Bounded-Depth Frege Complexity of Tseitin Formulas for All Graphs

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Bounded-depth Frege $\mathcal{F}_d$

Formula depth: 1+max number of alternations of connectives type over all paths

Frege: finite set of sound derivation rules : $\frac{\psi_1 \ldots \psi_k}{\psi}, k \geq 0$

- Proof of ψ: $\psi_1, \ldots \psi_m$ s.t. $\psi_m = \psi$ and $\frac{\psi_{i_1} \ldots \psi_{i_k}}{\psi_i}, i_j < i$;
- Proof depth: max formula depth;
- Proof size: sum of formula size;
- $\mathcal{F}_d$: proof depth bounded by d.

PB-Game for Frege

Initial Conditions: $\psi_i = a_i$ s.t. $F := \bigwedge_i(\psi_i = a_i)$ is UNSAT

Sam, the adversary, claims a SAT assignment for F.

Round: Pavel queries a formula ψ and Sam decides a value a

Termination: Sam's answers define a Trivial Contradiction with F:

$$((\psi_1 \wedge \psi_2) = 0, \psi_1 = 1, \psi_2 = 1)$$

PB-game

PB-proof: is a binary tree of the frms asked by P. Leaves are Trivial Contradictions

depth (of PB-proof): max depth of a formula used in the derivation

size (of PB-proof): sum of size of formulas used in the derivation

$\psi_1, \dots \psi_k \vdash \psi$: a PB-proof from Initial Conditions

$$[\psi = 0, \psi_1 = 1, \dots \psi_k = 1]$$

Lemma (F_d and PB-Game)

(1) If ψ has $\mathcal{F}_d$ proof of size S, then it has PB-proofs of size $O(S^2)$ and depth d.
(2) If ψ has PB-proofs of size S and depth d, then ψ has $\mathcal{F}_d$ proofs of size $O(S^3)$ and depth $d + O(1)$.

Tseitin Formulas

A vertex-charging for $G(V, E)$ is a mapping $f : V \longrightarrow \{0, 1\}$. We say that f is an *odd-charging* of G if $\sum_{v \in V} f(v) \equiv 1 \mod 2$

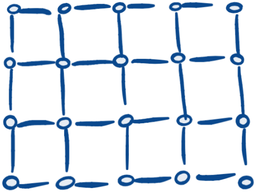
The *Tseitin contradictions* on G use variables x_e, $e \in E$: are the formulas:

$$\bigwedge_{v \in V} \mathrm{Par}(v) \qquad \text{where} \quad \mathrm{Par}(v) := \bigoplus_{e \in E(v)} (x_e = f(v))$$

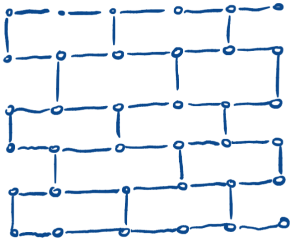
As tautology we use:

$$\mathrm{T}(G, f) := \neg \bigwedge_{v \in V} \mathrm{Par}(v)$$

Grids and Walls



$H_{5,5} = H_5$



W_6

Previous Results

- [Urquhart Fu 96] Exponential lower bounds for Tseitin formulas on complete graphs K_n. Depth $d = O(\log\log n)$.
- [Ben-Sasson 02] Superpolynomial lower bound for Tseitin on bounded-degree expanders graphs. $d = O(1)$ (reduction to PHP_n).
- [Pitassi Rossman Servedio Tan 16] Superpolynomial lower bounds for Tseitin formulas over 3-regular expanders, $d = O(\sqrt{\log n})$.
- [Håstad 17] Lower bound $2^{n^{\Omega(1/d)}}$ for Tseitin formulas over $\mathcal{H}_n$, $d \leq \frac{K \log n}{\log\log n}$.

Treewidth

Tree Decomposition of $G = (V, E)$: is a tree $T = (V_T, E_T)$ such that every vertex $u \in V_T$ corresponds to a set $X_u \subseteq V$ and

- $\bigcup_{u \in V_T} X_u = V$.
- For every edge $(a, b) \in E$ there exists $u \in V_T$ such that $a, b \in X_u$.
- If a vertex $a \in V$ is in the sets X_u and X_v for some $u, v \in V_T$, then it is also in X_w for all w on the path between u and v in T.

