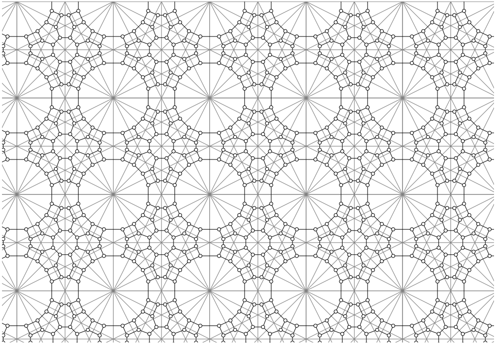
Stability conditions via Tits cone intersections



Michael Wemyss

www.maths.gla.ac.uk/∼mwemyss

Plan of Talk

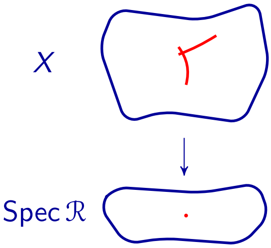
1. 3-fold flops: enhancing the movable cone.

2. Tits cone intersections.

3. Application: flops, mutation, and stability conditions.
   (plus: what is the picture on the first slide?)

Setting

Three-dimensional multi-curve flops, which are pictorially:



where X is Gorenstein terminal (e.g. smooth).

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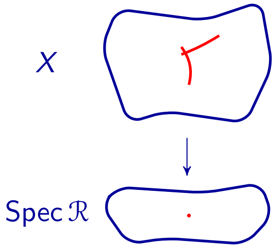
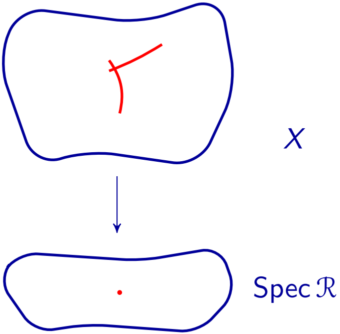


where X is Gorenstein terminal (e.g. smooth). We're interested in:

- Classification.
- Invariants, curve counting.
- Derived categories and stability conditions.
- Symmetries: derived autoequivalences.
- Noncommutative resolutions.

What combinatorics controls flops?

Take-home message: the combinatorics of flops, and to a large extent their homological algebra, is controlled by surfaces data.



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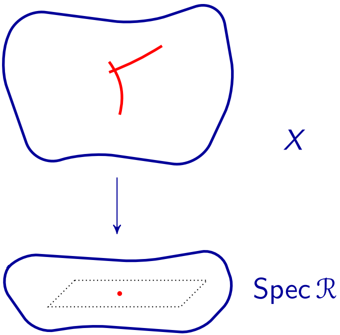

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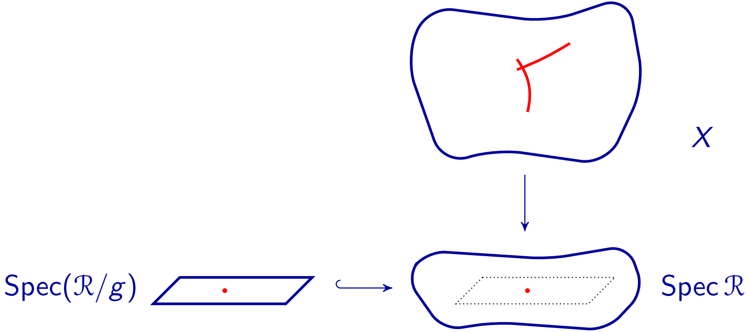

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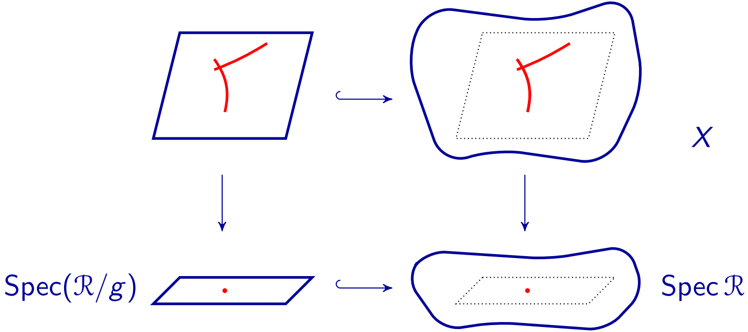

