Beyond Elliptic Polylogarithms

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1 Overview of the Field

As the Large Hadron Collider (LHC) continues to collect data, the need for particle physicists to make highly precise predictions has grown. A key component of these predictions involves perturbative calculations of quantities known as scattering amplitudes [1]. Recently, scattering amplitudes have also been used to predict gravitational waves in experiments like the Laser Interferometer Gravitational-Wave Observatory (LIGO) [2], highlighting their increasing importance in various areas of precision physics. However, computing these scattering amplitudes is a complex task. As physicists aim for higher precision or consider processes involving more particles, the analytic complexity grows exponentially. Addressing this challenge requires a deep mathematical understanding of the analytic structure of the functions involved in these calculations. One well-understood class of these functions, known as multiple polylogarithms [3, 4], has been fundamental to progress in the field over the past decade. These functions can be geometrically interpreted as iterated integrals over a punctured Riemann sphere. However, it has become clear that these functions alone cannot fully describe the space of scattering amplitudes. To account for various effects such as massive particles, multi-particle processes, or higher precision, more complex geometries must be considered.

2 Recent Developments and Open Problems

In recent years, there has been significant progress in understanding the simplest functions, known as elliptic polylogarithms, which are iterated integrals on a torus involving a single elliptic curve [5, 6, 7, 8]. This advancement has provided a powerful framework for understanding and manipulating these functions. Inspired by this rapid progress, the community is now striving to develop tools to comprehend the full range of mathematical functions that appear in scattering amplitude calculations [9]. The emergence of these functions suggests the presence of rich mathematical structures underlying scattering amplitudes. These structures are inherently geometrically complex, linking them to some of the most profound areas of modern geometry, number theory, and motivic theory.

The primary goal of our workshop was to explore functions in scattering amplitudes beyond elliptic polylogarithms and investigate the underlying structures that give rise to them. Various classes of these functions have been identified, each with distinct geometrical descriptions. Some involve multiple elliptic curves [10, 11, 12, 13], others involve Riemann surfaces of higher genus [14, 15, 16, 17], and the most enigmatic class involves higher-dimensional Calabi-Yau manifolds [18, 19, 20, 21, 22, 23]. Investigating these functions requires an interdisciplinary approach, bringing together particle physicists focused on scattering amplitudes, string theorists with expertise in Calabi-Yau manifolds, and mathematicians from diverse fields such as algebraic geometry and combinatorics. Our workshop initiated this interdisciplinary investigation, leading to a deeper mathematical understanding and the development of powerful tools to achieve the next level of precision in collider physics, string theory calculations as well as to better understand the connection between scattering amplitudes and gravity.

3 Workshop Highlights

Prior to the actual workshop, we began with a two-day mini-school, primarily for students and early-career post-docs in Latin America primarily interested in the physical and mathematical aspects of scattering amplitudes and computational techniques for Feynman diagrams. This mini school on ”Amplitudes & Feynman integrals” covered a set of topics, including lectures on mathematical and physical aspects of scattering amplitudes, as well as lectures on the geometry of elliptic curves and Calabi-Yau manifolds. This school was held in Puebla and locally organised by Lorenzo Diaz-Cruz. It attracted a wide range of students affiliated with a number of local Universities such as Centro de Investigacion y de Estudios Avanzados (Cinvestav), Physics Department Centro de Investigacion y de Estudios Avanzados (Cinvestav), Mathematics Department (Qro) Universidad Autonoma de Chiapas, Universidad Autonoma de Mexico, Institute of Nuclear Sciences Universidad Autonoma de Mexico, Institute of Mathematics, Cuernavaca Universidad Autonoma de Mexico, Institute of Physics Universidad Autonoma de Puebla. Following the mini-school, the workshop was kick-started in Oaxaca on 16th June. It consisted of a light schedule of talks interspersed with discussion sessions. Plenty of time was left between the talks, with the intent of soliciting new approaches to the challenges of scattering amplitudes beyond elliptic polylogarithms and fostering a frank exchange of ideas between different scientific communities. We believe this light schedule was in particular appreciated by the participants.

4 Scientific Progress Made

The schedule of the workshop was curated in a way such that there was a mixture of varied range of topics. The 14 talks covered topics from scattering amplitudes for phenomenology and string theory, understanding the mathematical structure of Feynman integrals for various genera, mastering the differential equations for Feynman integrals and scattering amplitudes and its connection to gravity.

