Automorphisms of projective surfaces: finite orbits of large groups

Based on a joint work with Romain Dujardin

Dynamics on a real K3 surface (C.T. McMullen, V. Pit)
Automorphisms of surfaces:

Examples
Surfaces and automorphisms

- $X$ = smooth complex projective surface (real dimension 4)
- Aut($X$) = group of holomorphic diffeomorphisms
  = group of (regular, algebraic) automorphisms
  = a complex Lie group.

**Example 1.** – $E = \mathbb{C}/\Lambda$, an elliptic curve.

\[ X = E \times E = \mathbb{C}^2/(\Lambda \times \Lambda). \]
\[ X = \text{translations} \subset \text{Aut}(X). \]
\[ \text{GL}_2(\mathbb{Z}) \subset \text{Aut}(X). \]

**Example 2.** – $\eta(x, y) = (-x, -y)$ on $X = E \times E$.

$\eta$ commutes to the action of $\text{GL}_2(\mathbb{Z})$.

\[ Y = \widehat{X}/\eta \text{ is a Kummer surface.} \]
• Example 3.– $X \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, smooth, degree $(2, 2, 2)$:

$$x^2 y^2 z^2 + (x^2 y^2 + y^2 z^2 + z^2 x^2)/200 + x^2 + y^2 + z^2 + xy + z - y = 6.$$
Cohomology:
Minkowski space and types of automorphisms.
• **Intersection form.**–
  \[ \langle C|D \rangle = \text{intersection number, with multiplicities}; \]
  \[ \langle \cdot|\cdot \rangle = \text{bilinear form on divisors}. \]

• **Néron-Severi group.**– Numerical classes of divisors.

  \[ \text{NS}(X; \mathbb{Z}) = H^2(X; \mathbb{Z}) \cap H^{1,1}(X, \mathbb{R}). \]

• **Picard number.**– \( \rho(X) = \dim_\mathbb{R} \text{NS}(X, \mathbb{R}) \).

• **Hodge index Theorem.**– On \( \text{NS}(X; \mathbb{R}) \), the intersection form is non-degenerate, of signature \( (1, \rho(X) - 1) \).
Three types of isometries

- Elliptic: $f^*$ has finite order,
- Parabolic: is virtually unipotent,
- Loxodromic: or $\lambda(f) > 1$. 
Tame automorphisms

• If $f$ elliptic, then some positive iterate $f^k$ is in $\text{Aut}(X)^0$.

• **Gizatullin’s Theorem.**—
  If $f^*$ is parabolic, then $f$ preserves a genus 1 fibration $\pi : X \to B$, and induces a finite order automorphism of $B$ if $X$ is not an abelian surface.

**Examples.**— Mordell-Weil groups of a genus 1 fibration $\pi : X \to B$: translations from one section of $\pi$ to another one.
— Break for Questions —

and Banff International Research Station

Thank You!
The invariant measure $\mu_f$:

stable manifolds, periodic points, equidistribution
Loxodromic automorphisms

- Two invariant isotropic lines

\[ R\theta^+_f \text{ and } R\theta^-_f, \text{ with } \langle \theta^+_f | \theta^-_f \rangle = 1. \]

- \( f^*\theta^+_f = \lambda(f)\theta^+_f. \)

- \( \theta^+_f, \theta^-_f \in \text{Ample cone}. \)

- \( \theta^+_f \) is represented by a closed positive current \( T^+_f \) with \( f^* T^+_f = \lambda(f) T^\pm_f. \)

**Fact.**— The current \( T^\pm_f \) is **unique** and has **Hölder continuous** potentials. The measure

\[ \mu_f = T^+_f \wedge T^-_f \]

is an invariant probability measure.
• **Theorem (Bedford, Lyubich, Smille; C.; Dujardin).**—
  The periodic points of $f$ of period $N$ become equidistributed with respect to $\mu_f$ as $N$ goes to $+\infty$:

  \[
  \frac{1}{|\text{Per}_f(N)|} \sum_{x \in \text{Per}_f(N)} \delta_x \longrightarrow \mu_f.
  \]

  Moreover, $|\text{Per}_f(N)| \sim \lambda(f)^N$.

