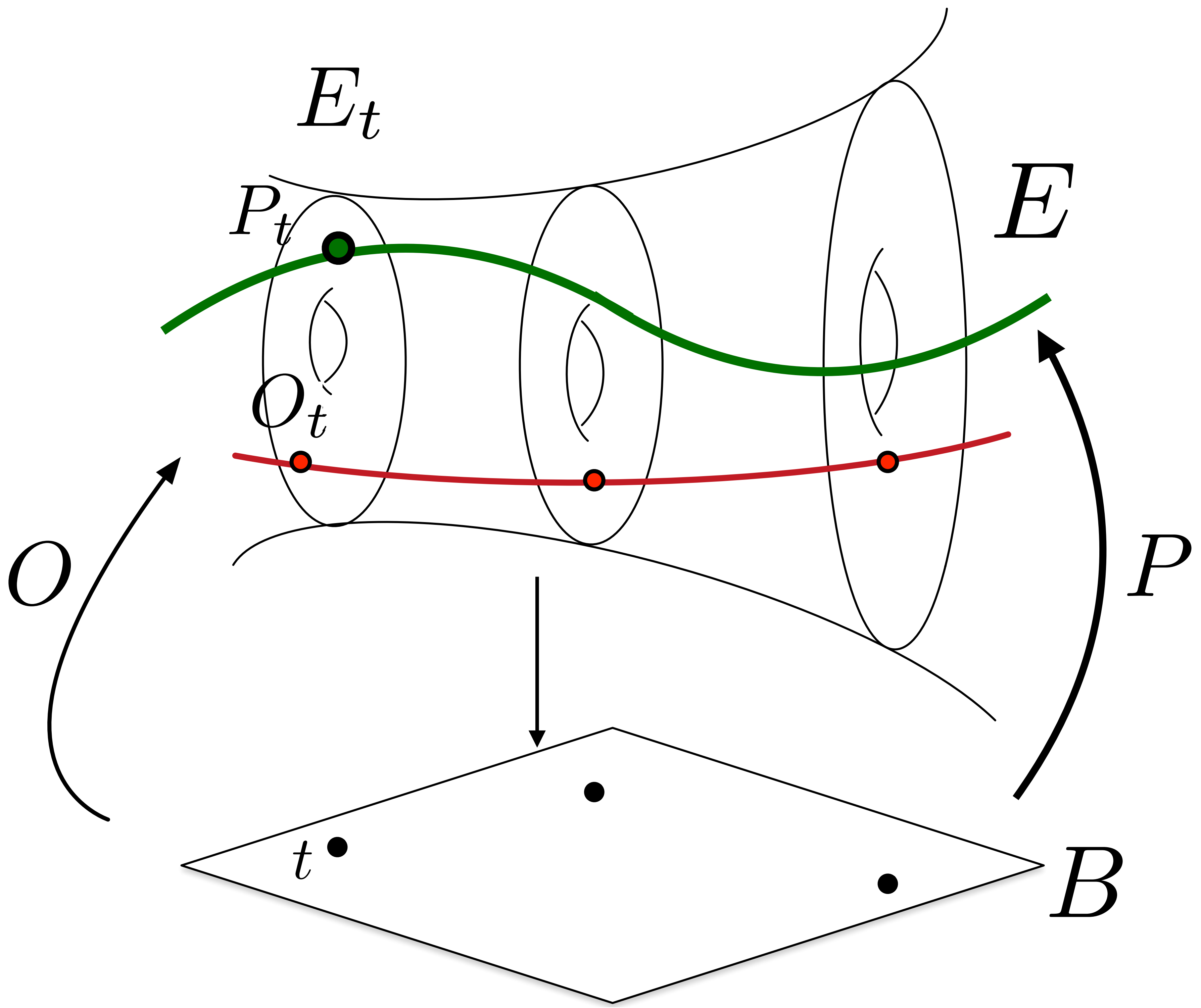
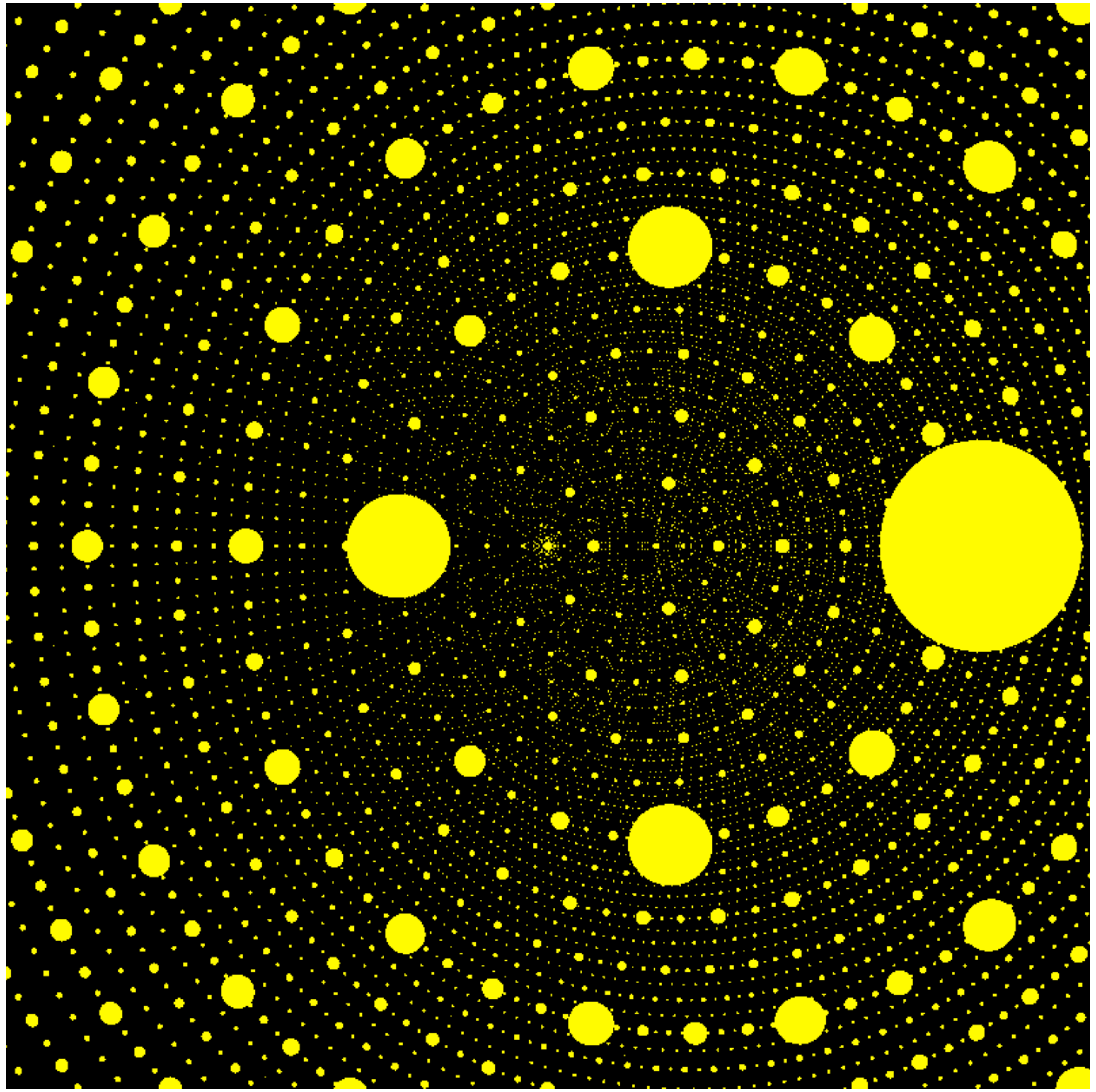
Elliptic surfaces and $\mathbb{R}$-divisors