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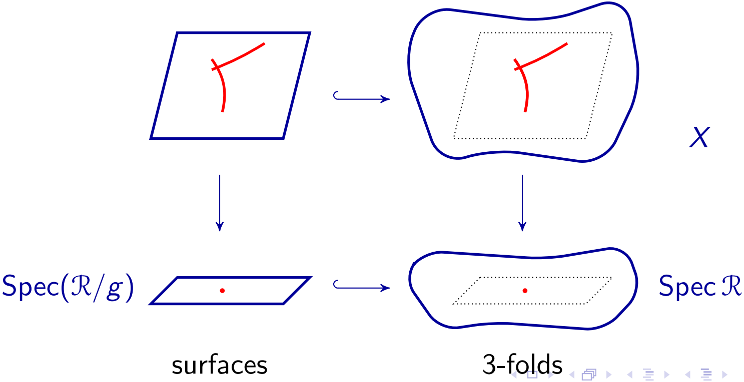

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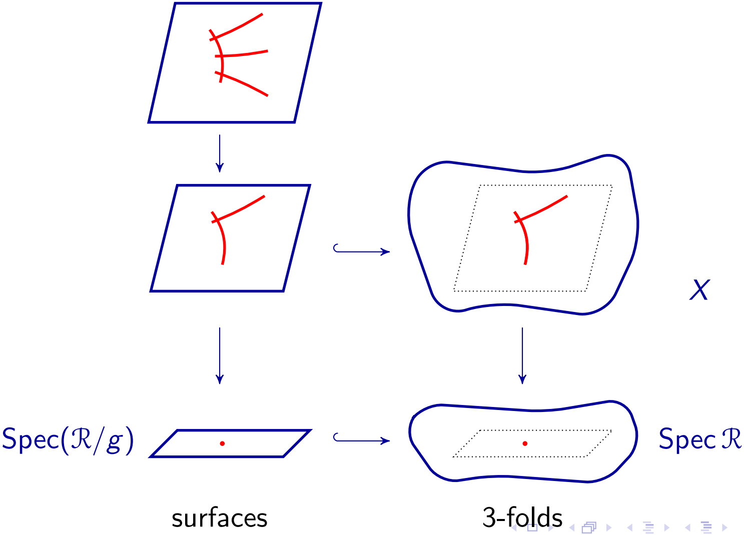

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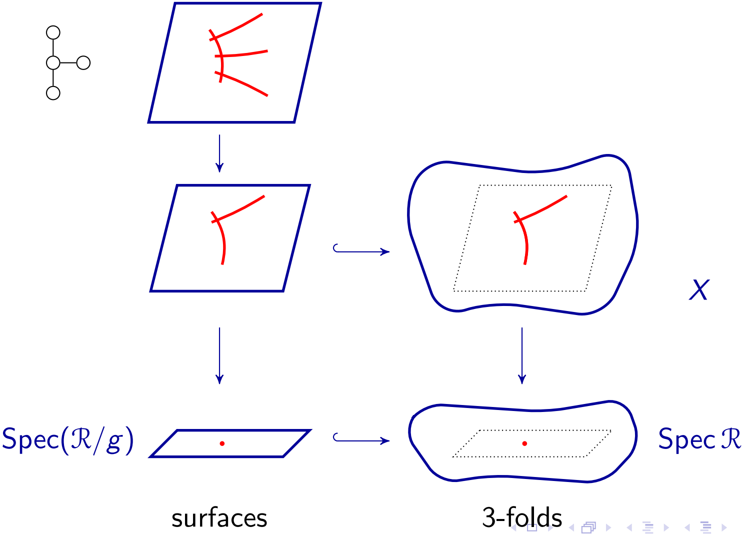

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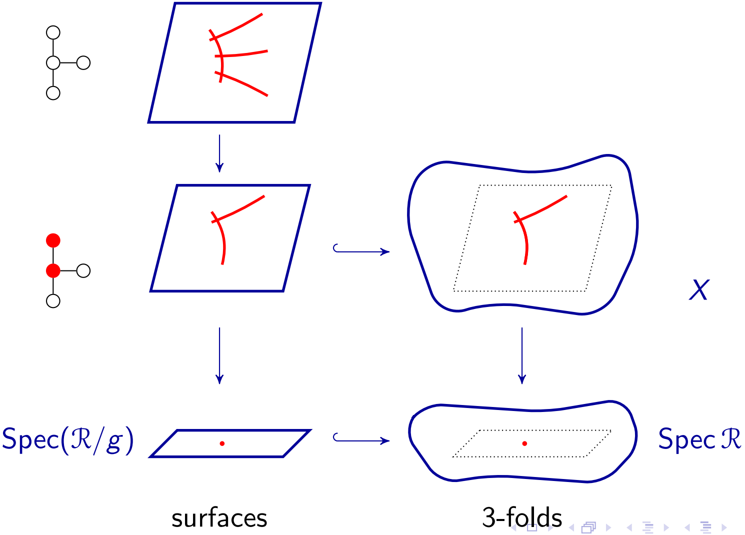

