a_{i+1}$. For any $k \leq n$, let $A_k = [a_1, \ldots, a_k]$. Then
\[
2\text{lindisc}(A_k) = \max(a_k, 2\text{lindisc}(A_{k-1}) - a_k).
\]

Proof. (ii) $a_k < 2\text{lindisc}(A_{k-1}) - a_k$
Lemma: Let \( A = [a_1, ..., a_n] \) with \( a_i \geq a_{i+1} \). For any \( k \leq n \), let \( A_k = [a_1, ..., a_k] \). Then
\[
2\text{lindisc}(A_k) = \max(a_k, 2\text{lindisc}(A_{k-1}) - a_k).
\]

Proof. (ii) \( a_k < 2\text{lindisc}(A_{k-1}) - a_k \)
Lemma: Let \( A = [a_1, \ldots, a_n] \) with \( a_i \geq a_{i+1} \). For any \( k \leq n \), let \( A_k = [a_1, \ldots, a_k] \). Then

\[
2 \text{lindisc}(A_k) = \max(a_k, 2 \text{lindisc}(A_{k-1}) - a_k).
\]

Proof. (ii) \( a_k < 2 \text{lindisc}(A_{k-1}) - a_k \)
Lemma: Let $A = [a_1, ..., a_n]$ with $a_i \geq a_{i+1}$. For any $k \leq n$, let $A_k = [a_1, ..., a_k]$. Then

$$2\text{lindisc}(A_k) = \max(a_k, 2\text{lindisc}(A_{k-1}) - a_k).$$

Proof. (ii) $a_k < 2\text{lindisc}(A_{k-1}) - a_k$
Lemma: Let $A = [a_1, ..., a_n]$ with $a_i \geq a_{i+1}$. For any $k \leq n$, let $A_k = [a_1, ..., a_k]$. Then

$$2\text{lindisc}(A_k) = \max(a_k, 2\text{lindisc}(A_{k-1}) - a_k).$$

Proof. (ii) $a_k < 2\text{lindisc}(A_{k-1}) - a_k$
Lemma: Let $A = [a_1, ..., a_n]$ with $a_i \geq a_{i+1}$. For any $k \leq n$, let $A_k = [a_1, ..., a_k]$. Then
\[ 2\text{lindisc}(A_k) = \max(a_k, 2\text{lindisc}(A_{k-1}) - a_k). \]

Proof. (ii) $a_k < 2\text{lindisc}(A_{k-1}) - a_k$
Lemma: Let $A = [a_1, \ldots, a_n]$ with $a_i \geq a_{i+1}$. For any $k \leq n$, let $A_k = [a_1, \ldots, a_k]$. Then

$$2\text{lindisc}(A_k) = \max(a_k, 2\text{lindisc}(A_{k-1}) - a_k).$$

Proof. (ii) $a_k < 2\text{lindisc}(A_{k-1}) - a_k$
Other results:
Other results:

• Linear discrepancy is **NP-Hard**

**CRP on Lattices:**

**[IN]** A lattice \( L = \mathcal{L}(b_1, ..., b_m) \) of \( n \) dimensional vectors.

**[OUT]** \( \max_{w \in \mathbb{R}^n} \min_{x \in \mathbb{Z}^n} \| L(w - x) \|_\infty \)
Other results:

- Linear discrepancy is **NP-Hard**
- $M \in \mathbb{Z}^{d \times n}$ with entries of magnitude bounded by $\delta$; exact linear discrepancy in time

$$O\left(d(n\delta)^{d^2+d}\right)$$
Other results:

• Linear discrepancy is **NP-Hard**

• $M \in \mathbb{Z}^{d \times n}$ with entries of magnitude bounded by $\delta$; exact linear discrepancy in time

$$O\left(d(n\delta)^{d^2+d}\right)$$

• $M \in \mathbb{Z}^{m \times n}$; $2^{n+1}$ approximation in polynomial time
Future work:
Future work:

• Is linear discrepancy $\textbf{NP}$-Complete? $\Pi_2$-Hard?
Future work:

• Is linear discrepancy \textbf{NP-Complete}? \textbf{\( \Pi_2 \)-Hard}?

• Improve approximation gap
Future work:

• Is linear discrepancy \textbf{NP}-Complete? $\Pi_2$-Hard?

• Improve approximation gap

• Extend exact algorithm; compute linear discrepancy for matrices $M \in \mathbb{R}^{d \times n}$ (small constant $d$) in polynomial time
Future work:

• Is linear discrepancy $\text{NP-Complete? } \Pi_2$-Hard?

• Improve approximation gap

• Extend exact algorithm; compute linear discrepancy for matrices $M \in \mathbb{R}^{d \times n}$ (small constant $d$) in polynomial time