joint work with N. Myrto Mavraki





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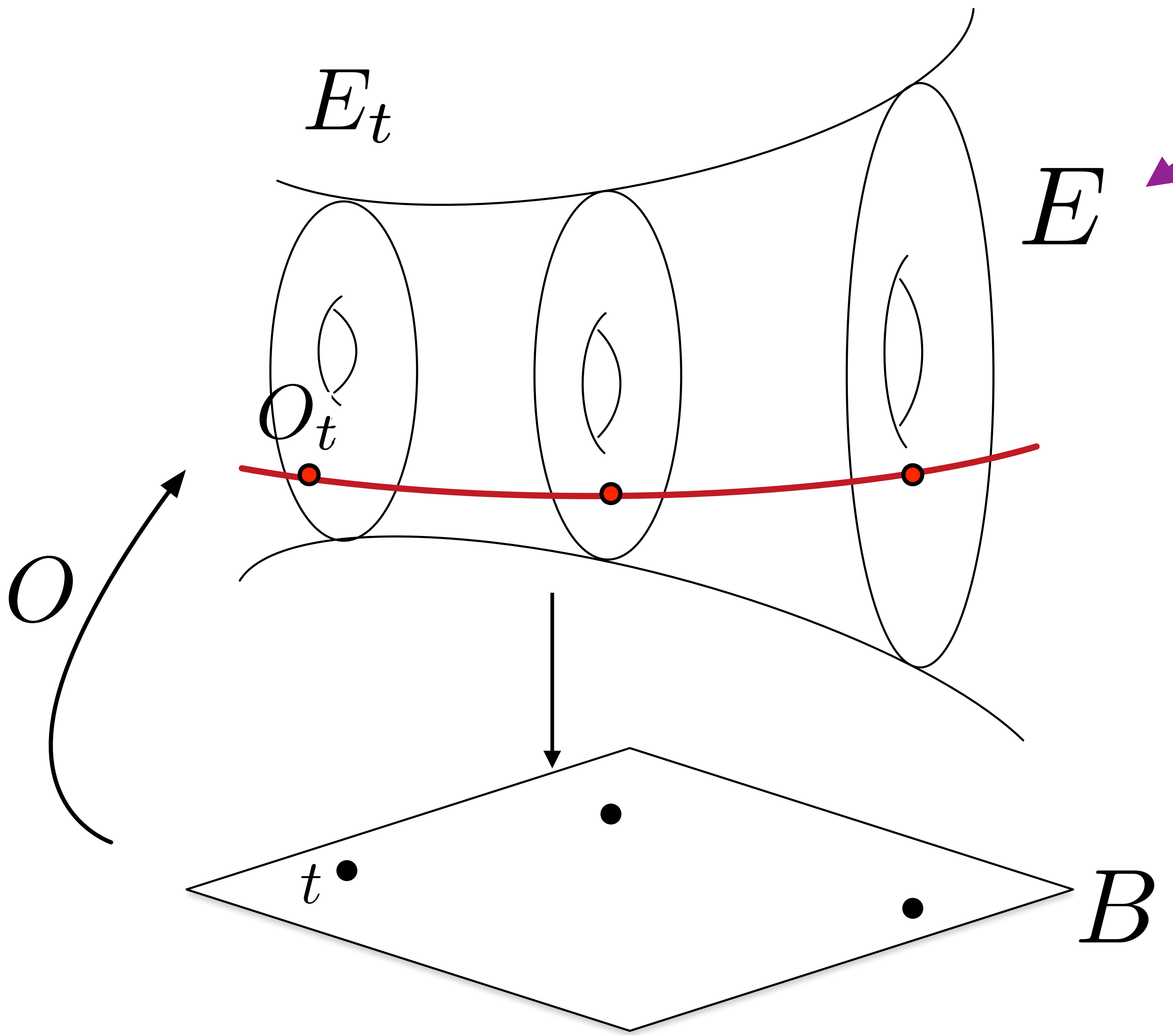

