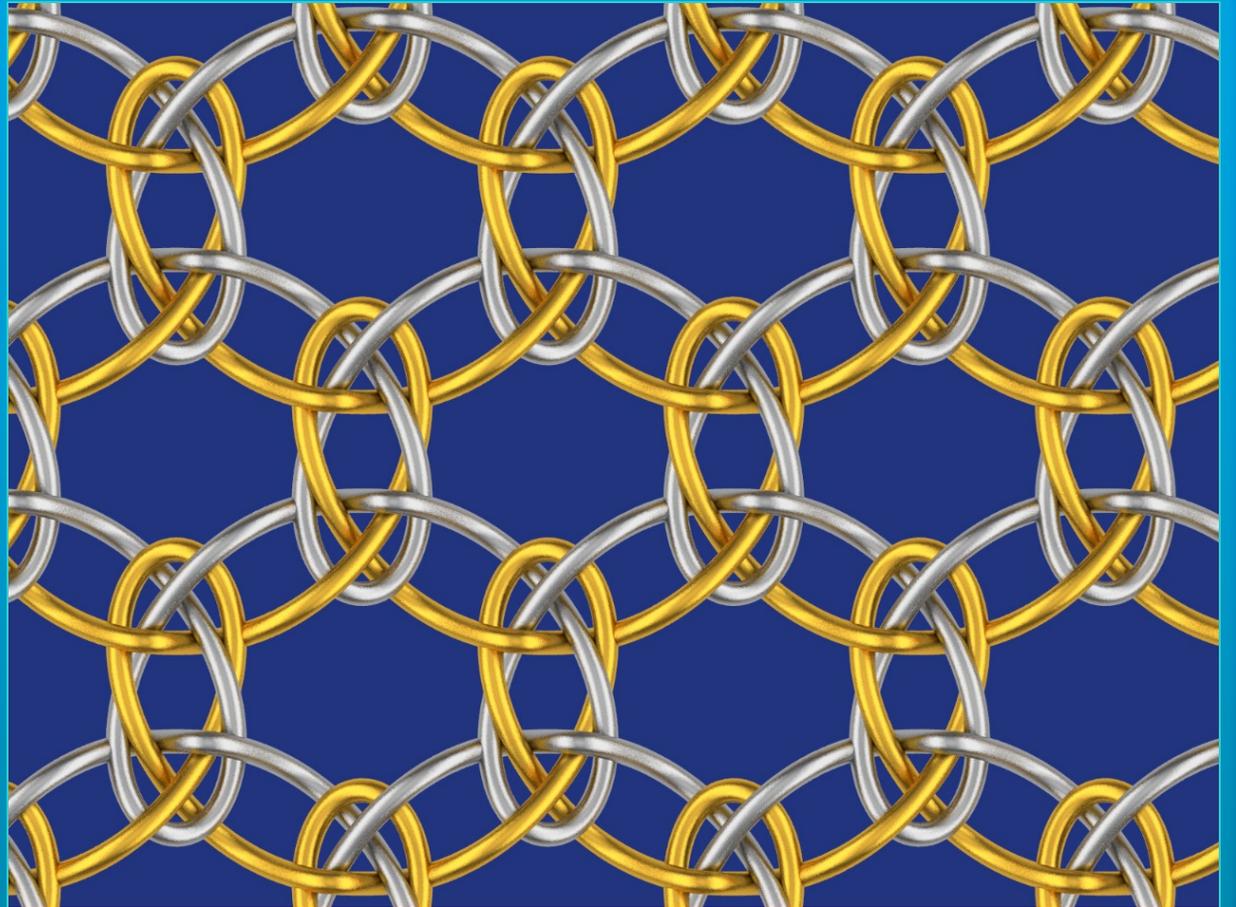


# Wallpaper Patterns from Looping Strands: The Layer Groups

Frank Farris,  
Santa Clara U

GEAR 2021 at Banff

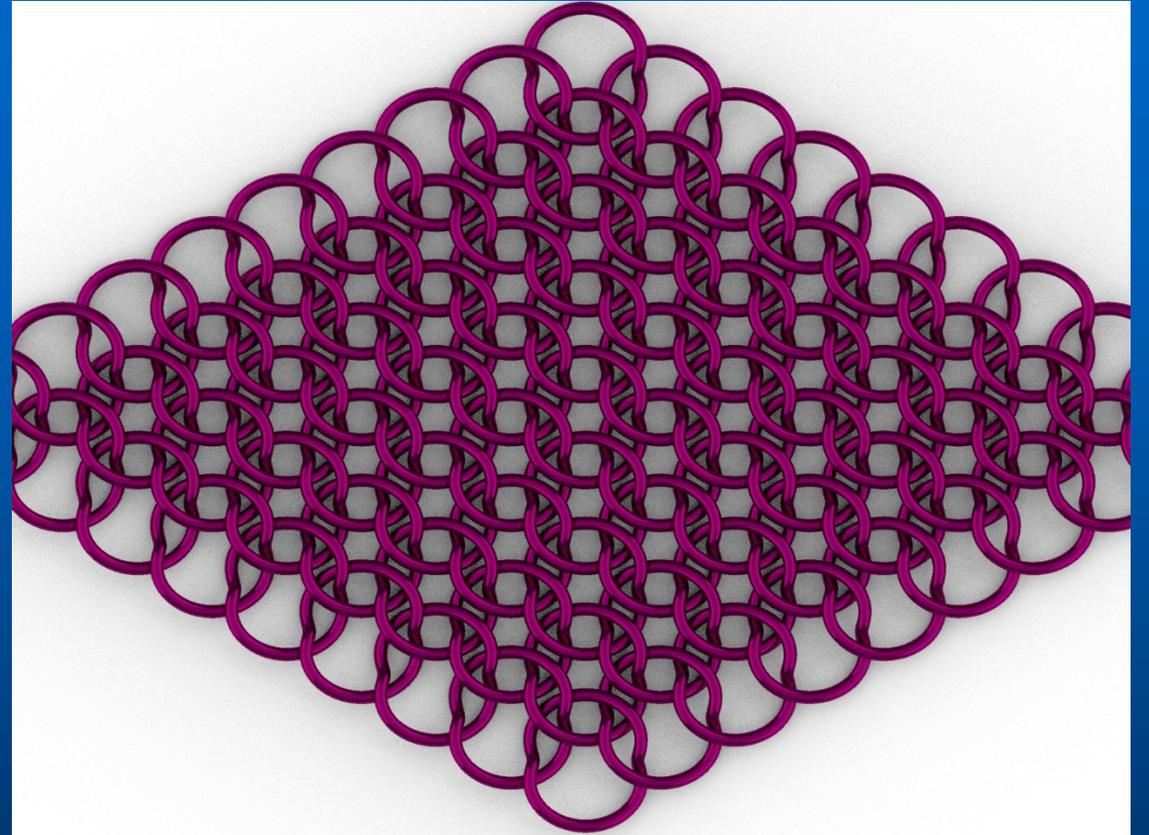
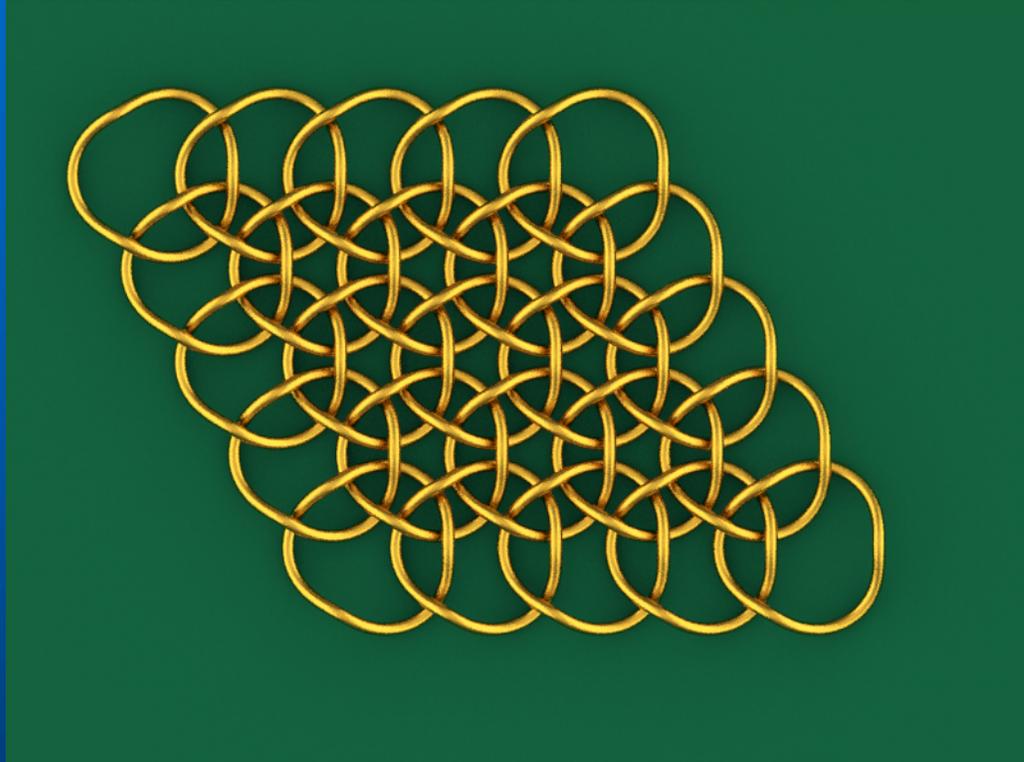


If you do anything with wallpaper patterns,  
you can generalize it to 3D.

# Before we begin

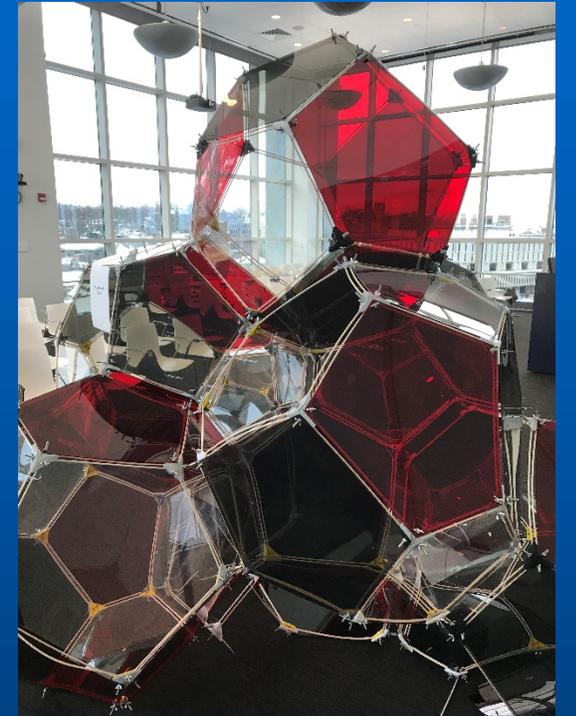
- *We pause to acknowledge that Santa Clara University sits on the land of the Ohlone and the Muwekma Ohlone people, who trace their ancestry through the Missions Dolores, Santa Clara, and San Jose. We remember their connection to this region and give thanks for the opportunity to live, work, learn and pray on their traditional homeland. (Juristac <http://www.protectjuristac.org/>)*
- The Golden Section of the MAA (Northern CA, NV, and HI) has its section meeting next weekend! Register and attend for free. Meeting held in Zoom and Gather.town

# Earlier project: Mathematical chain mail



“Wallpaper Patterns from Nonplanar Chain Mail Links” FF,  
Bridges, 2020.

