The Mathematics of Layers and Interfaces

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1 Overview of the Field

It has long been known that natural systems can, in some circumstances, host "layers" – structures that are statistically invariant in some directions and vary rapidly in others. This meeting was dedicated to the mathematical study of naturally layered systems.

One of the most common types of layered systems in geophysics are "thermohaline double-diffusive staircases", observed in the tropical and arctic oceans, and in volcanic lakes. The name staircase arises from the typical temperature and salinity profiles observed, whereby both fields increase or decrease step-wise with depth. The density of water, which depends on temperature and salinity, increases step-wise with depth accordingly, so that while the system is overall stably stratified, it takes the form of convective layers separated by thin, relatively stable interfaces. These staircases are called double-diffusive because they can occur more generally in any fluid whose density depends on two different quantities that diffuse at different rates[1] such as for instance a laboratory-mixed solution of salt and sugar, or a gas whose density depends both on temperature and composition. Density staircases have also been observed to form in stably stratified fluids even when density only depends on one quantity (e.g. salt for instance), under the influence of weak yet persistent mechanical mixing.

Further afield, layers are also observed in astrophysical objects, most notably in the giant planets of the solar system. Observations of the surface weather patterns of these planets reveal banded structures, with azimuthal winds of alternating directions (when viewed in the frame of reference rotating with the planet). Interestingly, graphs of a quantity proportional to the latitudinal derivative of the azimuthal velocity (namely the potential vorticity) also reveal the presence of staircases, with wide regions of nearly-uniform vorticity separated by thinner regions where the vorticity changes very rapidly. Despite arising in very different circumstances, the vorticity staircases have remarkable similarities with the double-diffusive ones: the steps are often roughly equidistant, and reveal thick turbulent and well-mixed layers, separated from one another by very thin interfaces whose dynamics are much less turbulent and act as a barrier to transport.

Finally, layers and interfaces are also seen in other natural applications such as magnetized plasmas and material science for example.

[1] In the case of salty water, the diffusivity of salt is about 1/100 of that of temperature.

2 Open Problems

The ubiquitous nature of staircases suggest that much might be learned by comparing the mathematical models used to describe their formation and evolution. However, this approach has not been attempted to date because the fields of astrophysics, ocean sciences, limnology, plasma physics and material science rarely interact. Progress in one area, whether numerical, analytical or experimental, rarely permeates to the other. The goal of this workshop was to gather applied mathematicians studying these problems, as well as experts in the respective application areas, to address some of the outstanding questions concerning staircases that are common to these fields. These questions we addressed were:

1. *How do layers and interfaces form?*

   Various mechanisms have, over the years, been proposed to explain the spontaneous or mechanically-driven formation of buoyancy and potential vorticity staircases. Common to both planetary jet formation and thermo-compositional staircases is the Phillips mechanism (Phillips 1972) and related theories, which invokes anti-diffusive mixing to amplify any initial horizontally invariant perturbation. Alternative ideas include wave-mean-flow interactions and large-scale instabilities, both of which effectively drive an inverse energy cascade that feeds energy from small-scale forcing into the largest-scale, horizontally invariant modes of the system (the layers). To date, however, the reasons for layer formation remain under debate, as no universally-accepted theory exists even within a given subject area.

2. *What sets the properties of a layered system?*

   Staircases are characterized by the layer and interface thicknesses, the relative increments of buoyancy/potential vorticity across each interface, and by the overall transport rate across the system. These quantities are related if the staircase is in equilibrium, as the rate of transport across a layer must be equal to the rate of transport across an interface, each of which depends on their thicknesses and the processes taking place within. To model a staircase, one must therefore model mixing within the layers, mixing across an interface, as well as the interplay between the two. This can be done using a number of techniques including dimensional analysis, asymptotic analysis, weakly nonlinear theory, direct statistical simulations, numerical simulations and laboratory experiments. However, a universally-accepted, "first-principles" theory for the structure of a staircase does not yet exist.

3. *What determines the evolution of a layered system?*

   Thermohaline staircases in the ocean, and potential vorticity staircases in giant planets appear to be stable on very long timescales, unless they are mechanically disrupted by strong storms. Layer mergers are rare, once the staircase is established. However, little is known about how a staircase evolves from formation to this equilibrium configuration. Furthermore, experimental and numerical evidence pointing to the existence of unstable staircase configurations also exists in some circumstances. In these unstable systems, layers progressively merge until the system is fully mixed. Under which conditions are there steady staircases, and under which conditions are they stable/unstable? What drives the evolution of a staircase, and what sets its evolution timescale? Do layers tend to merge, disappear or split? Is this evolution similar in the case of potential vorticity staircases and buoyancy staircases? Again, these questions still await a definitive answer.