• **Theorem (C., Dupont; see also Filip and Tosatti).**—
  If the measure $\mu_f$ is smooth, or absolutely continuous with respect to the Lebesgue measure on $X$, then $(X, f)$ is a Kummer example.
• **Kummer groups.** \( \Gamma \subset \text{Aut}(X) \) is a Kummer group if there exists

- an abelian surface \( A \); a subgroup \( \Gamma_A \subset \text{Aut}(A) \);
- a finite, normal subgroup \( G \) of \( \Gamma_A \);
- a birational morphism \( q_X : X \to A/G \);
- homomorphisms \( \tau_X : \Gamma \to \text{Aut}(A/G) \) and \( \tau_A : \Gamma_A \to \text{Aut}(A/G) \);

such that \( q_X \) and the quotient map \( q_A : A \to A/G \) are naturally equivariant and define the same groups:

- \( q_X \circ f = \tau(f) \circ q_X \) for every \( f \in \Gamma \);
- \( q_A \circ g = \tau(f) \circ q_A \) for every \( g \in \Gamma_A \);
- \( \tau_A(\Gamma_A) = \tau_X(\Gamma_X) \).
— IV —

Periodic orbits for large groups
• **Theorem A (C., Dujardin).** –
  - \( k \) = number field.
  - \( X \) = smooth projective surface defined over \( k \).
  - \( \Gamma = \text{subgroup of } \text{Aut}(X_k) \text{ containing parabolic elements with distinct invariant fibrations.} \)
  
  If \( \Gamma \) has a Zariski dense set of periodic points, then \((X, \Gamma)\) is a Kummer group.

• **Remarks.** –
  - Works also over the field \( \mathbb{C} \) if we assume that \( \Gamma \) has no periodic curve.
  - Related question: classify pairs of loxodromic elements with \( \mu_f = \mu_g \).
    (see the work of Dujardin and Favre for Hénon automorphisms)
• \( k = \) number field, \( \bar{k} \cong \bar{Q} \).
• \( X \) and \( \Gamma \) defined over \( k \).
• \( \text{Pic}(X; \mathbb{R}) = \text{Pic}(X_{\bar{k}}) \otimes_{\mathbb{Z}} \mathbb{R} \) (Picard group)
  
  \[ \text{NS}(X; \mathbb{R}) \text{ if } \text{Pic}^0(X_{\bar{k}}) \neq 0. \]

• **Definition (A. Baragar).**— A **canonical vector height** is a function

\[ h: \text{Pic}(X; \mathbb{R}) \times X(\bar{k}) \to \mathbb{R} \]

such that

(a) for \( D \in \text{Pic}(X; \mathbb{R}) \), \( h(D, \cdot) \) is a Weil height w.r.t. \( D \) on \( X(\bar{k}) \);

(b) \( h(D, x) \) is linear in \( D \): \( h(aD + bE, \cdot) = ah(D, \cdot) + bh(E, \cdot) \);

(c) \( h \) is equivariant: \( h(f^*D, x) = h(D, f(x)) \) for all \( f \in \Gamma \).
• **Example.**— The Néron-Tate height, for automorphisms fixing the neutral element.

• **Example.**— When $\rho(X) = 2$, and $\Gamma$ is generated by a loxodromic element (Baragar, after a construction of Silverman).

• **Example.**— Kawaguchi found examples of Wehler surfaces with no such height functions.

• **Theorem B (C., Dujardin).**— $\Gamma \subset \text{Aut}(X_k)$ as in Theorem A. If there exists a canonical vector height for $\Gamma$, then
  
  • $X$ is an abelian surface,
  
  • $\Gamma$ has a periodic point $y$,
  
  • and $h$ is derived from the Néron-Tate height:
    
    $$h(D, x + y) = h_{NT}(D, x) + \langle [E][D] \rangle \varphi(x).$$
Proof Strategy
Yuan’s equidistribution (following Kawaguchi)

• 1.A– Kawaguchi’s stationary height
  • \( \nu \) = probability measure on \( \Gamma \), with finite support
  • \( \sum_f \nu(f)f^*(D) = \alpha(\nu)D \), for some \( \alpha(\nu) > 1 \), and some \( D \) ample

Then there is a Weil height \( \hat{h}_D: X(\bar{k}) \to \mathbb{R}_+ \),

\[
\sum_f \nu(f)\hat{h}_D(f(x)) = \alpha(\nu)\hat{h}_D(x), \quad \forall x \in X(\bar{k}),
\]

with a decomposition as a sum of continuous local heights.

**Finite orbits correspond to points of height 0 for \( \hat{h}_D \).**

• 1.B– Yuan’s equidistribution theorem, for a sequence of periodic points \( x_i \):

\[
\frac{1}{|\Gamma(x_i)|} \sum_{y \in \Gamma(x_i)} \frac{1}{|\text{Gal}(\bar{k} : k)(y)|} \sum_{\sigma} \delta_{\sigma(y)} \longrightarrow \mu
\]

where \( \mu \) is a \( \Gamma \)-invariant probability measure.
• 2.– The limit $\mu$ does not depend on $\nu$

$$\nu_n \to \frac{1}{2} \delta_f + \frac{1}{2} \delta_{f^{-1}}$$

The measure $\mu$ coincides with $\mu_f$, for every loxodromic $f \in \Gamma$.

• 3.– Compose parabolic elements with distinct invariant fibrations

The measure $\mu$ has full support.
• 4.– The measure $\mu$ is smooth

• 5.– Every loxodromic element is a Kummer example. Then $(X, \Gamma)$ is a Kummer group.
What more?