projective surface

smooth
projective curve

Assumptions:
E is not isotrivial
(E, B, O) is defined over $\overline{\mathbb{Q}}$

viewed as elliptic curve / k

Theorem. (Mordell, Weil, 1920s, Tate, Manin, $\sim$ 1960)
Let k = function field $\overline{\mathbb{Q}}(B)$. The set of rational points $E(k)$ forms a finitely-generated group.

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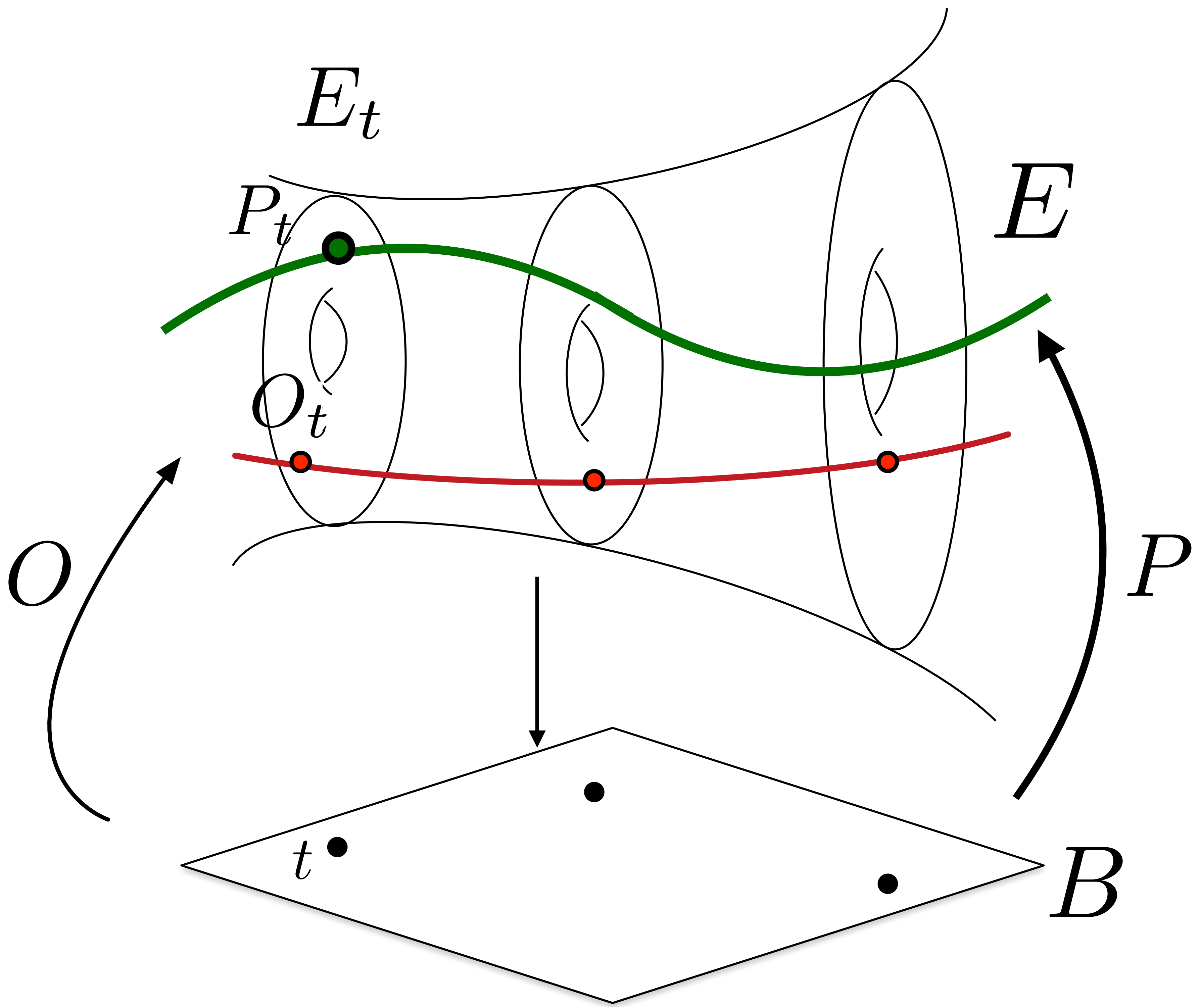