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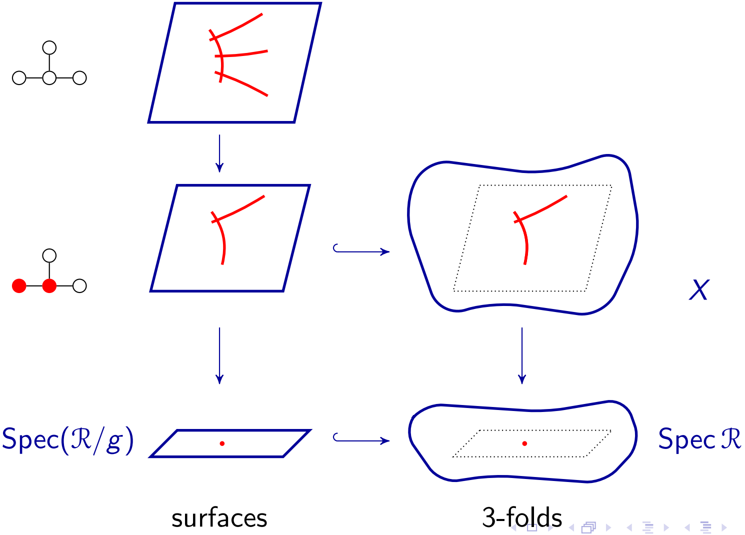
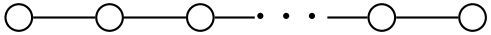
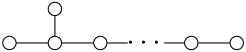


Recap on ADE Dynkin Diagrams

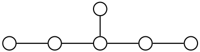
A_n
$n \geq 1$



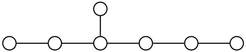
D_n
$n \geq 4$



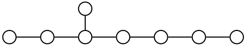
E_6



E_7



E_8



Recap on ADE Dynkin Diagrams + choice of nodes

A_n
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D_n
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E_6

E_7

E_8

Construction

Input

- Any choice of ADE Dynkin diagram Δ,
- and any choice of nodes $\mathcal{J} \subseteq \Delta$.

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Now, each such Δ has an associated *root system*. This is just a real vector space $\mathbb{R}^{|\Delta|}$, with basis given by the nodes, together with some *reflecting hyperplanes*.

This does not depend on the choice $\mathcal{J}$.

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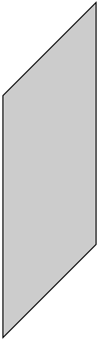
Aim

Want something similar, but which also depends on $\mathcal{J}$.

Intersection arrangements

The root system has a basis given by the nodes. Thus, the choice $\mathcal{J}$ gives *some* of these, so a *subspace* $\mathbb{R}^{|\mathcal{J}|}$. Picture for $|\mathcal{J}| = 2$ is:



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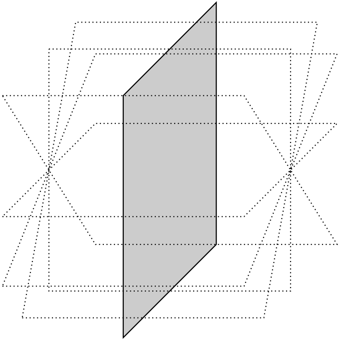

The reflecting hyperplanes slice the subspace

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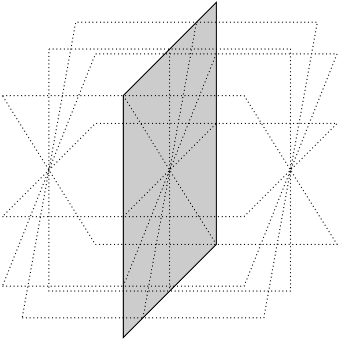

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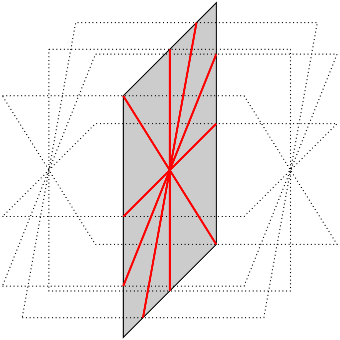


We *intersect* the reflecting hyperplanes with the subspace

Output

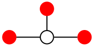
A finite collection of (red) hyperplanes, written $\mathrm{Cone}(\mathcal{J})$.