Some highlights of the talks presented at the workshop were as follows:

- **Scattering amplitudes for phenomenology**: *Manfred Kraus* talked about the recent progress on the computation of crucial elements needed to compute higher-order ttH scattering amplitudes. This process remains one of the highly demanded objective within the current precision era.

- **Understanding the properties of Feynman integrals**: *Andrew McLeod* talked about the geneological constraints on Feynman integrals. We learnt about a new method for working out the implications of the heirarchical principle, which places strong constraints on the analytic structure of Feynman integrals. We also heard from *Matija Tapuskovi* about motivic Galois theory of regularised Feynman integrals and from *Cristian Vergu* about homology and non-polylogarithmic integrals. All these talks provided us with deeper insights into the mathematical properties of Feynman integrals.

- **Feynman integrals & higher-genus surfaces**: We learnt more about Feynman integrals and string-amplitudes from talks from *Oliver Schlotterer*, who talked about integration kernels and Fay identities on higher-genus Riemann surfaces, and *Franziska Porkert*, who talked about coaction for iterated Eisentein integrals. Through the first talk we learnt how to build integration kernels of the flat connection from the convolutions of the Arakelov Green function and its derivatives. Whereas from the second talk, we learnt specifically about coaction for iterated Eisenstein integrals that acts on the generating series of these special functions and the interplay of the candidate-coaction with modular transformations, derivatives and evaluations. In this category, we had another interesting talk from *Sven Stawinski*, who shed important light on hyperelliptic polylogarithms for Feynman integrals.

- **Differential equations for Feynman integrals**: *Andrzej Pokraka*, *Leonardo de la Cruz* and *Sara Maggio* talked about how to better manipulate the differential equations of Feynman integrals in order to obtain analytic results of various Feynman integrals. Andrzej talked about a tale of two elliptic curves appearing in the 5-mass kite family of Feynman integrals and Sara talked about how to obtain a better set of differential equations for Feynman integrals with Calabi-Yau geometries. Leonardo talked about how to obtain differential equations for Feynman integrals in general dimensions.

- **Scattering amplitudes and gravity**: *Roger Morales* provided us information on how to classify post-Minkowskian (PM) geometries via the so-called loop-by-loop Baikov parametrization, giving us important information about Feynman integrals that contribute to the emission of gravitational waves through the PM expansion. *César Ramos* discussed an approach to the classical double copy in bigravity, *Christaian Schubert* told us about worldline integrals and related mathematical structures, whereas *Jonathan Reyes Pérez* talked about on-shell methods from SUGRA to top quark. Through these talks we were able to get a glimpse of the state-of-the-art of the Feynamn integrals & scattering amplitude computations in gravitational physics.

The timetable was structured such that the talks from senior researchers/ professors were held in the morning whereas the afternoon was always reserved for talks from junior researchers. Since questions were welcome at any time during the talks, there was a lot of discussion both during the after every talk. Additionally, there was a good representation from both women as well as local community amongst the participants.

5 Outcome of the Meeting

This 5-day workshop saw participation from particle physicists, string theorists, and mathematicians, who all share a common interest in functions associated with multiple elliptic curves and higher-genera object, as well as their underlying mathematical structure. There were plenty of discussions to exchange the latest developments and ideas in their respective fields. We achieved in

- identifying new methods to enable deeper understanding of scattering amplitudes required to push the precision frontier,
- exploring the rich mathematical structure arising from Feynman integrals in various genera,
- understanding the manipulation of the differential equations for multi-scale Feynman integrals, and
- establishing deeper connection between scattering amplitudes, string theory as well as gravitational physics.

Some new collaborations between the participatns were formed during the workshop, for instance, on *tubings of trees and graphs* between Karen Yeats and Andrzej Pokraka, and between Ekta Chaubey and Manfred Kraus on *overcoming the analytic complications due to many external scales for precision physics*. Interestingly, this workshop also provided a good opportunity to advertise open research positions to the participants of this workshop, for example in the group of Manfred Kraus, who is a newly appointed professor in Mexico city. We hope that through this workshop we were able to influence the scientific orientation of the community in Mexico towards the field of scattering amplitudes and Feynman integrals.

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