Silting in Representation Theory, Singularities, and Noncommutative Geometry

Lara Bossinger (IM-UNAM Oaxaca)
Henning Krause (Bielefeld University)
Osamu Iyama (The University of Tokyo)
Daniel Labardini-Fragoso (IM-UNAM)
Michael Wemyss (University of Glasgow)

September 10 – 15 2023

1 Introduction

The workshop 'Silting in Representation Theory, Singularities, and Noncommutative Geometry' took place in September 2023 at the Casa Matemática Oaxaca. The scientific programme consisted of 3 two-lecture talks, which surveyed and presented recent developments, 11 individual lectures, 5 short talks of 30 minutes each and given by more junior participants, plus a *gong show* consisting of presentations of 5 minutes each and featuring junior participants mostly. The topics of the lectures can be grouped into the following themes:

- Silting theory
- Surface Models and Higher structures
- Relationship to Cluster theory
- Algebraic Geometry and Noncommutative Applications

A summary of the lectures is provided below.

2 Scientific Activities

We summarize the activities during the week.

Day 1. *Geiss* opened the conference with a talk on the relationship between surfaces with marked points and skewed-gentle algebras. The key new concept of partial KRS-monoid was introduced, and from this structure the highlight was a bijection between the set of τ-reduced irreducible components of the decorated representation spaces of the Jacobi-algebra of a triangulation and the set of laminations of the surface was obtained, compatible with generic g-vectors on one side, and shear coordinates on the other. This enhanced a known bijection between the basic support τ-tilting modules of the Jacobi algebra, and the associated tagged triangulations.

Pauksztello gave a series of two lectures, outlining the fundamentals of silting theory. This included co-t-structures, of which Pauksztello discovered, and the bijection between silting objects and bounded co-t-structures was given. One of the key take-home messages was that various concepts in mutation, t-structures, and co-t-structures are all essentially the same, when viewed through the lens of formulae of perpendicular categories. The crescendo, in the second lecture, was the introduction of the key new concept of the heart fan. This solves the problem that the g-vector fan (perhaps the fundamental object of the workshop) in silting theory has "missing data", and this heart fan is broadly expected to have applications in representation theory and in algebraic geometry.

Opper gave a series of two lectures outlining recent developments in the relationship between geometric models and gentle algebras. The lectures began with the Assem-Skowronski's definition of a gentle algebra, which need not be finite dimensional, and Opper explained how each is glued from local A_n data. The main theorem in the first lecture was the Harden-Katzarkov-Kontsevich theorem between indecomposable objects in the perfect derived category, and curves (suitably interpreted!) in the corresponding surface. The second lecture overviewed this correspondence in detail, before outlining applications, including the landmark classification of gentle algebras using surface topology data. The derived Picard group, the symmetry group which then translates between the algebra and the topology, was described as a semi-direct product, importing for the first time the theory of mapping class groups into derived problems in representation theory.

Hanihara gave a 2-lecture overview of the still mysterious relationship between cluster categories and singularity categories. The first lecture overviewed the Amiot construction of the cluster category, following work of Keller, Ginzburg and others, with the main theorem being the existence of the cluster category with a canonical cluster-tiling object. The second lecture first began with a commuative algebra overview, and following Buchweitz introduced the singularity category as a quotient of the category of CM modules. The zenith of the lectures was recent work, including cases of dimer models, where the cluster and singularity categories coincide.

Barnard gave a lively overview of the combinatorics of semi-brick pairs, 2-simple minded collections and related concepts. The lecture focused on the case of the A_n quiver, where already the inability to complete a semi-brick pair to a 2-smc becomes stark. The combinatorics of the poset of the Weyl group (through its incarnation of torsion theories) controlled this process, in a very combinatorial and natural way.

Day 2. The morning included the second of two talks by Pauksztello, Opper and Hanihara explained above.

Hara then sketched a remarkable application of silting theory to algebraic and symplectic geometry, by classifying spherical objects in various categories in dimension two and three. Rather satisfyingly, the correct notion of objects to classify is that of a semi-brick complex, and Hara defined these, before stating the theorem that each such object is the image of a collection of simples under the action of the autoequivalence group. There are various corollaries, all adhering to the theme that every object which looks like a simple module is one, suitably interpretted. The techniques extend into t-structures, and Hara outlined how it is possible to classify all t-structures in the geometric categories under consideration, in analogy with silting discrete algebras.

