An Introduction to Cut Generating Functions

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The Cutting Plane paradigm

- Disjunctive Cuts.
- Closed Form Formulas for Cuts.

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  Advantage: Handles very general feasible regions.

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Confluence of two seminal ideas from the 1970s:

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Cut Generating Functions

S is a closed subset of $\mathbb{R}^n \setminus \{0\}$.

$$X_S(R, P) := \{(s, y) \in \mathbb{R}^k_+ \times \mathbb{Z}^\ell_+ : Rs + Py \in S\}$$

where $R \in \mathbb{R}^{n \times k}$, $P \in \mathbb{R}^{n \times \ell}$.

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Examples:

1. S = translation of integer/mixed-integer points in $\mathbb{R}^n_+$ (Mixed-Integer Linear Programming).
2. S = translation of integer/mixed-integer points in a closed convex set C (Mixed-Integer Conic/Convex Programming).

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Feasible region of a mixed-integer linear program

$$Ax = b,\ x \in \mathbb{R}^d_+,\ x_i \in \mathbb{Z} \quad \forall i \in I$$

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$$A_B x_B + A_N x_N = b,\; x \in \mathbb{R}^d_+,\; x_i \in \mathbb{Z} \quad \forall i \in I$$

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$$x_B + A_B^{-1}A_N x_N = A_B^{-1}b,\ x \in \mathbb{R}^d_+,\ x_i \in \mathbb{Z} \quad \forall i \in I$$

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$$x_B + A_B^{-1}A_N x_N = A_B^{-1}b,\ x \in \mathbb{R}^d_+,\ x_i \in \mathbb{Z} \quad \forall i \in I$$

$$x_B - A_B^{-1}b = (-A_B^{-1}A_{N\setminus I})x_{N\setminus I} + (-A_B^{-1}A_{N\cap I})x_{N\cap I},$$

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$$\begin{aligned}
S &:= (\mathbb{R}^{k'}_+ \times \mathbb{Z}^{\ell'}_+) - A_B^{-1}b \\
R &:= -A_B^{-1}A_{N\setminus I}, \qquad s := x_{N\setminus I} \quad (k = |N \setminus I|) \\
P &:= -A_B^{-1}A_{N\cap I}, \qquad y := x_{N\cap I} \quad (\ell = |N \cap I|)
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We seek pair of functions

$$\psi_S : \mathbb{R}^n \to \mathbb{R} \qquad \pi_S : \mathbb{R}^n \to \mathbb{R}$$

such that the inequality

$$\sum_{i=1}^{k} \psi_S(r^i)s_i + \sum_{j=1}^{\ell} \pi_S(p^j)y_j \geq 1$$

is valid for any k, ℓ, R, P.

Cut Generating Functions: Example

$n = 1$, $S = b + \mathbb{Z}$ where $b \notin \mathbb{Z}$.

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where $R = [r^1, \ldots, r^k], P = [p^1, \ldots, p^\ell]$.

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$$\psi_S(r) = \max\left\{\frac{r}{[b]}, \frac{-r}{1-[b]}\right\} \qquad \pi_S(r) = \min\left\{\frac{[r]}{[b]}, \frac{1-[r]}{1-[b]}\right\}$$

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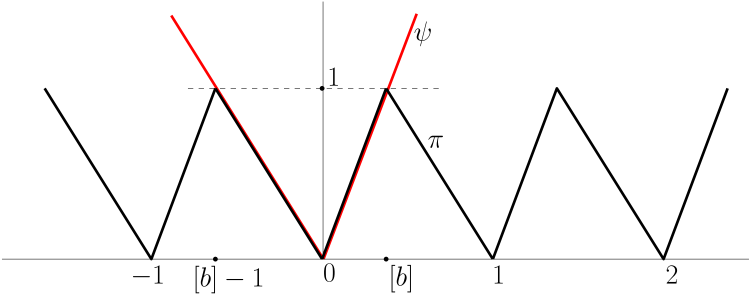

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Want minimal valid pairs to remove redundancies.

$$(\psi'_S, \pi'_S) \leq (\psi_S, \pi_S) \quad \Rightarrow \quad \psi'_S = \psi_S, \pi'_S = \pi_S$$