Néron-Tate canonical height

Geometric height $\hat{h}_E$ on $E(k)$

Arithmetic height $\hat{h}_{E_t}$ on fibers $E_t(\overline{\mathbb{Q}})$ for $t \in B(\overline{\mathbb{Q}})$

Theorem. (Silverman, 1983)
Assume $P \in E(k)$ is non-torsion. Then

$$\lim_{h_B(t)\to\infty} \frac{\hat{h}_{E_t}(P_t)}{h_B(t)} = \hat{h}_E(P)$$

any Weil height on base curve $B(\overline{\mathbb{Q}})$ of degree 1

Theorem. (Tate, 1983)
Assume $P \in E(k)$ is non-torsion. Then
there is a $\mathbb{Q}$-divisor D_P of degree $\hat{h}_E(P)$ on B so

$$\hat{h}_{E_t}(P_t) = h_{D_P}(t) + O(1)$$

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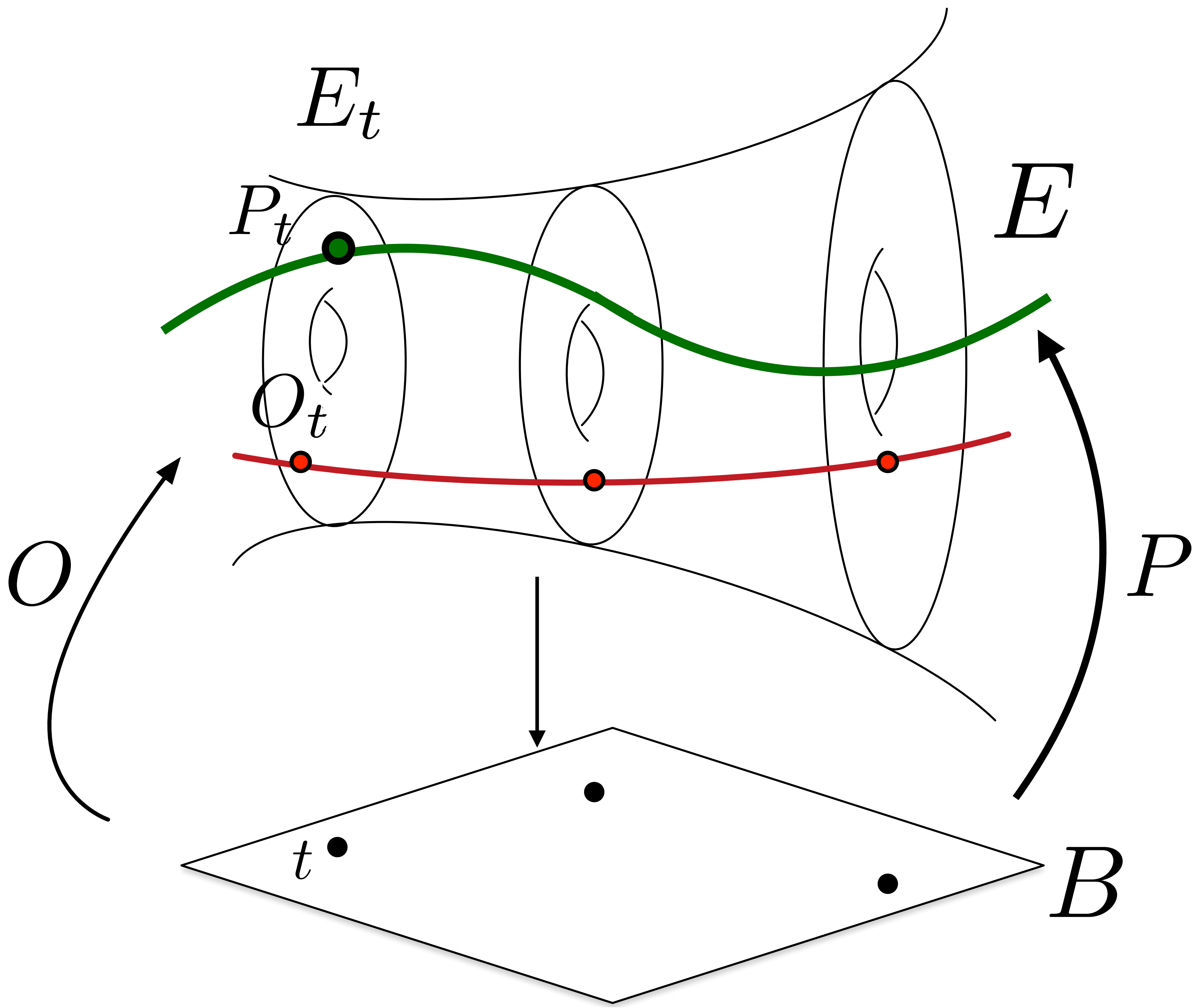

Néron-Tate canonical height

Geometric height $\hat{h}_E$ on $E(k)$

Arithmetic height $\hat{h}_{E_t}$ on fibers $E_t(\overline{\mathbb{Q}})$ for $t \in B(\overline{\mathbb{Q}})$

Fix number field K so every $P \in E(k)$ defined $/K$.
For each $P \in E(k)$, set $h_P(t) = \hat{h}_{E_t}(P_t)$ for $t \in B(\overline{\mathbb{Q}})$.