Treewidth of G: The *width* of a tree-decomposition $T = (V_T, E_T)$ is the $\max_{u \in V_T} |X_u| - 1$. The *treewidth* of G, $\mathrm{tw}(G)$ is the minimal width among all tree-decompositions of the graph G.

Our Results

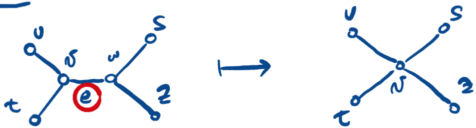
Lower bound: There is a constant K such that for any graph G over n nodes and for all $d \leq K \frac{\log n}{\log \log n}$, every $\mathcal{F}_d$ proof of $\mathrm{T}(G, f)$ has size at least $2^{\mathrm{tw}(G)^{\Omega(1/d)}}$.

Upper bound: For all large enough d there are $\mathcal{F}_d$ proofs of $\mathrm{T}(G, f)$ of size $2^{\mathrm{tw}(G)^{\mathcal{O}(1/d)}} \mathrm{poly}(|\mathrm{T}(G, f)|)$.

Edge removal

vertex removal

Edge contraction



Minors and Grid Minor Theorem

Minor of G: a graph that can be obtained from G by a sequence of edge contractions, edge and vertex deletion.

[Robertson Seymour 86, Chuzhoy 15]
Every graph G has $\mathcal{H}_r$ as a minor, where $r = \Omega(\mathrm{tw}(G)^{1/37})$.

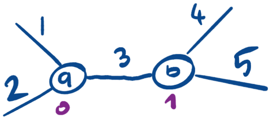
Naive idea. Use Grid Minor Theorem with Håstad Grid lower bound:

Hope: If H is a minor of G then small $\mathcal{F}_d$ proofs of of $\mathrm{T}(G, f)$ con be efficiently converted into small $\mathcal{F}_{d+O(1)}$ proofs of $\mathrm{T}(H, f)$.

$$\text{Sequence} : G \rightsquigarrow H_1 \rightsquigarrow \ldots \rightsquigarrow H_s \rightsquigarrow \mathcal{H}_r$$

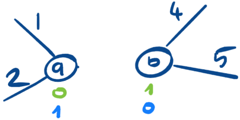
Naive Idea: problems

Edge removal on $T(G, f)$



$Par(a) := x_1 + x_2 + x_3 = 0$

$Par(b) := x_4 + x_5 + x_3 = 1$

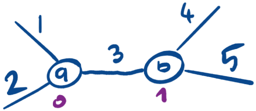


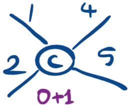
$x_3 = 0$: $Par(a) := x_1 + x_2 = 0$, $Par(b) := x_4 + x_5 = 1$

$x_3 = 1$: $Par(a) := x_1 + x_2 = 1$, $Par(b) := x_4 + x_5 = 0$

Edge removals on E $\equiv$ restrictions on x_e

Edge contraction on $T(G,f)$





$Par(a) := x_1 + x_2 + x_3 = 0$

$Par(b) := x_4 + x_5 + x_3 = 1$

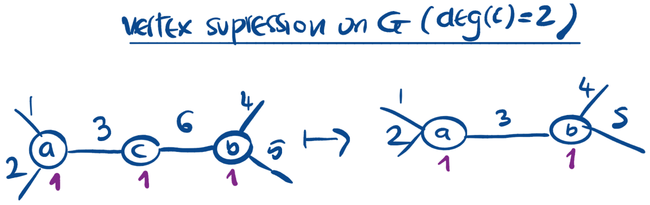
$Par(c) := x_1 + x_2 + x_4 + x_5 = 0+1$

replace x_3 with $x_4 + x_5$

but $x_4 + x_5 =$

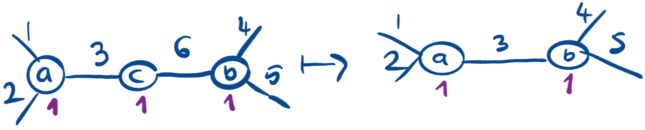
Depth increase by 1 at each contraction !