4. *What kinds of PDEs are most appropriate for modeling staircases?*

   In an attempt to understand the formation and structure of the various kinds of staircases listed above, one may wish to model them mathematically using PDEs that support layer-like or staircase-like solutions. Possible well-known candidates are the Cahn-Hilliard equation or the scalar Ginzburg-Landau equation, that are more appropriate in some cases (e.g. staircases in stratified fluids, phase transitions) than others (e.g. double-diffusive and vorticity staircases). One of the goals of this meeting was to determine whether new formalisms or new PDEs can similarly be created to model the remaining types of staircases.

3 Presentation Highlights

These are selected highlights of the presentations given during the meeting, day by day.

3.1 Monday

Monday's session started the meeting with a broad overview of double-diffusive layering in fluids, both from an observational perspective (i.e. what is known about double-diffusive layers in natural systems) and from a theoretical perspective (i.e. what mechanisms have been proposed to explain double-diffusive layering).

In the first 3 talks of the morning, Raymond (Ray) Schmitt (Woods Hole Oceanographic Institution), Mary-Louise Timmermans (Yale) and Alfred Wüest (EAWAG), presented reviews on thermohaline staircases, i.e. staircases where both the temperature and salinity profiles increase or decrease step-wise with depth.

Ray Schmitt presented the results of a series of field measurements taken in the western tropical Atlantic, where hotter salty surface water overlays colder fresh deep water, in a manner that is unstable to the fingering form of double-diffusive convection. The measurements clearly show the existence of a very tall staircase spanning a total of a few hundred meters, with individual steps ranging from 10-40 meters high, and interfaces about 1 meter high. The steps are fully convective, while the interfaces are mildly turbulent and probably undergoing active fingering convection. He reviewed a recent theory for the formation of these staircases proposed by Radko (2003), which attributes them to the presence of the so-called γ-instability. This instability, which turned out to become a recurring feature of many talks at the meeting, operates whenever γ, the ratio of the heat to salt flux induced by fingering convection, is a decreasing function of the density ratio (the ratio of the density gradient due to the temperature stratification, to the density gradient due to the salt stratification). Ray compared some of the predictions of the theory with the field results, and found good agreement. He also reported on field results on the mixing of passive tracers, released in the staircase.

An interesting comparison of these results could then be made with those presented in the talk of Alfred Wüest, who showed field measurements of staircases that occur when colder fresh water overlays warmer deep water, a situation that occurs in volcanic lakes. This kind of stratification is linearly stable but nonlinearly unstable to the "diffusive" form of double-diffusive convection. There, the staircases show turbulent steps that are much smaller (a few meters in height) and laminar interfaces that are even smaller (ten centimeters at most) and very stable. By comparing double-diffusive staircases observed in different lakes (Lake Nyos, Lake Kivu and Powell Lake), he was able to look at how different global properties of the staircase (e.g. the mean temperature and salinity gradients, for instance) affect the local properties of the steps and interfaces. Interestingly, he found that in two of the three cases the staircase was very stable, while in the third case (that of Lake Nyos), the staircase evolves relatively rapidly with time, appearing and disappearing over the course of a few years only.

Mary Louise Timmermans, finally, presented field measurements of the properties of various double-diffusive layers that are observed under the sea-ice in the Arctic ocean. Remarkably, no less than 3 types of staircases/layers are seen: intrusive layers due to lateral advection of warm atlantic water, a stable double-diffusive staircase above that, very similar to the one found in lakes, and another geothermally-heated double-diffusive staircase much deeper down. She discussed the salient properties of each set of layers, focussing in particular in their role in the exchange of heat between the surface and the warm atlantic intrusion, and the possible consequences for melting the sea-ice.

The close succession of these three talks was particularly effective in generating discussions, in which the meeting participants attempted to determine what salient similarities and differences the various kinds of staircases had. It was generally agreed that despite some superficial similarities, the formation mechanism, structures, evolution, and overall transport properties of "fingering"-type staircases and "diffusive"-type staircases were actually all quite different. In particular, while the γ-instability mechanism that appears to be quite successful at explaining the fingering staircase, cannot work as such to explain the formation the diffusive staircases.