Gnedin Glued silting complexes over almost gentle algebras. Almost gentle algebras in the sense of Green and Schroll generalize gentle algebras by allowing more arrows at each vertex. For a class of rings including such algebras, the classification problem of silting complexes can be reduced to the study of rigid modules in a certain hereditary category. This reduction is based on the gluing technique by Burban and Drozd and Simson's approach to matrix problems.

There was then a **Gong show**, with highlights:

- *Wemyss* sketched a conjecture relating the contractibility of curves to the finite dimensionality of algebras, and stated a theorem which gave the full A_∞ structure, and thus the noncommuative deformation algebra, in a large class of (-3,1)-curves.
- *Simental* summarized a result on the 'deep locus' of finite type cluster varieties which refers to the difference of the spectrum of the cluster algebra and the union of the cluster tori. Using braid variety realizations in collaboration with Speyer, M. Gorsky and Castelnuovo they classified which deep loci are non empty, irreducible, equidimensional, respectively
- *Santos* presented his ongoing PhD topic on the relationship of torsion classes and integral partitions in cluster type A.

- *Santiago* elaborated on homological epimorphisms and Hochschild-Mitchell cohomology.
- *Nakajima* presented a dimer model in the torus and explained how to obtain a quiver with potential from this data.
- *Melo López* presented the topic of her PhD thesis: the cluster complex for cluster X-varieties of finte skewsymmetric type and she is solving the problem using Auslander-Reiten quivers.
- *Krause* stated a recent result of Neeman, and then a generalisation which moved from the commutative to noncommutative realm by replacing schemes by coherent rings. Strikingly, the condition on finite global dimension was replaced by a new condition on the finiteness of the finistic dimension of a triangulated category, a notion that he has recently introduced.
- *Barnard* explained the combinatorics of pop-stack sorting permutations and how the concept can fruitfully be extended to torsion classes.
- *Argudin* presented on tilting classes for quiver representations.

Day 3. *Jasso* outlined his joint proof of the Donovan-Wemyss conjecture in birational geometry, which states that 3-fold flops are classified by the contraction algberas. Previous work of August, heavily using silting theory, had reduced the statement to an isomorphism problem, and between this and viewing contraction algebras as cluster tilted algebras the theorem was reduced to a very general statement involving A_∞ algebras. The spectacular new result is that a derived version of the Auslander-Iyama correspondence [iself a landmark in the area] exists, and the proof boils down to a question on uniqueness of A_∞ structures, under the assumption that a certain Hoschild class is a unit (viewed in Tate cohomology). This was a remarkable lecture, which will have many further uses.

Laking outlined her approach to a very general homological problem in both representation theory and algebraic geometry, namely that of understanding when two t-structures, both obtained from a fixed one by tilting at torsion pairs, are themselves related by a simple tilt. When one has finite length, this is known and related to 'small' silting mutation. When the t-structures do not have finite length, the question is much harder, and the answer corresponds to cover relations in torsion theories, and to brick labelling. One of the take-home messages was the 'big' cosilting objects control 'small' torsion theories, and the theory of mutation on these big cosilting objects are the key to answering the motivating question.

Nájera Chávez gave a short talk on the cluster complex for cluster Poisson varieties initiating the study of these combinatorial object which encode theta functions. When the cluster variety is associated to an acyclic quiver his joint results with Melo López give an explicit description in terms of c- and g-vectors.

Day 4. *Dyckerhoff* presented a far-reaching higher homological algebra generalisation of additive categories, via 'lax matrices'. The main point is that under very general conditions four natural limits and colimits all coincide, and this allows us to mimick some of the arguments of additive categories in much greater generality. The main applications of this new construction are to gluing categories along a functor, and to producing autoequivalences via d-periodic semi-othogonal decompositons.

Pressland gave his proof of the Muller-Speyer conjecture, which asserts that two cluster structures coincide. The plays the (non-equivalent) Gorenstein projective modules and CM modules off against each other, in the process produceing tiled Gorenstein orders (of Krull dimension one but injective dimension two), and a derived equivalence between these exact categories that then de-categorifies, via cluster characters, to obtain the conjecture.