Vertex Suppressions



Vertex Suppressions

Vertex supression on G (deg(c)=2)



Vertex suppressions on T(G,f)

$Par(b) := x_4 + x_5 + x_6 = 1$

$Par(c) := x_3 + x_6 = 1$

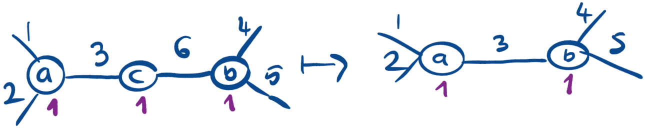
$x_6 + x_3 = 1$

$x_6 = \bar{x}_3$

$Par(b) := x_4 + x_5 + \bar{x}_3 = 1$

Vertex Suppressions

Vertex supression on G ($deg(c)=2$)



Vertex suppressions on $T(G,f)$

$Par(b) := x_4 + x_5 + x_6 = 1$

$Par(c) := x_3 + x_6 = 1$

$x_6 + x_3 = 1$

$x_6 = \bar{x}_3$

$Par(b) := x_4 + x_5 + \bar{x}_3 = 1$

Vertex suppressions captured by substitutions of variables by literals: projections !

Topological Minor Theorem

Topological Minor of G: a graph that can be obtained by a sequence of suppressions, edge and vertex deletion.

Theorem

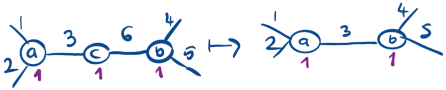
Every graph G has $\mathcal{W}_r$ as a topological minor, where $r = \Omega(\mathrm{tw}(G)^{1/37})$.

$\mathcal{W}_r$ is a subgraph (hence a minor) of $\mathcal{H}_r$. Hence $\mathcal{H}_r \rightsquigarrow \ldots \rightsquigarrow \mathcal{W}_r$ by removals. By Grid Minor theorem $G \rightsquigarrow \ldots \rightsquigarrow \mathcal{H}_r$. Take the sequence $G \rightsquigarrow \ldots \rightsquigarrow \mathcal{W}_r$ with the *minimal number of contractions. By rearranging (operations commute)*

$$\mathcal{S} : \overbrace{G \rightsquigarrow H_1 \rightsquigarrow \ldots \rightsquigarrow H_s}^{\textit{removals}} \rightsquigarrow \overbrace{G_1 \rightsquigarrow \ldots \rightsquigarrow \mathcal{W}_r}^{\textit{contractions}}$$

Topological Minor Theorem

Contractions on degree 2 vertices can be replaced by suppressions



We claim that all contractions on $\mathcal{S}$ involve a degree 2 vertex and hence can be suppressions.

Notice that contractions decrease the degree only on nodes of degree 1. But then an edge removal would work and this is not possible since $\mathcal{S}$ has minimal number of contractions)

If, by contradiction a contraction is on an edge $e = (u, v)$ and $deg(u) \geq 3$ and $deg(v) \geq 3$, this would create a node of degree ≥ 4. This is impossible since in $\mathcal{S}$ after contractions degree cannot decrease (contractions follows removals) and $\mathcal{W}_r$ does not have degree 4 nodes.

Proof of the lower bound

$$\text{Sequence}: G \rightsquigarrow \overbrace{H_1 \rightsquigarrow \ldots \rightsquigarrow H_s}^{\text{edge removals}} \rightsquigarrow \overbrace{G_1 \rightsquigarrow \ldots \rightsquigarrow \mathcal{W}_r}^{\text{vertex suppressions}}$$

Lemma (Removals)

H subgraph of G. If there are $\mathcal{F}_d$ proof of $\mathrm{T}(G, f)$ of size S, then there are $\mathcal{F}_{d+O(1)}$ proofs for $\mathrm{T}(H, f)$ of size $S + \mathrm{poly}(|\mathrm{T}(G, f)|$.

Easy. Removals in G are restrictions on $\mathrm{T}(G, f)$.