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Non minimal π

$$\sum_{i=1}^{k} \psi_S(r^i) s_i + \sum_{j=1}^{\ell} \pi_S(p^j) y_j < 1$$

$$\Downarrow$$

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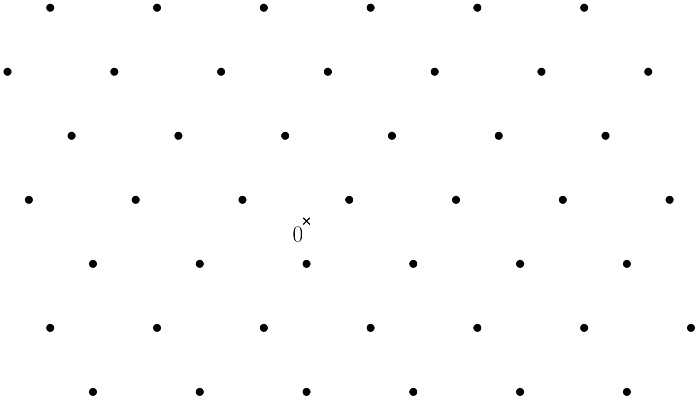
Let $B \in \mathbb{R}^n$ be a maximal S-free convex set with $0 \in \mathbf{int}(B)$.

Maximal S-free Convex Sets

Let $S = b + \mathbb{Z}^n$ be a translated lattice.



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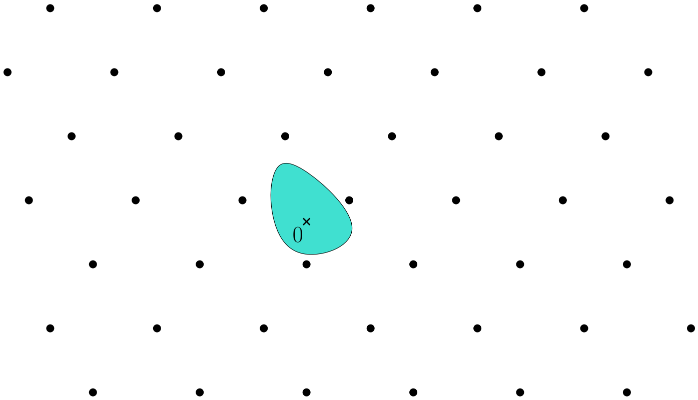

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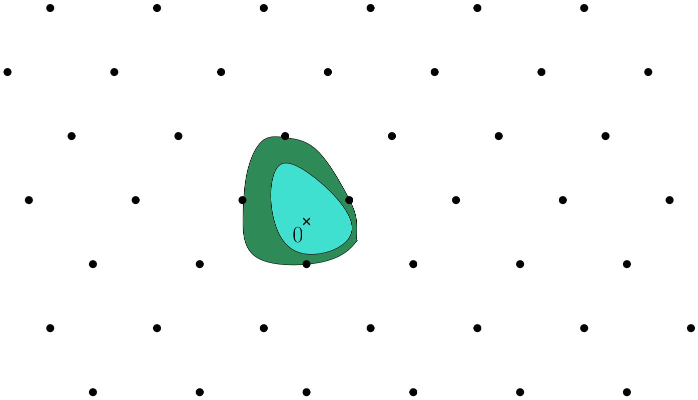

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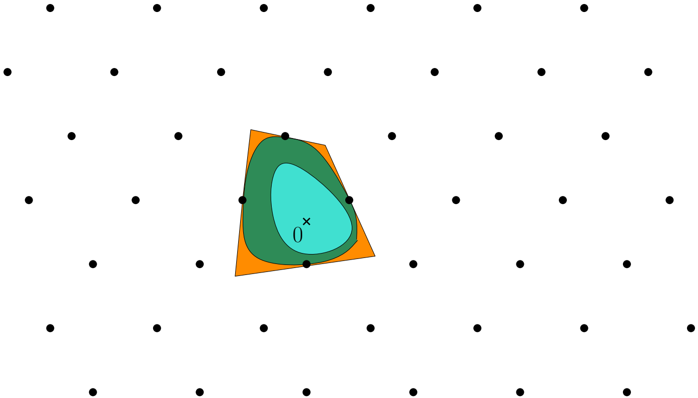

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THEOREM: Maximal S-free convex sets are polyhedra. Thus,

$$B = \{x \in \mathbb{R}^n : a_i \cdot x \leq 1,\ i \in I\}$$

Lovasz 1989
Basu, Conforti, Cornuéjols, Zambelli 2010
Dey and Moran 2011
Averkov 2013