Building on Silverman & Tate, we showed a few years ago:

Theorem. (D.-Mavraki)
Assume $P \in E(k)$ is non-torsion. Then h_P is the height induced from an adelic, continuous, semipositive metric $\{\|\cdot\|_{P,v}\}_{v \in M_K}$ on ample $\mathcal{L}_P$ (associated to the $\mathbb{Q}$-divisor D_P). Moreover,

$$\overline{\mathcal{L}}_P \cdot \overline{\mathcal{L}}_P = 0$$

Arakelov-Zhang intersection number

Equidistribution theorems of Chambert-Loir, Thuillier, Yuan (2008) and work of Masser-Zannier (2008-2012)

$\Longleftarrow$

There exists $\epsilon > 0$ so that

$$\{h_P(t) \leq \epsilon\} \cap \{h_Q(t) \leq \epsilon\}$$

is finite if and only if P and Q are independent.

Main Theorem. (D.-Mavraki) Fix $E \to B$ defined over a number field. There are constants $c_1, c_2 > 0$ so that

$$c_1 \, \hat{R}_{NT}(P,Q) \quad \leq \quad \overline{\mathcal{L}}_P \cdot \overline{\mathcal{L}}_Q \quad \leq \quad c_2 \, \hat{R}_{NT}(P,Q)$$

for all P and Q in $E(k)$.

Arakelov-Zhang intersection number

Néron-Tate regulator
$\hat{h}_E(P)\hat{h}_E(Q) - \langle P, Q\rangle^2$

$$\boxed{\begin{aligned}&\overline{\mathcal{L}}_P \cdot \overline{\mathcal{L}}_Q := \\ &\quad h_P(D_Q) + \sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \int_{B_v} \lambda_{Q,v} \, d\omega_{P,v}\end{aligned}}$$

Remarks

this is well known:
Néron-Tate height is a
positive definite quad form

Our earlier theorem implies

$$\overline{\mathcal{L}}_P \cdot \overline{\mathcal{L}}_Q = 0 \iff P \text{ and } Q \text{ are dependent} \iff \hat{R}_{NT}(P,Q) = 0$$

The above theorem means that

The biquadratic form $(P,Q) \mapsto \overline{\mathcal{L}}_P \cdot \overline{\mathcal{L}}_Q$ is also nondegenerate on real vector space $E(k) \otimes \mathbb{R}$.

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why should we care?

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Observation. Geometric height is known to control growth of arithmetic height, i.e., large height phenomena. This shows that geometric height also controls small height phenomena. Via Zhang's Inequality (1995), the intersection number can be replaced with

$$(\hat{h}_E(P) + \hat{h}_E(Q)) \ \text{ess.min.}_{t \in B(\overline{\mathbb{Q}})} (\hat{h}_{E_t}(P_t) + \hat{h}_{E_t}(Q_t))$$

Our earlier result:

$$\approx \frac{\overline{\mathcal{L}}_P \cdot \overline{\mathcal{L}}_Q}{\hat{h}_E(P) + \hat{h}_E(Q)}$$

There exists $\epsilon > 0$ so that

$$\{h_P(t) \leq \epsilon\} \cap \{h_Q(t) \leq \epsilon\}$$

is finite if and only if P and Q are independent.

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for all P and Q in $E(k)$.

This proves a case of Shouwu Zhang's Conjecture (1998)

$$\Updownarrow$$

Theorem. (D.-Mavraki) Fix nonisotrivial $E \to B$ defined over a number field. Let C be any irreducible curve in the fiber product $\pi : E^m \to B$ that dominates B, with $m \geq 2$. Let $E^{m,2}$ be union of flat subgroup schemes of codimension ≥ 2. There is $\epsilon > 0$ so that the intersection of C and

$$T(E^{m,2}, \epsilon) := \left\{ p \in E^m \;\; : \; \exists \, p' \in E^{m,2}(\overline{\mathbb{Q}}) \text{ with } t = \pi(p) = \pi(p') \text{ and } \hat{h}_t(p - p') \leq \epsilon \right\}$$

is contained in a finite union of flat subgroup schemes of positive codimension.

- For isotrivial E, proved by Viada (2009), Galateau (2010).
- For nonisotrivial E with $\epsilon = 0$, proved by Barroero-Capuano (2016). $m = 2$ case with $\epsilon = 0$: Masser-Zannier (2012)
- Our previous result covered $C \cap T(E^{m,m}, \epsilon)$, the torsion part.

Compare: Pink Zilber Conjectures, Bombieri-Masser-Zannier, Raynaud (Manin-Mumford Conj.), Habegger, Rémond-Viada, Habegger-Pila, ...
Zhang, Ullmo (Bogomolov Conj.)

Main Theorem. (D.-Mavraki) Fix $E \to B$ defined over a number field.
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$$c_1\, \hat{R}_{NT}(P,Q) \quad \leq \quad \overline{\mathcal{L}}_P \cdot \overline{\mathcal{L}}_Q \quad \leq \quad c_2\, \hat{R}_{NT}(P,Q)$$

for all P and Q in $E(k)$.

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$$\Updownarrow$$

Theorem. (D.-Mavraki) Fix nonisotrivial $E \to B$ defined over a number field.
Let $P_1, \ldots, P_m$ be $m \geq 2$ points in $E(k)$ over $k = \overline{\mathbb{Q}}(B)$.
Then $P_1, \ldots, P_m$ are **linearly dependent** if and only if
there is a sequence $t_n \in B$ and small-height perturbations P'_{i,t_n} of P_{i,t_n}
so that

$$\{P'_{1,t_n}, \ldots, P'_{m,t_n}\} \subset E_{t_n}(\overline{\mathbb{Q}})$$

satisfy **at least two** independent linear relations in $E_{t_n}(\overline{\mathbb{Q}})$.