Later on in the same day, Francesco Paparella (University of Salento) argued that there may in fact be an alternative explanation for the formation of fingering staircases. He presented numerical results which show the apparent self-organization of double-diffusive fingers into larger structures whose transport properties are notably different from the basic fingering convection. Measuring the relationship between the local turbulent fluxes and the local buoyancy gradient he found that in the presence of these structures the fingering convection acted in an anti-diffusive manner. This kind of idea is reminiscent of the Phillips mechanism normally discussed in the context of singly-stratified fluids (Phillips 1972) and was also a recurrent theme of the meeting. Anti-diffusion naturally leads to the formation of layers from small initial perturbations. This talk generated interesting discussions as to the relationship between this proposed mechanism and Radko's mechanism (Radko 2003), and as to the nature of the observed self-organization of the fingers.

Jo Fawna Reali (a graduate student at UC Santa Cruz), also discussed another alternative staircase formation mechanism, albeit for *sedimentary* fingering convection rather than thermohaline fingering convection. In sedimentary fingering convection, sedimenting particles (such as clay, organic matter, etc..) take the place of salt, but as long as the speed of sedimentation is not too large compared with the typical velocity of fingering structures, the dynamics of fingering convection are relatively unchanged. Curiously, however, she argued using semi-analytical work and direct numerical simulations that layer formation is much more ubiquitous in the sedimentary case that in the normal thermohaline case. In particular, layer formation is possible even when the system is otherwise stable to Radko's γ-instability (Radko 2003). Her work adds yet another candidate for layer formation to the growing list discussed at the meeting.

Meanwhile, David Hughes (Leeds University) took us to more astrophysical applications, and introduced the very peculiar phenomenon of magnetic double-diffusive convection. With a judicious change of variables, the governing equations can be made to look exactly like those for thermohaline double-diffusive convection, suggesting that magnetic layering may occur. In fact, the analogy with the thermohaline system is so strong that Radko's γ-instability directly applies, in the right regions of parameter space. David then presented numerical simulations of 2D magnetic double-diffusive convection, that clearly show layer formation.

Finally, Monday also saw two additional talks on double-diffusive layers, by Grae Worster (University of Cambridge) and Bruce Sutherland (University of Alberta). Bruce discussed the propagation and dissipation of gravity waves through a density staircase. Indeed, internal gravity waves are generated in a number of different ways in the ocean, in particular by wind-induced forcing and by tides flowing over bottom topography. Bruce attempted to determine whether these waves can pass, or at least partially pass through a staircase, given that they are evanescent in the well-mixed layers. He found that waves whose wavelengths are larger than individual steps can pass more-or-less unimpeded through the staircase, but short wavelength waves are reflected instead. Grae, finally, presented a very interesting talk on the formation of inverted fresh water pools just below the Arctic sea-ice, and discussed their role in controlling the rate of sea-ice melt. He reported on a series of laboratory experiments and theoretical models of these pools, whose lower interface with the Arctic water is a strong double-diffusive interface similar to the ones discussed earlier during the day. He outlined a model for the structure of the interface. Additional complications arising from the non-monotonicity of the equation of state for water, together with the need to model the melting/solidification of ice, makes this question a formidable problem indeed.

3.2 Tuesday

The Tuesday morning session continued Monday's discussion on layer formation by double-diffusive convection (talks by Nic Brummell and Neil Balmforth), then moved on to discuss layer formation resulting from the effect of turbulent mixing in stratified fluids (talks by William Young and Richard Peltier).

1. Nic Brummell (UC Santa Cruz): *2D or not 2D (in Fingering Double-Diffusive Convection)*

   The talk by Nic Brummell examined the less-well understood circumstance of fingering convection in astrophysical flows for which $\mathrm{Pr} \ll 1$ and double diffusion is the result of increasing heat and mass (eg due to levitation of heavy elements) with radius within a star. Earlier work of Radko (2003,2010) showed that momentum diffusion of the rising and falling fingers collectively act to transferred their momentum to horizontal shear bands of alternating rightward and leftward horizontal flow. The presence of these bands then act to wipe out the fingers, which turns off their driving force. Fingering then kicks in once more and the bands strengthen again through a relaxation oscillation. But all of this is the result of simulations restricted to two dimensions.