# Fall '19 semester at ICERM! “Illustrating Mathematics”



# Grasshopper as Rhino plug-in

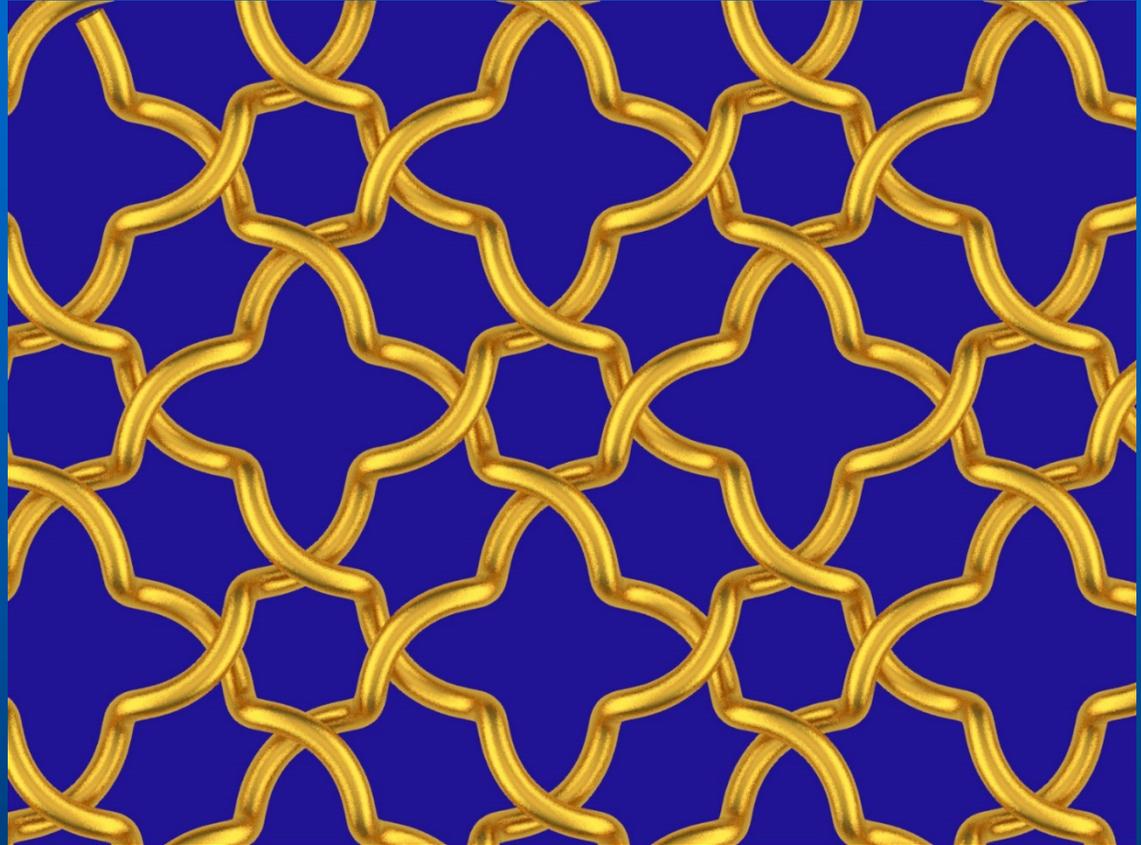
Loading Rhino Render, version 1.50, Jun 24 2020, 05:23:56

Command: \_Grasshopper

Command:

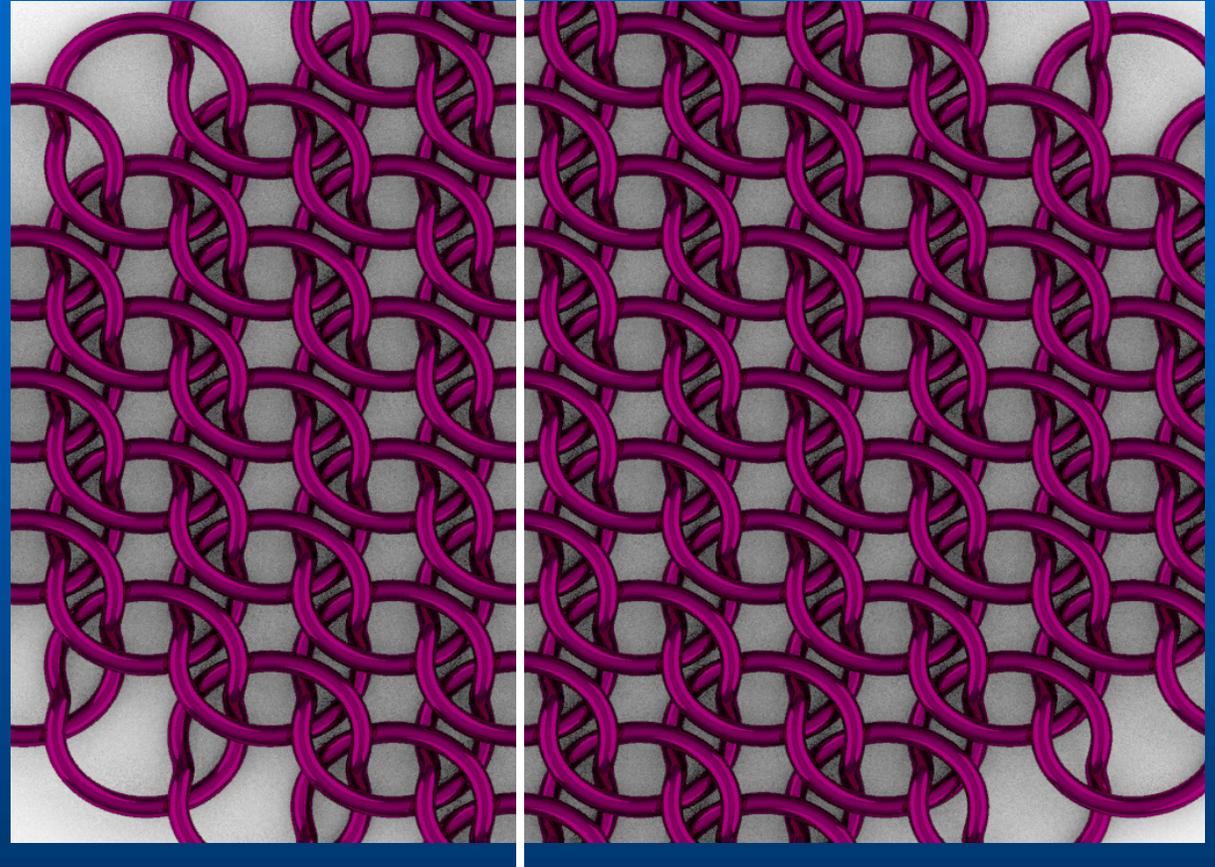
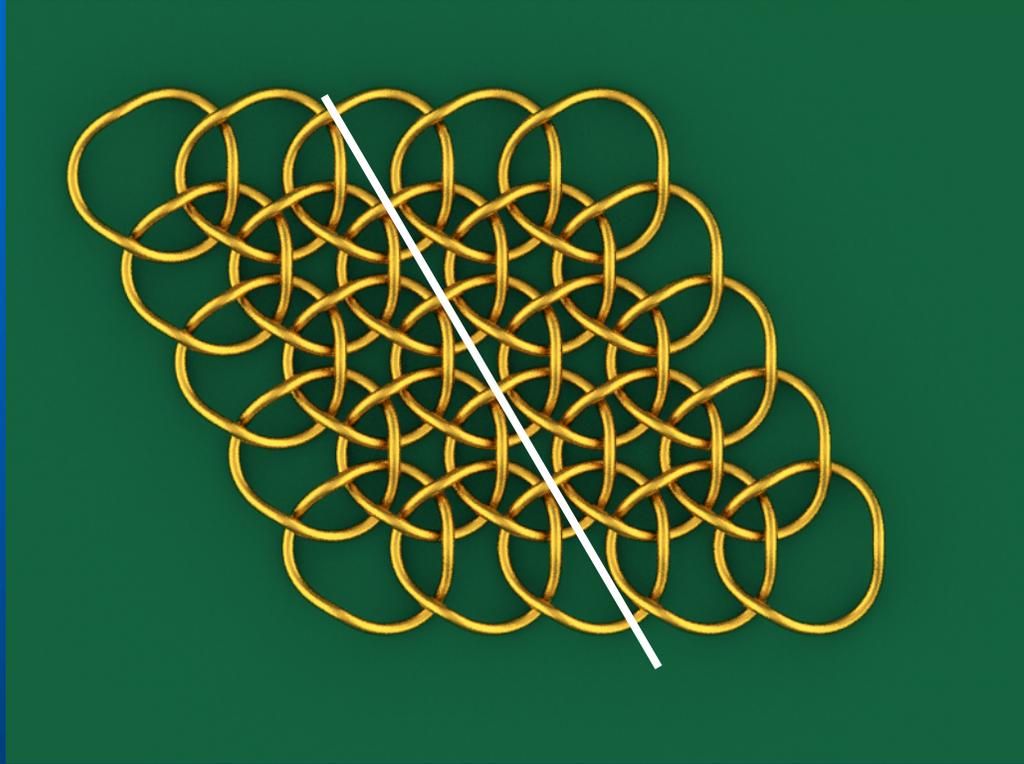
The image displays the Rhino software interface with the Grasshopper plug-in. The top menu bar includes options like Standard, CPlanes, Set View, Display, Select, Viewport Layout, Visibility, Transform, Curve Tools, Surface Tools, Mesh Tools, Solid Tools, Render Tools, Drafting, and New in V6. The Grasshopper window, titled 'Grasshopper - p3m1A1\*', shows a complex script with various components. A blue circle highlights a specific section of the script, which includes a 'N' component (value 3), a 'Radius' component (value 0.724), and a 'Pt' component. The script contains mathematical formulas for calculating points in a 3D space, such as  $((r, 0) * \exp(i * t)) .z + ((r1, 0) * \exp(i * (N+1) * (t+ph))) .r$  for the X-axis,  $((r, 0) * \exp(i * t)) .i + ((r1, 0) * \exp(i * (N+1) * (t+ph))) .i$  for the Y-axis, and  $(1 + \sin(N * (t+ph))) * r_z$  for the Z-axis. The 3D view on the right shows a perspective view of a green, braided structure, and a right-side view below it. The structure is composed of multiple intertwined tubes, creating a complex, woven pattern.