Theorem (Pinkham)

The intersection arrangement Cone($\mathcal{J}$) is the movable cone of the flopping contraction.

...can also prove this by tracking the skyscrapers around under the flop functors, then de-categorifying.

Some Examples



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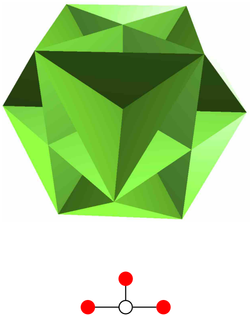
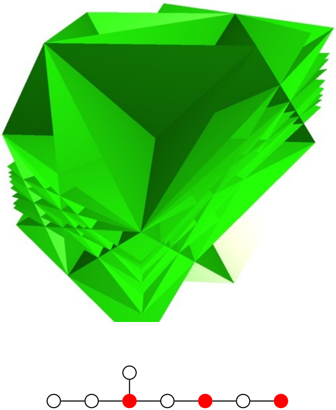
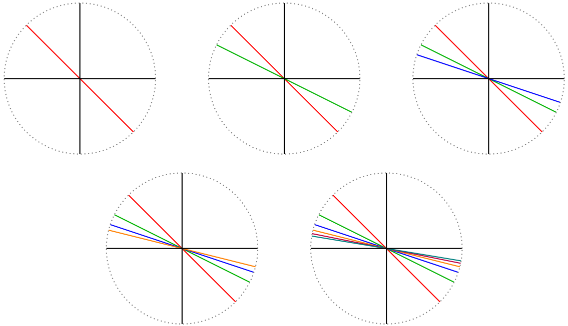




Proposition (Iyama–W)

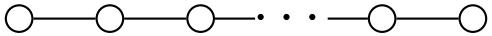
Consider any $\mathcal{J} \subseteq \Delta$ with Δ ADE Dynkin and $|\mathcal{J}| = 2$. Then, up to changing the slopes of the lines, $\mathrm{Cone}(\mathcal{J})$ is one of:



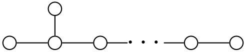
The number of chambers is $6, 8, 10, 12$ and 16 respectively.

Better: extended ADE Dynkin Diagrams

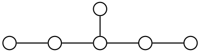
A_n
$n \geq 1$



D_n
$n \geq 4$



E_6



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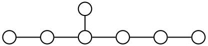
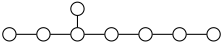


E_8



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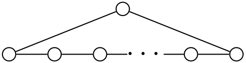

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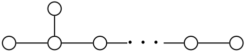

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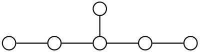

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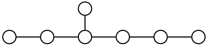

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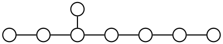

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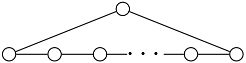

$\tilde{D}_n$
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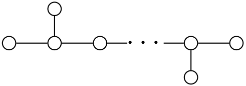

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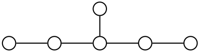

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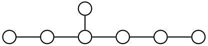

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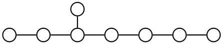

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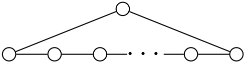

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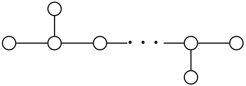

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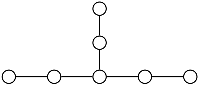

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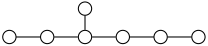

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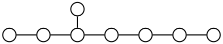

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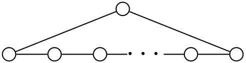

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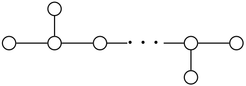

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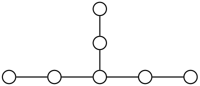

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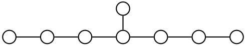

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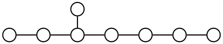