Simental presented his joint work equipping the coordinate ring of the braid variety of *any* positive braid with a cluster structure, in the process recovering as a special case almost all of the motivating examples in cluster algebras in the past 20 years. This unifying construction was striking in its ambition, and used weaves as the combinatorial tool to produce a quiver with frozen vertices and a torus with cluster coordinates for the associated A-cluster variety. The construction is of relevance in contact and symplectic geometry as the weaves combinatorucs yield new constructions of embedded exact Lagrangian fillings for Legendrian links associated with the braids.

Mousavand, inspired by various classical theorems in the context of finite, tame and wild representation type, outlined various remarkable conjectures and presented many recent theorems all of which would be consequences of these conjectures. The main idea is that bricks are a class of objects which are (conjecturally)

large enough to detect generic behaviour, and as such under appropriate conditions control the behaviour of whole module category.

Kvamme outlined his joint work which produced examples of silting complexes from higher torsion classes in homological algebra. From the work of Adachi, Iyama and Reiten we know that there is a bijection between functorially finite torsion classes, support tau-tilting pairs, and 2-term silting complexes. In this talk Kvamme investigated to which extent this can be generalized to higher Auslander—Reiten theory. In particular, he explained how any functorially finite d-torsion class gives rise to a (d+1)-term silting complex. This talk was based on joint work with Jenny August, Johanne Haugland, Karin M. Jacobsen, Yann Palu, and Hipolito Treffinger.

Allegretti studies the 3-CY categories associated to quivers with potentials, particular those from surfaces, and introduced quadradic differentials, following Bridgeland–Smith. The harmonic maps were introduced, as the solutions to a certain PDE, and for type A the connection between stability conditions, quadratic differentials and Teichmuller space was explained. The main result, in the surfaces setting, was that there is a map from the stability manifold (modulo the action of spherical twists on the canonical simples) to the R-positive points of the cluster Poisson variety. The proof goes via quadratic differentials, then harmonic maps, but the remarkable aspect is that the theorem is formulated purely algebraically. Allegretti posed the question as to whether the result holds for any general quiver with potential, under suitable nondegenerate assumptions.

Day 5. *Bautista* presented a tame wild dichotomy for modules filtered by homological systems. This is in analogy to the classical tame wild dichotomy for representations of finite dimensional algebras, but covers for example the category of Δ-filtered modules over a quasi-hereditary algebra. The proofs use Bocses with relation and reduction techniques as in the classical case (and developed by Drozd and Crawley-Boevey). Moreover, generic modules provide an elegant method for characterizing infinite representation type for the category of Δ-filtered modules.

Burban discussed the Hall algebra of a regular non-commutative curve of half ruled type. Following the philosophy of Lenzing, an exceptional non-commutative curve is a non-commutative hereditary projective curve, whose category of coherent sheaves admits a tilting object. Examples are weighted projective lines in the sense of Geigle and Lenzing, but there are more. The talk presented a specific example of an exceptional curve over a finite field, which is half ruled regular in the sense of Artin and de Jong. In that case the composition Hall algebra of the corresponding category of coherent sheaves can be identified with the Drinfeld realization of the quantum affine algebra of type A_2^2.

Todorov finished the conference by summarising the classical Auslander correspondence between algebras of finite representation type and 'Auslander algebras', namely those of global dimension ≤ 2, and dominant dimension ≥ 2. She then summarised the higher version of the theorem due to Iyama, before applying it in three separate cases to produce examples d-representation finite algebras.

3 Summary

We had the following feedback from Sebastian Opper, on site participant:

"The workshop presented a very fruitful opportunity to talk to several other participants (some of whom I had not previously met) and gave me the possibility to ask questions to other mathematicians in relation to ongoing research projects of mine. This greatly helped my understanding of certain mathematical problems I am currently working on."

Finally, we had the following feedback from Lidia Angeleri, who participated online:

"I attended the workshop only online, and due to the time shift I did not attend all talks. Nevertheless I benefited from the event, learning about new results which are relevant for my research. I can also confirm that the quality of the streaming was excellent."

References

[1] T. Adachi, O. Iyama and I. Reiten, *τ-tilting theory*, Compos. Math. 150 (2014), no. 3, 415–452.