- For isotrivial E, proved by Viada (2009), Galateau (2010).
- For nonisotrivial E with $\epsilon = 0$, proved by Barroero-Capuano (2016).
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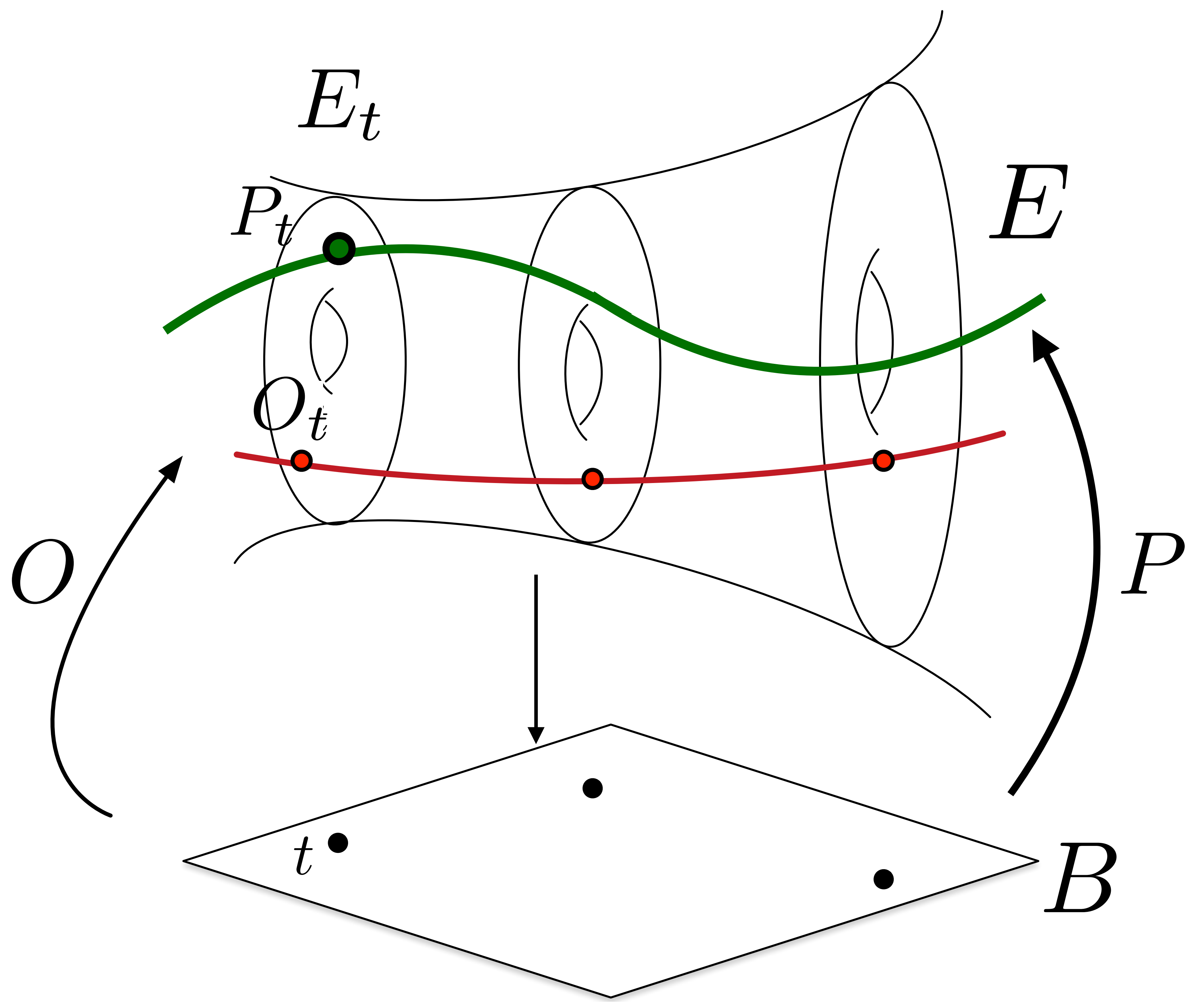

Our approach follows ideas of Szpiro-Ullmo-Zhang (1995) and Zhang, Ullmo (1998) in their proof of Bogomolov Conjecture.

Theorem 1. Every element $X \in E(k) \otimes \mathbb{R}$ gives rise to an $\mathbb{R}$-divisor D_X on B over K, equipped with an adelic, continuous, semipositive metrization satisfying $\overline{D}_X \cdot \overline{D}_X = 0$.