Better: extended ADE Dynkin Diagrams

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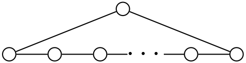

$\tilde{D}_n$
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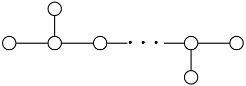

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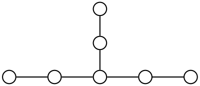

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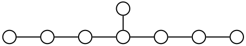

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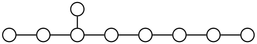
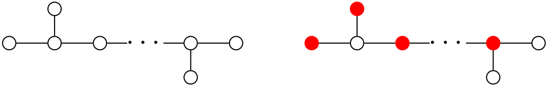


Better: extended ADE Dynkin Diagrams + choice of nodes

$\tilde{A}_n$
$n \geq 1$

$\tilde{D}_n$
$n \geq 4$



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Tits Cone Intersections

Input

- Any choice of extended ADE Dynkin diagram Δ_{aff},
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Tits Cone Intersections

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- Any choice of extended ADE Dynkin diagram Δ_{aff},
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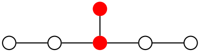
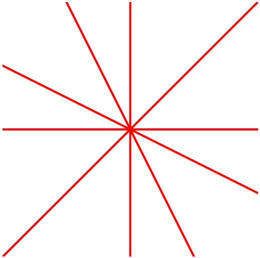
A similar story as to before, intersecting now inside the Tits Cone (instead of the root system) gives an *infinite* hyperplane arrangement, written Level($\mathcal{K}$).

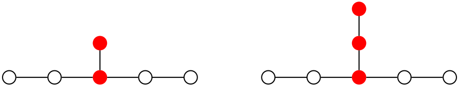
This lives in $\mathbb{R}^{|\mathcal{K}|-1}$.

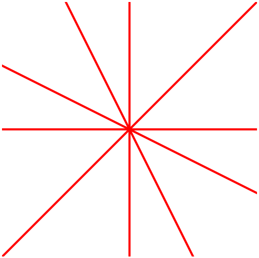
Finite Inside Infinite





Finite Inside Infinite





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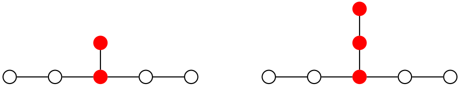

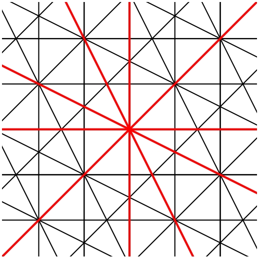

May as well develop the infinite theory; finite theory comes for free.

Labels and Wall Crossing

Question

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The key is what we actually do is the following:

- Every chamber is labelled by a pair $(w, \mathcal{I})$, where w is an element in some group, and $\mathcal{I}$ is a subset of nodes.
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The rule is a bit technical, but it allows us to start anywhere, and iterate. The rule is also important for geometric applications.

The Wall Crossing Rule

Number of wall crossings = number of red nodes in subset.

To cross one of these walls, choose red node. Temporarily delete *all other* red nodes, apply Dynkin involution, then put back in the deleted vertices.

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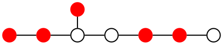

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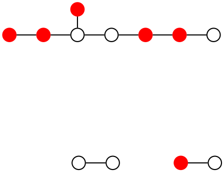

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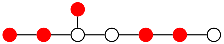

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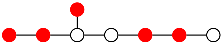

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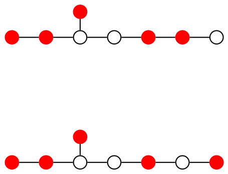

Theorem (Iyama–W)

If $\mathcal{K} \subseteq \Delta_{\mathsf{aff}}$ *satisfies* $|\mathcal{K}| = 3$, *then up to changing the slopes of some of the hyperplanes,* $\mathrm{Level}(\mathcal{K})$ *is one of:*