   Only recently have computers become sufficiently powerful to simulate the dynamics in three dimensions. But herein lay contradictory results. Whereas Radko's group (2012) suggested that shear bands do not develop in a 3D simulation, others had found there was no difference between the two cases.

The high resolution simulations presented by Nic showed that Radko's prediction was correct: in a sufficiently wide domain shear bands do not develop. On the one hand, this is a discouraging result because it means one must employ computationally expensive 3D simulations in order to gain a better understanding of convective processes in stars. But Nic had some good news. If the domain was just twice as wide as a finger width, the simulation behaved quantitatively the same as a simulation in a very wide domain. So one does not have to go to the full expense of a cubical domain in order to gain insight into double diffusive processes in stars.

The study of double diffusion in astrophysics promises many more fundamental breakthroughs as the wide parameter space is explored more thoroughly. As a tantalizing example of this, Nic also presented the results of 2D and 3D simulations of fingering in circumstances where the instability is just above critical, meaning that the density change associated with the background temperature gradient was moderately larger than that due to the background compositional gradient. In 2D the fingers merged to form "fat" fingers, about 10 times the size of the individual fingers. However, in 3D the fingers excited internal gravity waves that modulated the evolution of the fingers. The consequent energy spectrum measured in the two cases differed significantly with energy in 2D saturating at the fat-finger scale and the (lower) energy in 3D saturating at the domain-scale of low mode internal gravity waves.

2. Neil Balmforth (University of British Columbia): *Slots and Staircases*

   Although double-diffusive geophysical flows (with $\mathrm{Pr} > 1$) evolve to form density staircase structures, Neil showed that staircase formation is generally inhibited for double-diffusive convection in a porous medium. The only exception was a study in which a salt-stratified fluid was heated from below. In this case the density profile evolved to form step structures, as had been observed in experiments by Huppert and Linden (1979).

   In a separate study, Neil examined the evolution of a staircase subjected to background shear. This is an extension of the Taylor-Caulfield instability (Taylor 1931, Caulfield 1994, Caulfield et al. 1995), which examined the evolution of a uniform shear spanning two density steps. With multiple density steps, the shear could mediate resonant interactions between different pairs of interfaces leading to a complex modulation of the interfaces. With some cuddlefuddling, one might believe that these processes could influence the development as well as evolution of observed thermohaline staircases that are coincident with shear.

3. William (Bill) Young (Scripps): *Layering and Mixing in Stratified Flows*

   Another way to form layers in a stratified fluid is through mechanical mixing rather than through convection. Several laboratory experiments have shown that a vertical rod that repeatedly moves horizontally back and forth along a tank filled with salt-stratified fluid acts to mix the fluid so that a staircase develops. At early times, there are many small steps with small density jumps between them. Adjacent pairs of steps merge over time leaving successively wider steps with larger density jumps in between. The basic understanding of these dynamics is due to Phillips (1972) who posited that density staircases should develop if the turbulent buoyancy flux decreases in response to increasing background density gradient. The model was improved by Balmforth et al. (1998) in an attempt to put limits on the steepening of interfaces and to introduce a high wavenumber cut-off. The model included consideration of the evolution of energy as well as buoyancy. The model reproduced the development of multiple small steps that then go on to merge.

   Bill proposed a simpler model that included the additional dynamics of Balmforth et al. (1998), but expressed the evolution equation in terms of a single variable: a generalized Cahn-Hilliard equation. The proposed framework had the minimum set of necessary ingredients to give the development of staircase and step-merger.

4. W. Richard (Dick) Peltier (University of Toronto): *Stratified Turbulent Layers, Diapycnal Diffusivity, and the Low Frequency Variability of the MOC*

   Dick presented observations and simulations that explained the relaxation-oscillations associated with fast time-scale Heinrich events (involving the rapid calving of icebergs from the eastern flank of the Laurentide ice sheet during past ice ages) and the slow time-scale evolution of the Meridional Overturning Circulation in the Atlantic Ocean. Despite the global ocean and millennial time scale of the problem, the understanding of the phenomena was shown to rely sensitively upon good representation of diapycnal diffusivity in the ocean including the effects of shear.

The Tuesday afternoon session of the conference continued from the morning one, focussing on instability, turbulence and mixing in stratified flows, with the inevitable appearance of layer-interface dynamics. Four talks were presented, by Karan Venayagamoorthy, Paul Linden, Cristobal Arratia and Colm-cille Caulfield.