Seen last summer



What is the symmetry of this pattern?

# *Wallpaper* is not exactly the right term



These patterns are invariant under *layer group* actions.

# Wallpaper symmetry is for plane patterns



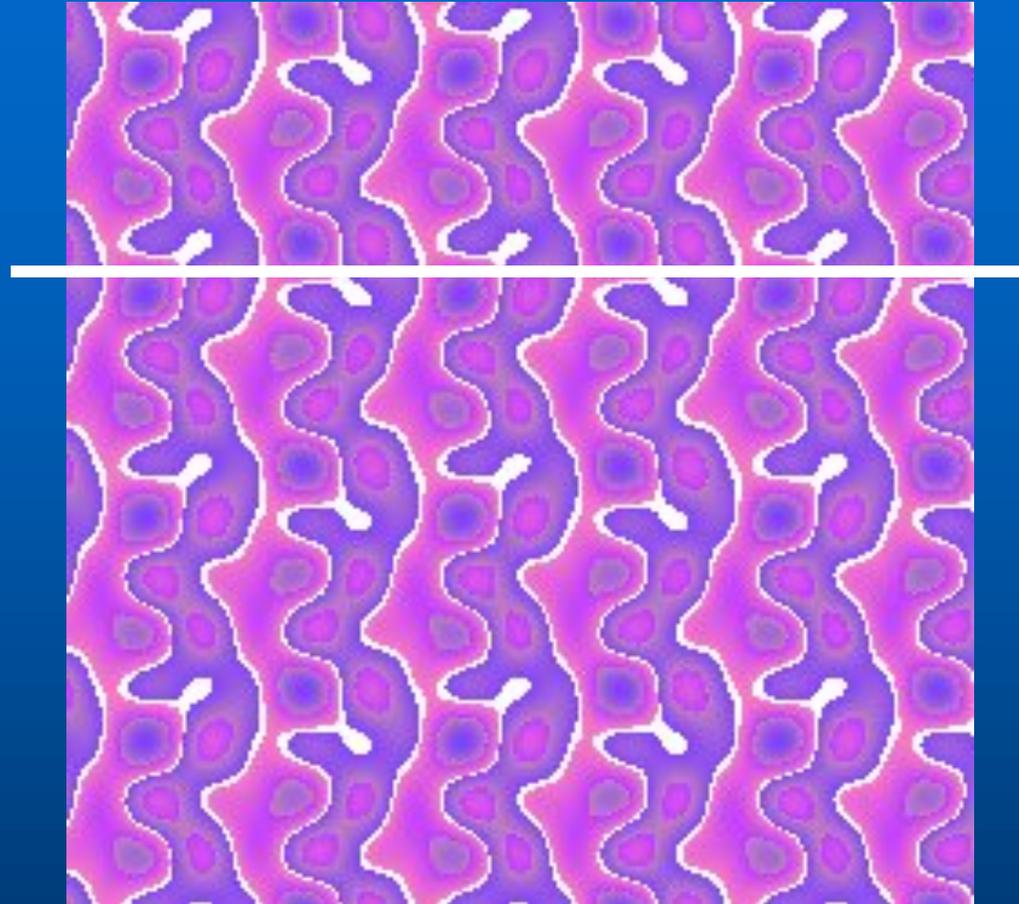
A *symmetry* of a pattern is a transformation that leaves it unchanged.

The symmetries of a pattern form a *group*. (Closed under composition)

There are only 17  
(isomorphism classes of)  
wallpaper groups!

This pattern has symmetry group  $pg$ .