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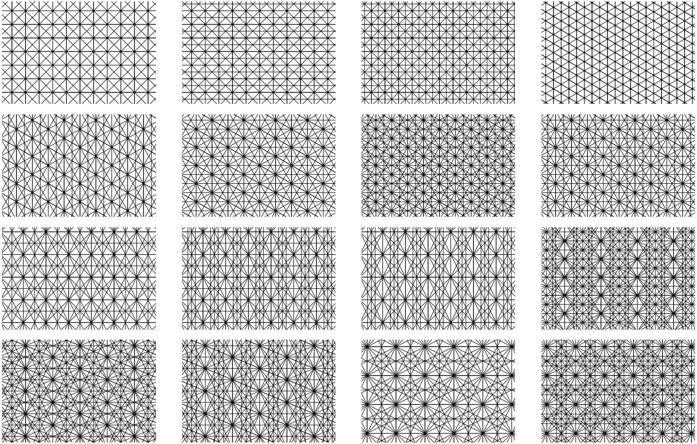
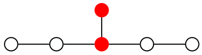


Upshot

For every 3-fold flop $X \to \operatorname{Spec} \mathcal{R}$, obtain a pair $(\Delta, \mathcal{J})$, namely a shaded ADE Dynkin diagram.

As before, from this we can always just add in the extended vertex:



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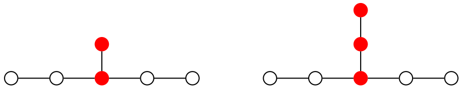

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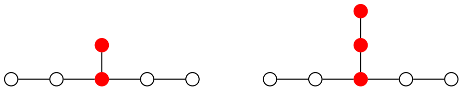

Start of talk: the left one gives us a finite hyperplane arrangement $\mathcal{H}$, the right hand one gives us an infinite arrangement $\mathcal{H}_{\mathsf{aff}}$.

Enter Noncommutative Resolutions (and variants)

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For finite story, consider $M \in \operatorname{mod} \mathcal{R}$ such that

- M is Cohen–Macaulay, namely $\operatorname{Ext}^i_{\mathcal{R}}(M, \mathcal{R}) = 0$ for all $i > 0$.
- M is rigid, namely $\operatorname{Ext}^1_{\mathcal{R}}(M, M) = 0$.
- M is maximal with respect to the above property.

In the lingo, 'maximal rigid objects in the category $\mathrm{CM}\mathcal{R}$'.

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...will turn out to only be finitely many of them.

For the infinite arrangement story, need more. Consider those $M \in \operatorname{mod} \mathcal{R}$ such that:

- M is reflexive, namely there is an isomorphism

$$M \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{R}}(\operatorname{Hom}_{\mathcal{R}}(M, \mathcal{R}), \mathcal{R})$$

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In the lingo, 'maximal modifying modules'. These are the building blocks of *noncommutative resolutions* (and their variants).

Theorem (Iyama–W)

Suppose that $X \to \operatorname{Spec} \mathcal{R}$ is a smooth flopping contraction.

Associate $\mathcal{J} \subseteq \Delta$ by slicing, which gives a finite arrangement $\mathcal{H}$ and an infinite arrangement $\mathcal{H}_{\mathsf{aff}}$.

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...in particular, we get a *complete* classification of noncommutative resolutions in this setting!

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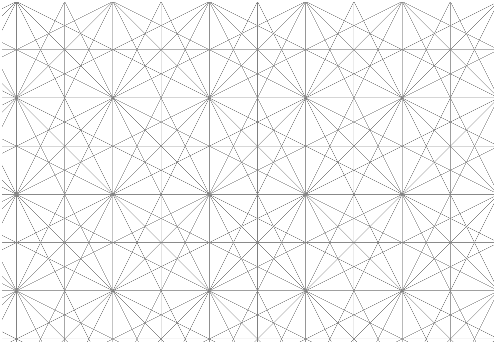

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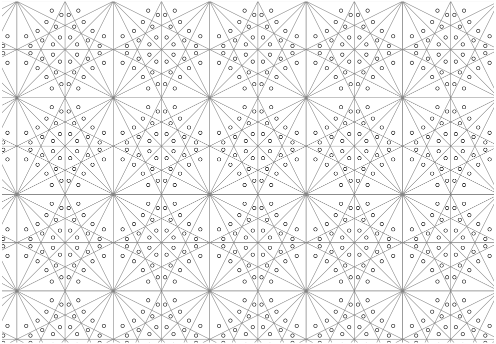

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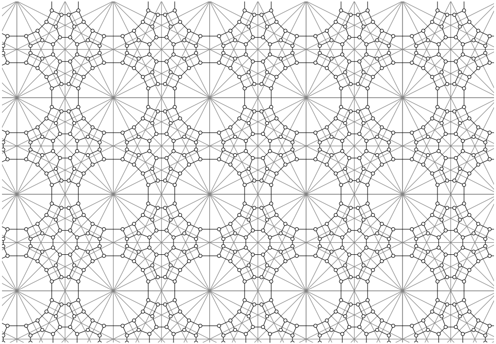