1. Karan Venayagamoorthy (Colorado State University): *On the prediction of turbulent diapycnal mixing in stably stratified geophysical flows*

   Karan discussed the appropriate modelling of turbulent diapycnal irreversible mixing in stably stratified geophysical flows, with particular focus on modelling mixing in the oceans. He discussed critically classical methods used to infer mixing from standard density profile measurements in the ocean using Conductivity-Temperature-Depth (CTD) measurements. Conventionally, a particular measure of the vertical extent of density overturns (known as the "Thorpe scale") is used to infer turbulent dissipation and hence mixing, through two key assumptions: a linear relationship between the Thorpe scale and the "Ozmidov scale" (i.e. the largest vertical scale of the turbulence unaffected by the ambient stratification) and an approximately constant flux coefficient (i.e. the ratio of mixing rate to dissipation rate). Karan demonstrated convincingly that both these assumptions are by no means always consistent with the data, and presented an alternative, promising, parameterisation approach, explicitly taking into account the relative strengths of the buoyancy and the shear to the flow turbulence.

2. Paul Linden (University of Cambridge): *Interface dynamics in stratified shear flows*

   Paul discussed a recent set of experiments in a stratified inclined duct exchange flow which exhibits rich instability and mixing dynamics. The inclined duct connects two large reservoirs of different salinity and hence density fluid, and can in general be tilted at a (small) angle to the horizontal. Paul demonstrated that this flow geometry could exhibit a wide range of flow states, ranging from laminar flow to flows unstable to periodic Holmboe-wave instabilities, to intermittently turbulent and indeed sustained turbulent flows. A criterion for turbulent flow was presented, in terms of the hydraulically controlled energetics of the flow within the channel. Particularly interesting were the experimental observations of the qualitatively different flow dynamics for thermally stratified flows, showing the leading order significance of the fluid's Prandtl number for the instability and mixing within this canonical, and well-controlled flow.

3. Cristobal Arratia (Universidad de Chile): *Transient mechanisms of vertical scale selection for the layering process in unstationary stratified flows*

   Cristobal presented a detailed non-modal linear stability analysis of vertically invariant non-stationary stratified shear flows. He demonstrated that the so-called zig-zag instability typically only occurred when the base flow has relatively widely separated vertical vortices. Through the use of a toroidal-poloidal decomposition, Cristobal presented a detailed energy analysis, showing the key mechanisms by which transient optimal perturbations can extract energy from the base flow to drive perturbation growth. A central insight was that energy tends to be extracted from vertical vorticity and transferred into vertical velocity and density perturbations in such flows.

4. Colm-cille Caulfield (University of Cambridge): *Spontaneous layer formation and interface dynamics in stratified Taylor-Couette flow*

   Finally, Colm-cille discussed the spontaneous appearance of layers and the associated mixing mechanisms in a sequence of experiments (and some numerical simulations) of stratified Taylor-Couette flow, i.e. the flow in the annular region between two concentric cylinders, driven by the rotation of the inner cylinder at a constant rotation rate. Colm-cille showed that an initial linear stratification spontaneously broke up into relatively well-mixed layers of a characteristic vertical extent, separated by relatively sharp interfaces. Furthermore, he showed that the interfaces were prone to destruction and reformation due to relatively long-lived nonlinear structures localised at each interface, which appeared to be triggered by strong ejections from the inner boundary layer. Interestingly, he also reported that a strong coupling between these structures on neighbouring interfaces could occur, with a robust phase-lag of the nonlinear structures appearing between neighbouring interfaces. Colm-cille eliminated linear instability mechanisms as the cause of these nonlinear structures, but it remained unclear what led to their development. The associated mixing appeared to be non-monotonic with stratification, suggestive of a Phillips or BaLLSY-type (paying appropriate homage to Balmforth, Llewellyn Smith and Young) staircase formation mechanism, suggesting a potentially deep connection with the talks of Balmforth and Young.

3.3 Wednesday

Wednesday morning featured seminars by Edgar Knobloch (UC Berkeley), Pascale Lelong (NWRA), Claudia Cenedese (WHOI) and Eckart Meiburg (UC Santa Barbara). The main themes were how turbulence and waves interacted with larger-scale eddying motion (including the classical vision of energy cascades between scales) and how a layer of heavy fluid slumps under gravity beneath a lighter fluid (i.e. a gravity current). Both are classical problems in geophysical fluid dynamics and the talks focussed on relating recent developments in the two areas.