Theorem 2 (Equidistribution). Given any two adelic, continuous, metrizations $\overline{D}_1$ and $\overline{D}_2$ of $\mathbb{R}$-divisors on smooth projective B over K, with $\overline{D}_1$ semipositive and $\overline{D}_1 \cdot \overline{D}_1 = 0$, if $h_1(t_n) \to 0$, then

$$h_2(t_n) \longrightarrow \frac{\overline{D}_1 \cdot \overline{D}_2}{\deg D_1}$$

The proof of Theorem 2 follows the methods of Yuan, Chambert-Loir, Thuillier (2008) with the extra input of Moriwaki (2016)

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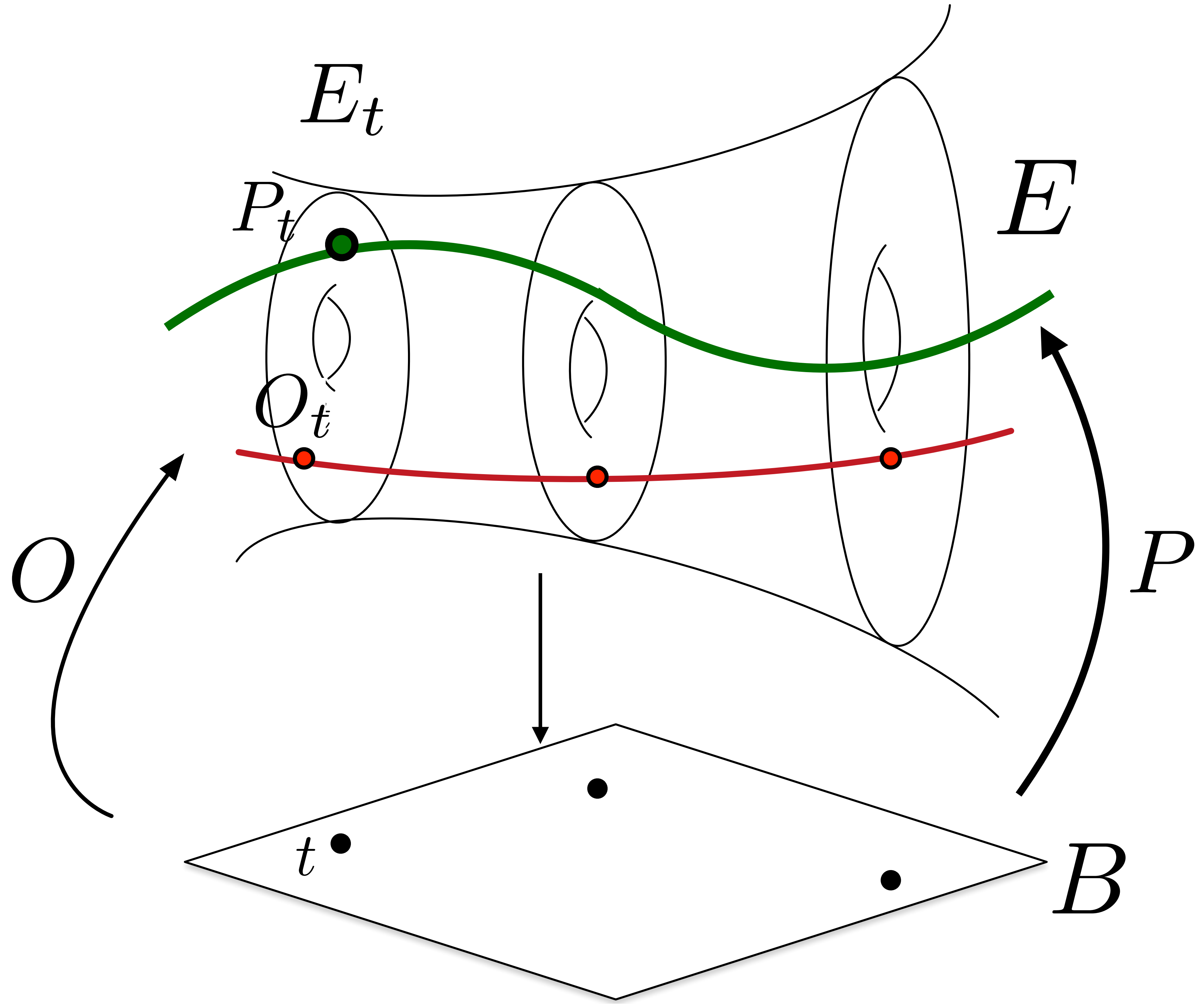

Theorem. (Barroero-Capuano, 2016)
Fix points $P_1, \ldots, P_m$ in $E(k)$
Suppose they satisfy 2 independent linear relations

$$a_{1,n}P_1 + \cdots + a_{m,n}P_m = O$$
$$b_{1,n}P_1 + \cdots + b_{m,n}P_m = O \qquad a_{i,n}, b_{i,n} \in \mathbb{Z}$$

at infinitely many points $t_n \in B(\overline{\mathbb{Q}})$.
Then $P_1, \ldots, P_m$ are dependent in $E(k)$.

Our proof.
Rearranging terms and rescaling, we have

$$a_{i,n}/A_n \to x_i \quad \text{and} \quad b_{i,n}/B_n \to y_i$$

and we get points $X = \sum_i x_i P_i$ and $Y = \sum_i y_i P_i$
in $E(k) \otimes \mathbb{R}$ with independent coefficients $\vec{x}, \vec{y} \in \mathbb{R}^m$.

$$h_X(t_n) \to 0 \quad \text{and} \quad h_Y(t_n) \to 0$$

Equidistribution then implies that $\overline{D}_X \cdot \overline{D}_Y = 0$.
Main Theorem implies X and Y must be dependent.