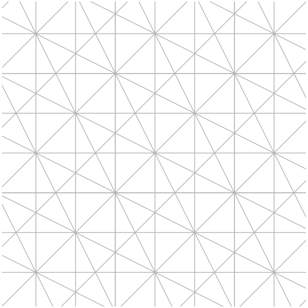
The dots are those $M \in \operatorname{ref} \mathcal{R}$ which give NCCRs. The edges connecting dots are the *mutations* of these; the above is really a picture of the exchange graph.

To have such highly regular structure is very unusual.

Now categorify...

The *mutation functors* lift the above combinatorial statements.

Consider the following groupoid:



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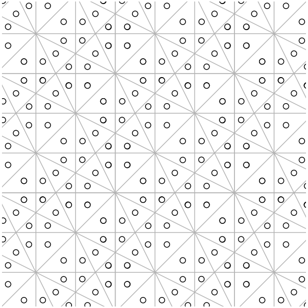

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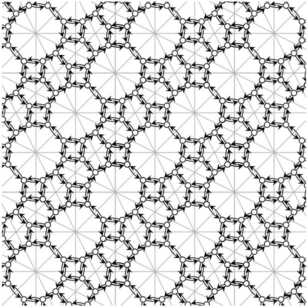

with relations give by identifying shortest paths. This is called the *Deligne groupoid*.

There is another way to build a groupoid. By last theorem:

- Each chamber has associated M, thus $\mathrm{D}^{\mathrm{b}}(\operatorname{mod}\operatorname{End}_{\mathcal{R}}(M))$.
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Theorem (Iyama–W)

There exists a functor from the Deligne groupoid to the groupoid described above.

Corollary (Iyama–W)

$\pi_1(\mathbb{C}^n \backslash (\mathcal{H}_{\mathrm{aff}})_{\mathbb{C}})$ acts on $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$.

And categorify again...

Consider the following two subcategories of $\mathrm{D^b}(\operatorname{coh} X)$.

$$\mathcal{C} = \{\mathcal{F} \in \mathrm{D^b}(\operatorname{coh} X) \mid \mathbf{R}f_*\mathcal{F} = 0\}$$
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Main Theorem (Hirano–W)

Given flopping contraction $X \to \operatorname{Spec}\mathcal{R}$, associate finite $\mathcal{H}$ and infinite $\mathcal{H}_{\mathsf{aff}}$ by slicing. Then the forgetful maps

$$\mathrm{Stab}^{\circ}\mathcal{C} \to \mathbb{C}^n \backslash \mathcal{H}_{\mathbb{C}}$$
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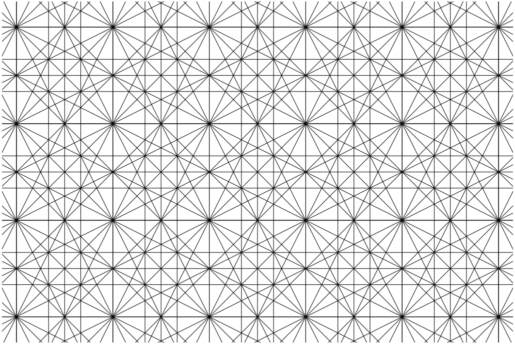
Autoequivalences of the last slide are the deck transformations.

Conclusion

The t-structures needed to describe the stability conditions *do not* all come from the movable cone and its images under tensoring by a line bundle.



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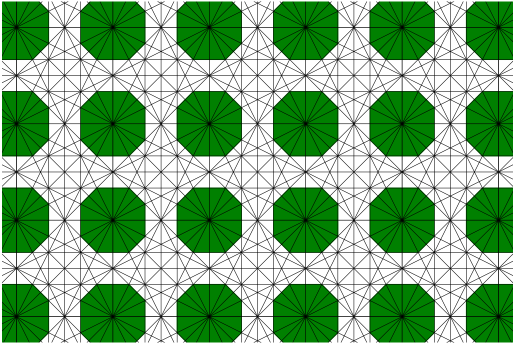

You need lots more: the others are noncommutative. The autoequivalence group is much larger than you expect.