For example, prominent current discussion of atmospheric and oceanic flows considers how the loss of a detailed balance between the dominant forces of rotation, gravity and pressure might generate weaker motions of different spatial scale, thereby transporting momentum and contributing to mixing. As discussed by Pascale, a common example of this is the generation of certain kinds of internal waves by eddies for which the primary balance of Coriolis forces and pressure gradients has been upset. Edgar's talk centred on establishing a generation theoretical formulation for such a cascade of energy between scales using ideas from asymptotic analysis.

The classical geophysical gravity current travels at sufficient speed to excite a cascade of turbulent motion all the way from the head of the current to its far wake. This mixes up the interface between the fluids, blurring the distinction between the two and transferring mass and momentum. It has been a longstanding problem to capture such transfer in simple models that seek to predict the current speed owing to the complexities of turbulent dynamics. Eckart's talk focussed on recent improvements in such models and their verification with direct numerical simulations, which have lately become sufficiently advanced to access the relevant physical regime. The situation is complicated yet further in real geophysical settings by the topography of an underlying surface. In this circumstance, the interaction of the flow with the surface irregularities can increase substantially the generation of turbulence and mixing. Claudia described a suite of elegant laboratory experiments on this problem that summarized the possible range of phenomenology and captured the main characteristics is simple scaling laws.

Although both problems seem somewhat removed from the main theme of the workshop, there is a deep connection between them and many other problems in geophysical fluid dynamics identified by the interaction and transfer of energy between very different spatial scales. Common mathematical approaches are then demanded and it was interesting to view parallel explorations from these other areas.

3.4 Thursday

The last day of the meeting was dedicated to the exploration of applications of layers and interfaces beyond geophysical flows, and to the introduction of new mathematical modeling techniques.

On Thursday morning, we began the session with an nice introduction by Eliot Fried (Okinawa Institute of Science and Technology) to the theory of interface formation and motion in the field of phase transitions. Interfaces in multi-phase systems in material sciences are traditionally described either from a microscopic point of view using energy considerations and statistical mechanics, or from a macroscopic point of view using a simple discontinuity. Eliot described a rigorous procedure that unifies these two approaches that results in the derivation of a scalar Ginzburg-Landau equation, which he then solved by means of matched asymptotic expansions. This procedure yields the evolution equation for the apparent motion of the interface between the two phases as the phase transition proceeds.

Ryan Moll (a graduate student at UC Santa Cruz) then gave a review talk on what is known about thermo-compositional double-diffusive staircases in the interior of giant planets. In these planets, global gradients of Helium and other heavier elements play the role that salt gradients play in oceanography. He explained that many of the salient features of double-diffusive layering in geophysics apply in this astrophysical context.

For instance, he showed how Radko's layer formation mechanism (Radko 2003) directly applies to the planetary case, and identified under which physical conditions it is expected to take place. He also reported on numerical experiments to measure the transport of heat and composition through the staircases, and compared the derived flux laws with those measured in geophysical laboratory and field experiments. One of the main differences between astrophysical staircases and geophysical staircases is that the interfaces in the former are much more turbulent than in the latter. As a result, layers in astrophysical conditions have a strong tendency to merge with one another, a phenomenon that is much less common in geophysics.

Gerardo Hernandez-Duenas (UNAM) presented a very interesting and pedagogical talk dissecting the Boussinesq equations of fluid dynamics and their various approximations (e.g. the simple standard Quasi-Geostrophic (QG) approximation that filter out internal gravity waves, models of successively higher-realism that include the interaction between gravity waves and the QG modes, and finally the full equations). The main question raised is what is the minimal model that can satisfactorily reproduce the general features of the solution of the full equations in a few selected standard problems (namely the propagation of a vortex-dipole, and the decay of random initial conditions in 2D rotating flows). He proposes various intermediate solutions depending on the dynamics that one wishes to correctly account for.