Theorem 2 (Equidistribution). Given any two adelic, continuous, metrizations $\overline{D}_1$ and $\overline{D}_2$ of $\mathbb{R}$-divisors on smooth projective B over K, with $\overline{D}_1$ semipositive and $\overline{D}_1 \cdot \overline{D}_1 = 0$, if $h_1(t_n) \to 0$, then

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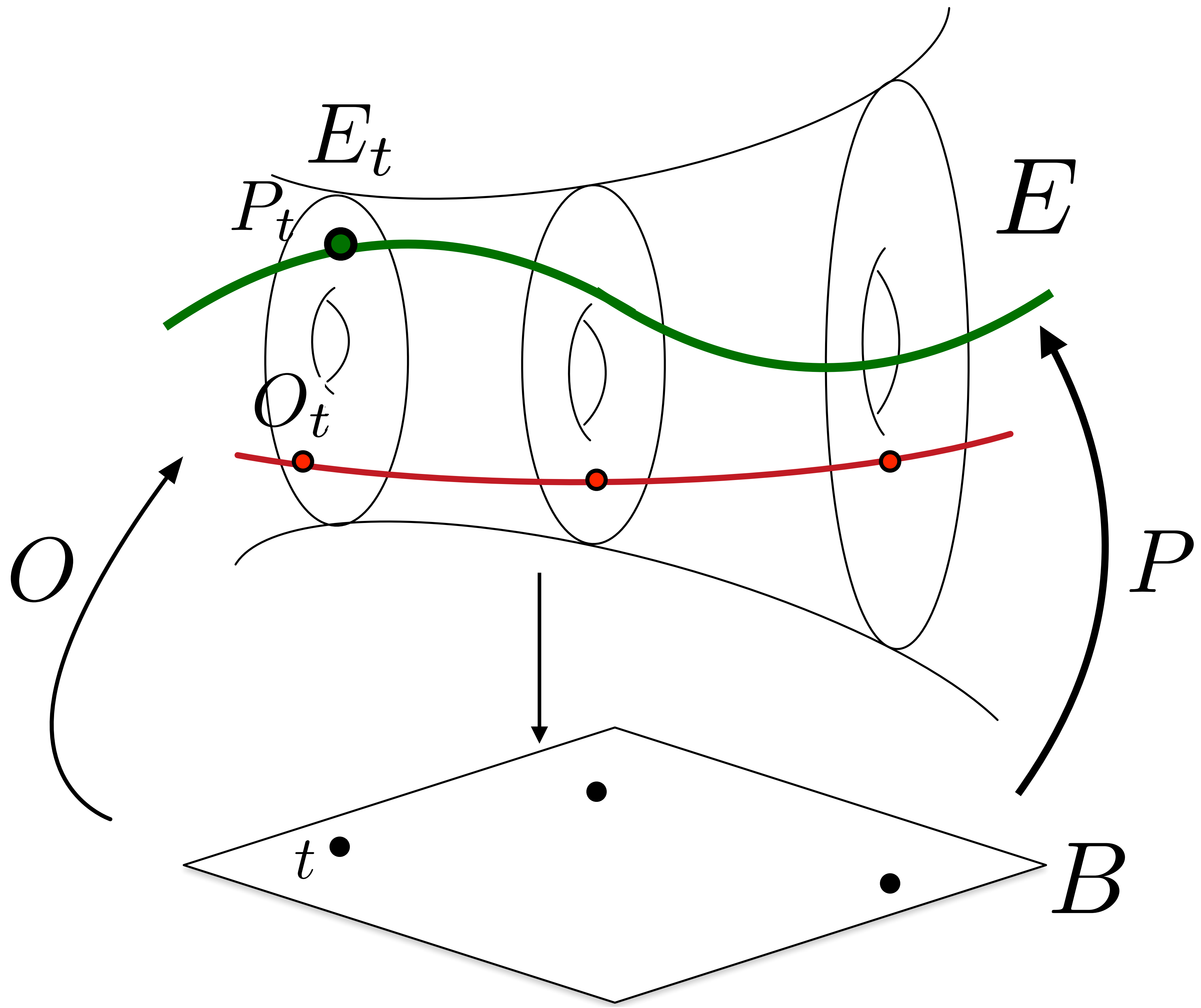
To complete proof of **Main Theorem**: show that $\overline{D}_X \cdot \overline{D}_Y > 0$ for all independent $X, Y \in E(k) \otimes \mathbb{R}$.



Theorem. (D.-Mavraki, 2020)
Assume P is non-torsion. The torsion points $\mathcal{T}_P = \{t : nP_t = O_t \text{ for some } n\}$ are equidistributed w.r.t. ω_P.

Theorem. (Corvaja-Demeio-Masser-Zannier, preprint) Then $\omega_P = d\beta_1 \wedge d\beta_2$ in the Betti coordinates $(\beta_1(t), \beta_2(t)) \in \mathbb{T}^2$ of P_t

There is a $(1,1)$-form ω on (smooth part of) E with ω_t on fibers $=$ Haar measure.

$$\boxed{\omega_P := P^*\omega = \text{curvature form on } B(\mathbb{C})}$$

$$\omega_P = 0 \iff P \text{ is torsion}$$

Final step to prove our Main Theorem:

Theorem. Fix an archimedean place v of K. For points $X, Y \in E(k) \otimes \mathbb{R}$, we have

$$\omega_{X,v} = \omega_{Y,v} \iff X = \pm Y$$

We use the holomorphic-antiholomorphic "trick" of André-Corvaja-Zannier (2020) and a transcendence result of Bertrand (1989)