# Wallpaper symmetry is for plane patterns

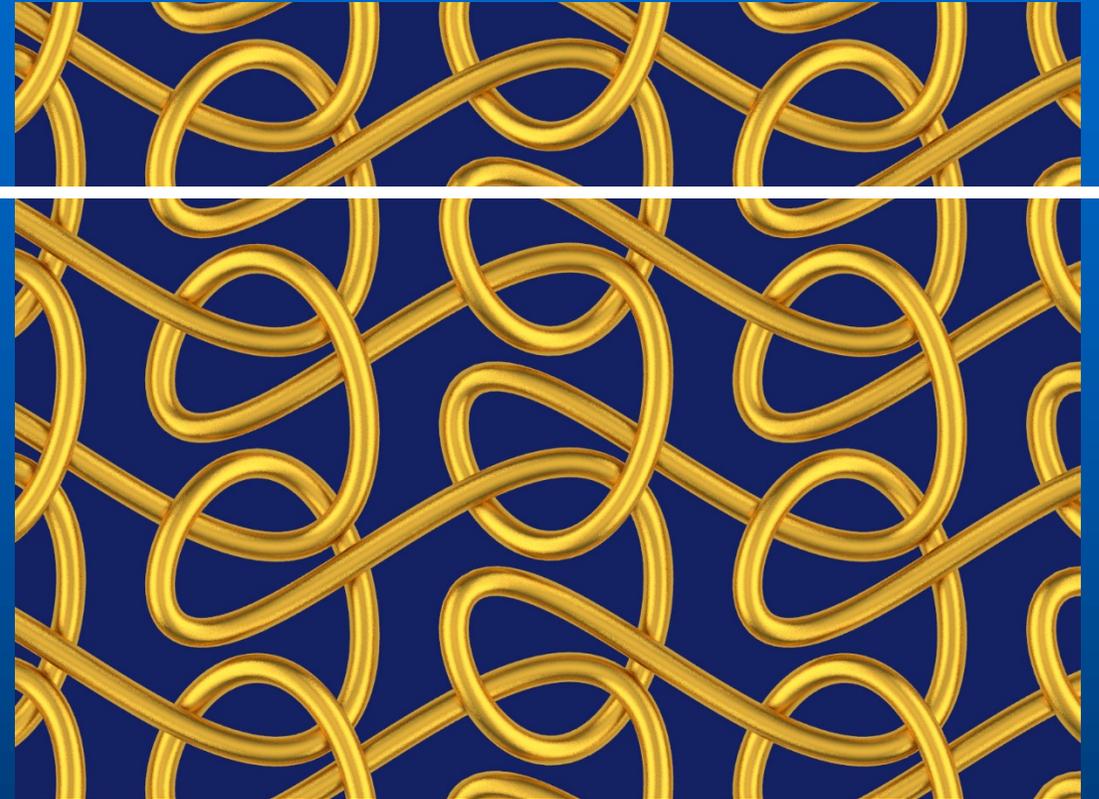
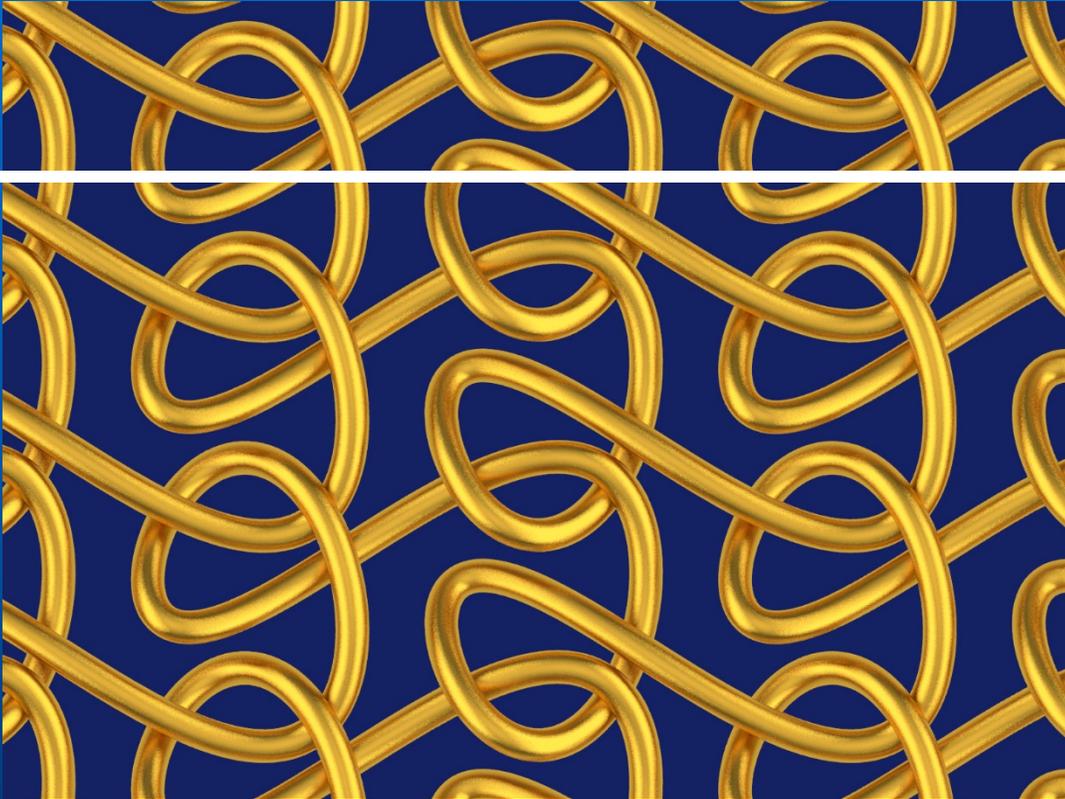


Cited in Wikipedia  
article "Layer Groups"



A pg pattern from "Vibrating Wallpaper" FF, 1998, CVM.

# 3D patterns have “layer group” symmetry



Two different layer group extensions of wallpaper group  $pg$

# From wallpaper groups to layer groups

- A *wallpaper group* is a gp of Euclidean isometries of the plane whose translations can be generated by 2 linearly independent translations. (17 isomorphism classes.)
- A *layer group* is a gp of Euclidean isometries of space whose translations can be generated by 2 linearly independent translations (and hence has an invariant plane). 80 of these!
- The restriction of a layer group to its invariant plane is a wallpaper group.

# Wallpaper group notation

- Individual transformations are written in complex notation.
- For instance, the glide reflection in our examples is

$$\gamma_x(z) = \bar{z} + 1/2$$

- Groups are named by the International Crystallographic Union symbols, such as p3m1 (\*333 in orbifold notation).

# Extending transformations to space

- The trivial extension of a planar transformation is

$$\check{\alpha}(z, w) = (\alpha(z), w)$$

- The only other possible extension is a composition with

$$\sigma_z(z, w) = (z, -w)$$

- Call that the *flip extension*.