In a talk that had a similar flavor to Gerardo's talk (but using a very different approach) John (Brad) Marston (Brown University), introduced the novel mathematical tool of direct statistical simulations to solve, at least approximately, the equations of fluid dynamics, the idea being to model the evolution of statistical moments of the flow rather than evolving the fluid equations themselves. In this talk, he focused on systems that are known to drive the formation of jets, namely that of stochastically-driven 2D turbulence on a rotating sphere (e.g. an example of the planetary jet formation problem discussed later in the day by James Cho). While different types of approaches for Direct Statistical Simulations have been proposed in the past, in this talk he introduced a new idea that consists of filtering out from the cumulant expansions the wave-wave interactions that would normally result in the generation of higher-wavenumber waves, but keeping the ones that may affect the generation of the mean flow. He compared the reduced model solutions with the solutions of the full equations, finding under which conditions the reduced model could successfully reproduce the properties of the jets that are observed to form in the full equations.

On Thursday afternoon, we held the final formal session of the conference, which could perhaps be characterized as concentrating on slightly more unusual applications of layering and staircase formation where often the phenomenology is cast in terms of potential vorticity. Three talks were presented, with two on plasma physics and one on giant planetary physics by Guilhelm Dif-Pradalier (CEA, France), Patrick Diamond (UCSD, USA) and James Cho (Queen Mary College, London, UK). Unfortunately, the final speaker, Stephan Stellmach (Muenster, Germany) became ill and was unable to speak.

1. Guilhelm Dif-Pradalier (CEA): *Plasma E×B Staircase*

   Guilhelm discussed turbulence in hot magnetized plasmas where permeable localized transport barriers globally organize into a so-called "E×B staircase". This structure is a spontaneously formed, self-organizing pattern of quasi-regular, long-lived, localized shear flow and stress layers coinciding with pressure corrugations. They are interspersed between regions of turbulent avalanching (cascading ionization induced by the acceleration of electrons by the electric field). The plasma staircase exemplifies how a systematic organization of turbulent fluctuations may lead to the onset of strongly correlated flows on magnetic flux surfaces. Guilhelm explained the possible reasons for the existence of the ExB staircase, elucidating the analogies and differences with other known "staircases" in geophysical fluid dynamics, and showed its dependence on the key plasma parameters via GYSELA (gyrokinetic) simulations. Guilhelm also described the first experimental observations of these staircases. These facts strongly emphasize the critical role of mesoscale self-organization in plasma turbulence and may have far-reaching consequences for turbulent transport models and for their validation.

2. Patrick (Pat) Diamond (UC San Diego): *On What We Can Learn From Reduced Models of Staircase Formation in QG Fluids and Magnetized Plasmas*

   Pat presented work on reduced models of the E×B and PV staircase formation and dynamics, in terms of drift-wave (E×B driven) turbulence. Here, the key feature is that there is a feedback loop between the thermodynamic fluctuations that lead to E×B forces and the shear flows that are driven, which can subsequently quench part of the heat flux allowing the residual collisional part to dominate. This leads to a non-monotonic dependence of the heat flux on the temperature differences ("S-curve") such that there is a region of "negative diffusivity". This explains phenomena, like the L-H transition, known as transport bifurcations. Pat drew the comparison between these situations and the geophysical staircases as a sequence of transport barriers, exhibiting reduced models (both the simplest for one field - quasi-geostrophic or Hasegawa-Mima - and the more complex two-field Hasegawa-Wakatani). These all rely on potential vorticity gradient feedback via a number of different mechanisms (shear suppression, Rossby wave elasticity, particle-vorticity flux, etc) and yet all show staircase formation and merger.

3. James Cho (Queen Mary University of London): *PV staircases and jet formation on giant planets*

   James reviewed the dynamics of potential vorticity and the formation of staircases and jets on giant planets, including giant planets outside the Solar System. The review centered on one-layer, rotating, stratified turbulent dynamics in homogenous, inhomogeneous, and magnetized cases and mostly for spherical geometry. Focus was also given to a background on exoplanets, in general, since the topic was not the main staple of many of the meeting's participants. James again emphasized the role of PV gradient steepening as a transport barrier, as the previous two speakers had also done, lending a nice cohesive theme to the session.

4 Outcome of the Meeting

The meeting was very much appreciated by the participants, with the majority being very pleased with the variety of subjects approached, and overwhelmingly pleased with the facilities provided. The long coffee breaks and the long lunch breaks provided many opportunities for small-group meetings where many of the talks and overall topics of the day were discussed in much more depth. The only downside was the unfortunate clash with the meeting of the AAS Division of Planetary Sciences the very same week, so that only two of the many planetary scientists we had originally invited could come. Our original plan of using this meeting to compare the formation and evolution of density and vorticity staircases could not be fulfilled.