[2] D. Allegretti, *Stability conditions and Teichmüller space*, arXiv preprint, arXiv:2112.07912

[3] L. Angeleri Hügel, R. Laking, J. Šťovíček, J. Vitória, *Mutation and torsion pairs*, arXiv preprint, arXiv:2201.02147

[4] J. August, J. Haugland, K. M. Jacobsen, S. Kvamme, Y. Palu, H. Treffinger, *A characterisation of higher torsion classes*, arXiv preprint, arXiv:2301.10463

[5] E. Barnard and E. J. Hanson, *Pairwise Compatibility for 2-Simple Minded Collections II: Preprojective Algebras and Semibrick Pairs of Full Rank*, Annals of Combinatorics 26 (2022), 803–855.

[6] R. Bautista, E. Pérez, L. Salmerón, *Tame and wild theorem for the category of filtered by standard modules*, arXiv preprint, arXiv:1706.07386

[7] R. Bautista, E. Pérez, L. Salmerón, *Homological systems and bocses* J. Algebra 617 (2023), 192–274.

[8] N. Broomhead, D. Pauksztello, D. Ploog and J. Woolf, *The heart fan of an abelian category*, arXiv preprint, arXiv:2310.02844

[9] W. Donovan and M. Wemyss, *Noncommutative deformations and flops*, Duke Math. J., 165(8):1397–1474, 2016.

[10] W. Donovan and M. Wemyss, *Contractions and deformations*, Amer. J. Math., 141(3):563–592, 2019.

[11] C. Geiss, D. Labardini-Fragoso and J. Wilson, *Laminations of punctured surfaces as τ-reduced irreducible components*, arXiv preprint, arXiv:2308.00792

[12] C. Geiss, D. Labardini-Fragoso and J. Wilson, *Bangle functions are the generic basis for cluster algebras from punctured surfaces with boundary*, arXiv preprint, arXiv:2310.03306

[13] F. Haiden, L. Katzarkov and M. Kontsevich, *Flat surfaces and stability structures*, Publ. Math. Inst. Hautes Études Sci. 126 (2017), 247–318.

[14] N. Hanihara, *Cluster categories of formal DG algebras and singularity categories*, Forum Math. Sigma 10 (2022), Paper No. e35, 50 pp.

[15] N. Hanihara and O. Iyama, *Enhanced Auslander-Reiten duality and tilting theory for singularity categories*, arXiv preprint, arXiv:2209.14090

[16] E. J. Hanson and K. Igusa, *Pairwise Compatibility for 2-Simple Minded Collections*, Journal of Pure and Applied Algebra 225 (2021), no. 6.

[17] W. Hara, M. Wemyss, *Spherical objects in dimension two and three*, arXiv preprint, arXiv:2205.11552

[18] A. Higashitani, Y. Nakajima, *Deformations of dimer models*, SIGMA Symmetry Integrability Geom. Methods Appl. 18 (2022), Paper No. 030, 53 pp.

[19] O. Iyama, *Auslander-Reiten theory revisited*, Trends in representation theory of algebras and related topics, 349–397, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2008.

[20] O. Iyama, *Cluster tilting for higher Auslander algebras*. Adv. Math. 226 (2011), no. 1, 1–61.

[21] G. Jasso, B. Keller, F. Muro, *The Donovan–Wemyss Conjecture via the Derived Auslander–Iyama Correspondence*, arXiv preprint, arXiv:2301.11593

[22] C. Melo, A. Nájera Chávez, *The cluster complex for cluster Poisson varieties and representations of acyclic quivers*, arXiv preprint, arXiv:2310.03626

[23] K. Mousavand, C. Paquette, *Minimal (τ-)tilting infinite algebras*, Nagoya Mathematical Journal, 249, 221–238. doi:10.1017/nmj.2022.28

[24] K. Mousavand, C. Paquette, *Biserial algebras and generic bricks*, arXiv preprint, arXiv:2209.05696

[25] Y. Nakajima, *Variations of GIT quotients and dimer combinatorics for toric compound Du Val singularities*, arXiv preprint, arXiv:2309.16112

[26] P.-G. Plamondon, S. Opper and S. Schroll, *A geometric model of the derived category of a gentle algebra*, arXiv preprint, arXiv:1801:09659

[27] M. Pressland, *Quasi-coincidence of cluster structures on positroid varieties*, arXiv preprint, arXiv:2307.13369

[28] E. Sen, *Nakayama Algebras which are Higher Auslander Algebras*, arXiv preprint, arXiv:2009.03383

[29] E. Sen, *Higher Auslander Algebras arising from Dynkin Quivers and n-Representation Finite Algebras*, arXiv preprint, arXiv:2307.13262