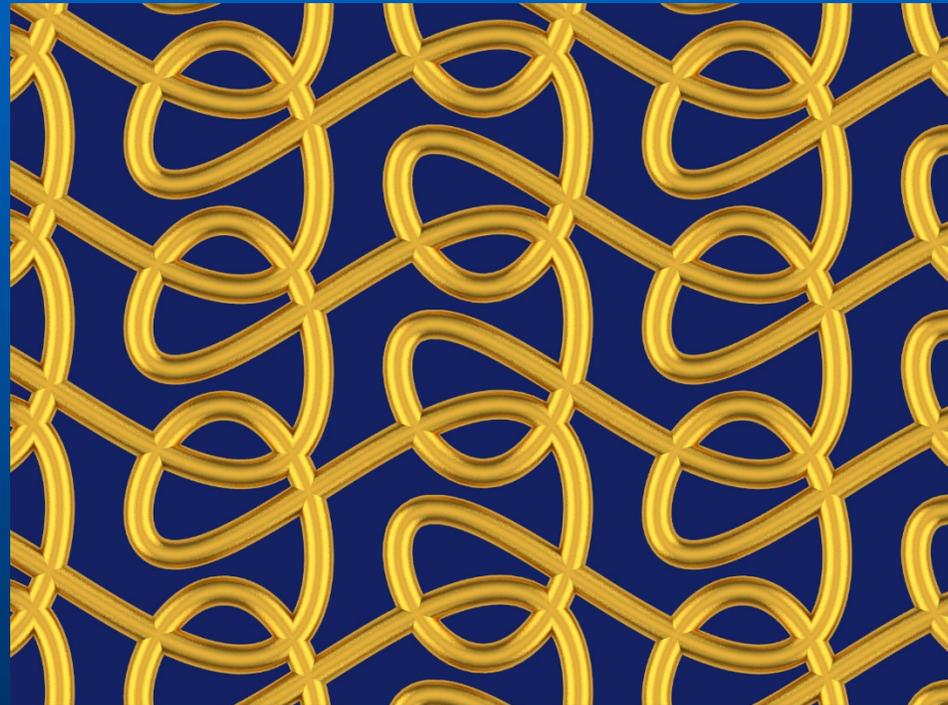
# Counting the 80 layer groups

- Each of 17 wallpaper gps has a trivial extension.
- Each of 17 wallpaper gps has a double extension.

The pg pattern flattened.

If round on top, flat on bottom: trivial extension of pg

If round on top and bottom: double extension of pg



# That leaves 46 layer groups

- Theory exactly the same as that of color symmetry
- To extend  $G$ , tag some elements for trivial extension, some for flip extension.

Given a homomorphism

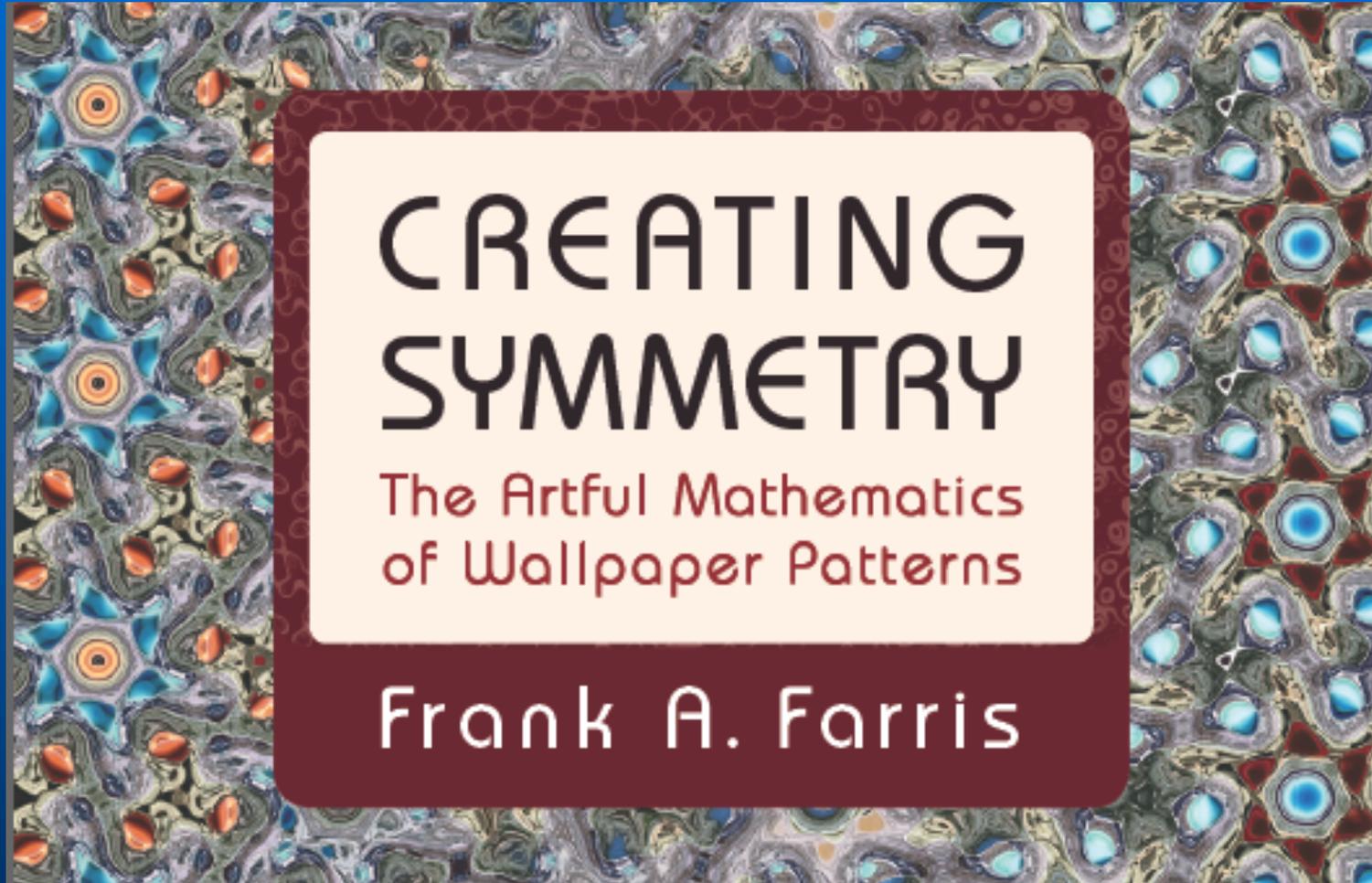
$$\phi : G \rightarrow \{0, 1\}$$

Extend elements of  $G$  by

$$g \rightarrow g \circ \sigma_z^{\phi(g)}$$

There are 46 equivalence classes of such homomorphisms

Read about color symmetry here:

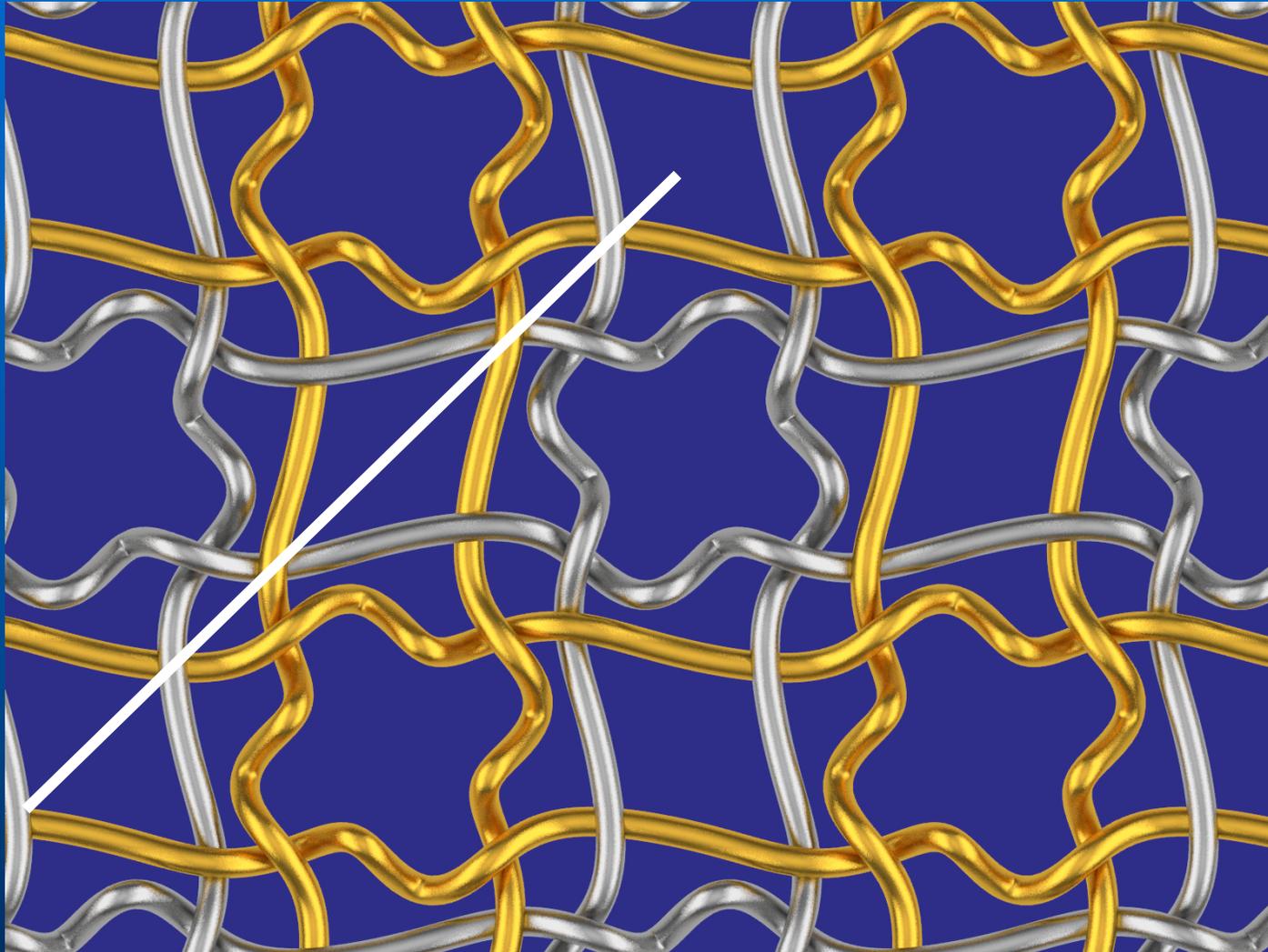


# Working through an example

Turn gold strands over to create silver strands

New sym. gp is an extension of  $p4g$

Coxeter notation  $p4g/p4$



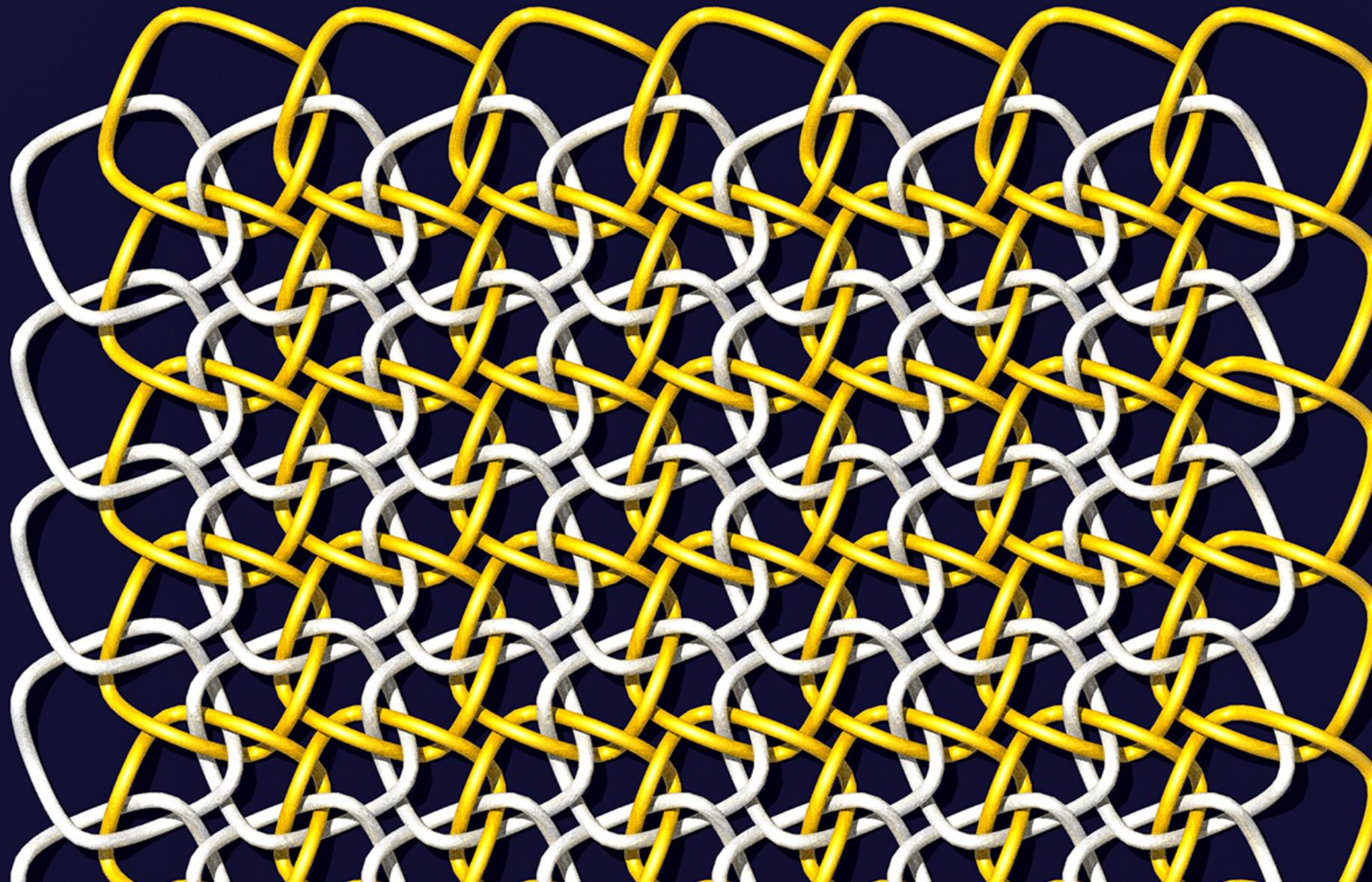
Focus on gold strands

Symmetry group is  $p4$

$p42_12$

in IUC notation

The same  
symmetry  
type in chain  
mail



# Tables to translate between terminologies

Supplement  
to Bridges  
'21 paper

Post to  
Wikipedia?

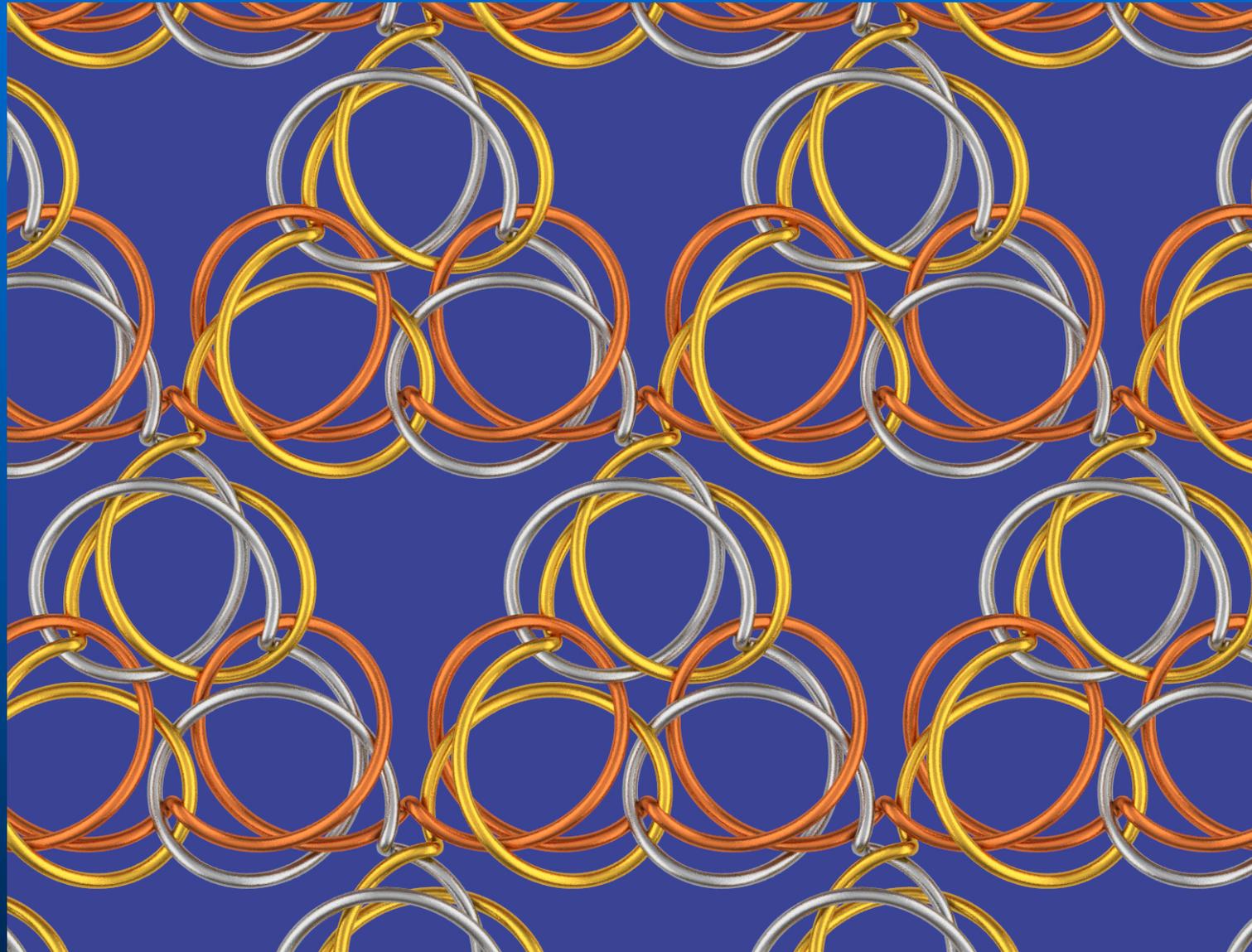
**Table 3:** *A key to the layer groups, part 3: lattices with higher symmetry*

Lattice	Group #	IUC Symbol	Wallpaper group interpretation
Tetragonal	49	$p4$	$p4$
	50	$p\bar{4}$	$p4/p2$
	51	$p4/m$	$p4 \times \mathbb{Z}_2$
	52	$p4/n$	$p4/p4$
	53	$p422$	$p4m/p4$
	54	$p42_12$	$p4g/p4$
	55	$p4mm$	$p4m$
	56	$p4bm$	$p4g$
	57	$p\bar{4}2m$	$p4m/cmm$
	58	$p\bar{4}2_1m$	$p4g/cmm$
59	$p\bar{4}m2$	$p4m/pmm$	
60	$p\bar{4}b2$	$p4g/pgg$	

# See you at Bridges 2021?

Each strand  
made with  
Fourier series  
that has the  
desired  
symmetry

Wiggle  
parameters  
until the  
pattern  
weaves



“Rod”  
symmetry  
 $p1m1/p111$

# Out of left field: A 7-color torus