In order to compensate for this problem, we broadened the scope of the meeting by inviting scientists with various interests in the dynamics of stratified fluids, both experimental and theoretical. As discussed earlier, a number of recurring themes emerged during the meeting, notably regarding the topic of turbulent transport laws that have a non-monotonous dependence on local gradients. Facetiously dubbed BaLLSY in reference to Balmforth, Llewellyn Smith and Young, such transport laws have anti-diffusive properties, as discussed by Phillips (1972) and Poismentier (1977). Indeed, assuming that the flux F of a quantity b only depends on the vertical gradient of b, as in $F = f(b_z)$ where $b_z = \partial b/\partial z$, then the conservation law

$$\frac{\partial b}{\partial t} = -\frac{\partial F}{\partial z} = -\frac{\partial f}{\partial b_z}\frac{\partial^2 b}{\partial z^2} \tag{1}$$

which has negative diffusion if f increases with b_z. Generally for microscopic behavior, $f = -\alpha b_z$ where $\alpha > 0$, as in Fourier's law of conduction for instance. However, it became clear throughout the meeting that in most instances of fluid systems where layers are known to form (e.g. mechanically stirred stratified fluids, double-diffusive convection, etc..), theoretical arguments or experimental evidence show that f can indeed increase with b_z, and that this is indeed the principal reason for layer formation in a number of otherwise unrelated examples. Despite the variety of topics addressed in the meeting, it was nice to see this common thread emerge.

Finally, in mid-February 2016, that is, 3 months after the meeting, we polled the participants to see what other more direct outcomes of the meeting they benefitted from. Here are some of the answers we received.

- From Colm-cille Caulfield: *The meeting reinvigorated my collaboration with Dick Peltier, and I visited Toronto again to follow up. We are embedding some of these layering ideas in parameterisation of ocean mixing, in several papers which are close to submission. I've been invited to write one of the new JFM "Perspectives", considering transition, turbulence and transport in stratified fluids, and the discussions in Oaxaca will definitely inform my review. Mixing, of ideas, of traditional beverages and especially of hats of various proportions proved very conducive to advancing collaborations across disciplines.*

- From Eckart Meiburg: *I had several good interaction with colleagues in Oaxaca. Maybe one to report is with Claudia Cenedese. We had a long discussion about mixing in gravity currents. We continued those discussions at the APS meeting in Boston. We plan to write a joint paper, and Claudia will come to UCSB for a few days in early March, so that we can make progress on this. All of this would not have happened without the wonderful meeting in Oaxaca.*

- From Paul Linden: *As a result of Bill Young's talk, I have resurrected an old experiment of mine on towing a grid through a stratified fluid. I have a French intern from Grenoble working on it.*

- From David Hughes: *Pat Diamond, Guilhem Dif-Pradalier and I are collaborating on models of layering, in the style of the Balmforth et al. paper. The collaboration was actually started pre-Oaxaca, but it certainly benefitted from that meeting, both from various presentations and our own discussions.*

- From Pascale Garaud: *The meeting helped me see the links between layer formation in double-diffusive convection (a topic I have been working on for a long time), and layer formation in stratified shear flows, a topic I have recently started to work on. This has really helped solidify some of my basic theoretical background in the field.*

- From Nic Brummell: *The meeting, as well as efficiently informing me of the latest developments across the general field, allowed plenty of time for interactions with my colleagues in the plasma/ astrophysical realm (especially David Hughes, with whom I am writing a paper on this topic), and with others such as Neil Balmforth and Bruce Sutherland from other areas.*

- From Bruce Sutherland: *Thanks to the questions after my talk on internal wave tunnelling through density staircases and in several stimulating discussions throughout the week, I was inspired to take my work further by including stochastic effects (uneven staircases). The resulting research has now been written up and is under review with Physical Review Fluids. Many of the talks inspired me to consider new research on staircase formation in the laboratory. Overall it was an excellent and productive meeting.*

- From Pascale Lelong: *As a result of the meeting, Gerardo Duenas-Hernandez and I have started a collaboration. We're going to investigate the role of triad interactions involving waves and vortical modes in shaping energy cascades by using Gerardo's decomposition of nonlinear terms in the Boussinesq equations. This will be combined with my physically based parameterization for introducing vortical modes produced by wave breaking events.*

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