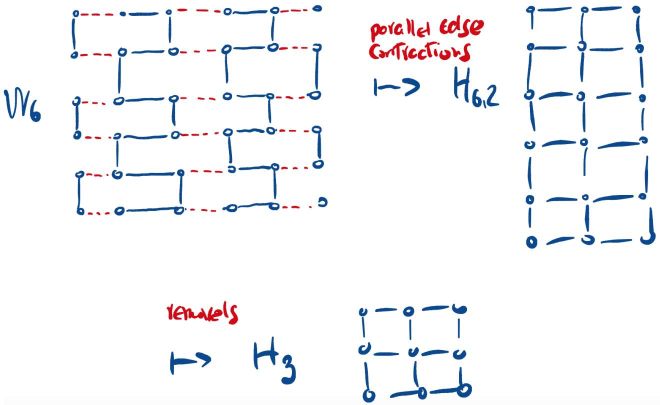
Lemma (Suppressions)

H obtained from G by suppressions. If there are $\mathcal{F}_d$ proof of $\mathrm{T}(G, f)$ of size S, then there are $\mathcal{F}_{d+O(1)}$ proofs for $\mathrm{T}(H, f)$ of size $S + \mathrm{poly}(|\mathrm{T}(G, f)|$.

Main point. Projections are closed under composition: $\sigma_1 := [y \mapsto \neg z]$ and $\sigma_2 := [x \mapsto \neg y]$, then $\sigma_1 \circ \sigma_2 := [x \mapsto z, y \mapsto \neg z]$.

From $\mathcal{W}_r$ to $\mathcal{H}_{\frac{r-1}{2}}$

$\mathcal{H}_{\frac{r-1}{2}}$ is a minor of $\mathcal{W}_r$. Using independent edge contractions. Hence proof depth is increasing by 1 since contractions can be performed in parallel.



Upper bound: Proof Idea

- We found a compact representation of $\bigoplus_i x_i = a$ into a d-depth formula $[\![\bigoplus_i x_i = a]\!]$ of size $2^{n^{O(1/d)}}$
- there are $\mathcal{F}_d$ efficient derivations of

$$\frac{[\![\bigoplus_{i\in A} x_i = a]\!] \qquad [\![\bigoplus_{i\in B} x_i = b]\!]}{[\![\bigoplus_{i\in A\triangle B} x_i = a \oplus b]\!]}$$

Lemma

There are $\mathcal{F}_d$ proofs of size $\mathrm{poly}(|\mathsf{T}(G,f)|)2^{m^{O(1/d)}}$ for $\mathsf{T}(G,f)$, where m is the number of edges in G.

But we need: $\mathrm{poly}(|\mathsf{T}(G,f)|)2^{\mathrm{tw}(G)^{O(1/d)}}$

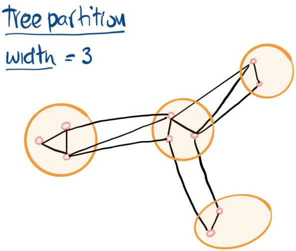
Tree partition width

a graph H is a partition of a graph G if:

- each vertex of H is a set of vertex of G (a *bag);*
- *each vertex of G is exactly in one bag*
- *distinct bags A_i,A_j are adjacent in H iff there is an an edge in G with one endpoint in A_i and the other in A_j*

When H is a tree T is called a a tree partition of G.

$w(T) = \max_i |A_i|$
$\mathrm{tpw}(G) = \min_T w(T)$.



In each bag A there are at most $\mathrm{tpw}(G)\Delta(G)$ edges.

Lemma

For each bag A, there are $\mathcal{F}_d$ proofs of $\mathrm{T}(G,f)$ *of size* $\mathrm{poly}(|\mathrm{T}(G,f)|)2^{(\mathrm{tpw}(G)\Delta(G))^{O(1/d)}}$.

By previous Lemma For each bag A are there are $\mathcal{F}_{3d}$ proofs of $\mathrm{T}(G\restriction A,f)$ of size $\mathrm{poly}(|\mathrm{T}(G,f)|)2^{(\mathrm{tpw}(G)\Delta(G))^{O(1/d)}}$

Proof Algorithm(high-level idea):

1. Search in T for a bag A such that

$$\bigoplus_{e\in E(A,V-A)} x_e \neq \bigoplus_{v\in A} f(v)$$

   This bag must exits since $\mathrm{T}(G,f)$ is unsatisfiable
2. produce proofs of $\mathrm{T}(G\restriction A,f)$

From tree-partition-width to tree-width

Theorem (Wood 09)

$\Omega(\mathrm{tw}(G)) \leq \mathrm{tpw}(G) \leq O(\mathrm{tw}(G)\Delta(G))$.

Thanks for the attention !