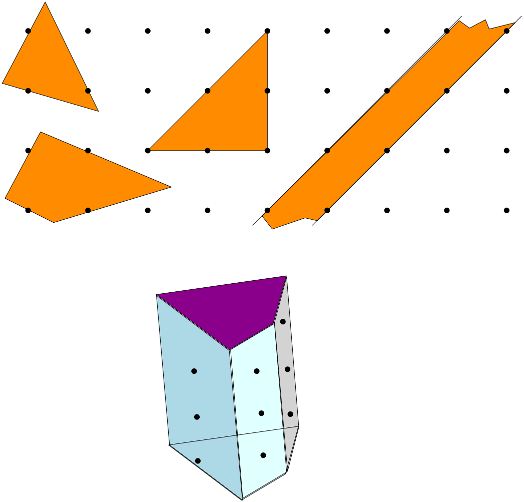
Maximal S-free Convex Sets

$S = b + \mathbb{Z}^n$ is a translated lattice.

THEOREM

Every bounded maximal S-free set is a polytope.

Every unbounded maximal S-free convex set is a cylinder above a polytope $P + L$ where L is a lattice-subspace.



Computing with Cut Generating Functions

MAIN GOAL: Closed form formulas for these minimal pairs.
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$$B = \{x \in \mathbb{R}^n : a_i \cdot x \leq 1,\ i \in I\}.$$

Define the function

$$\phi_{S,B}(r) = \max_{i \in I}\ a_i \cdot r, \quad \forall r \in \mathbb{R}^n.$$

THEOREM: $\psi_S = \pi_S = \phi_{S,B}$ is a valid pair. Moreover, (ψ_S, π_S) is "partially" minimal, i.e.,

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Properties of $\phi_{S,B}(r)$

1. Subadditivity: $\phi_{S,B}(r_1 + r_2) \leq \phi_{S,B}(r_1) + \phi_{S,B}(r_2)$ for all $r_1, r_2 \in \mathbb{R}^n$
2. Positive Homogeneity: $\phi_{S,B}(\lambda r) = \lambda \phi_{S,B}(r)$ for all $\lambda \in \mathbb{R}_+,\ \ r \in \mathbb{R}^n$
3. Validity: $B = \{r \in \mathbb{R}^n : \phi_{S,B}(r) \leq 1\}$ and $\mathbf{int}(B) = \{r \in \mathbb{R}^n : \phi_{S,B}(r) < 1\}$

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Subadditivity of $\phi_{S,B}$

$$\begin{aligned} \phi_{S,B}(r_1 + r_2) &= \max_{i\in I}\ a_i \cdot (r_1 + r_2) \\ &= \max_{i\in I}\ (a_i \cdot r_1 + a_i \cdot r_2) \\ &\leq \max_{i\in I}\ a_i \cdot r_1 + \max_{i\in I}\ a_i \cdot r_2 \\ &= \phi_{S,B}(r_1) + \phi_{S,B}(r_2) \end{aligned}$$

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Positive Homogeneity of $\phi_{S,B}$

$$\begin{aligned} \phi_{S,B}(\lambda r) &= \max_{i\in I}\ a_i \cdot (\lambda r) \\ &= \max_{i \in I}\ \lambda(a_i \cdot r) \\ &= \lambda \max_{i\in I}\ a_i \cdot r \\ &= \lambda \phi_{S,B}(r) \end{aligned}$$

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Validity of $\phi_{S,B}$

$$\begin{aligned} r \in B \quad &\Leftrightarrow \quad \forall i \in I \quad a_i \cdot r \le 1 \\ &\Leftrightarrow \quad \max_{i \in I}\ a_i \cdot r \le 1 \\ &\Leftrightarrow \quad \phi_{S,B}(r) \le 1 \end{aligned}$$

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(ψ_S, π_S) is a partially minimal valid pair

$$\psi_S(r) = \pi_S(r) = \phi_{S,B}(r) = \max_{i \in I} \; a_i \cdot r$$

Need to show:
$(s, y) \in X_S(R, P) := \{(s, y) \in \mathbb{R}^k_+ \times \mathbb{Z}^\ell_+ : Rs + Py \in S\}$ implies

$$\sum_{i=1}^{k} \psi_S(r^i) s_i + \sum_{j=1}^{\ell} \pi_S(p^j) y_j \geq 1$$

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Validity of (ψ_S, π_S)

$$\begin{aligned}
Rs + Py \in S \quad &\Rightarrow \quad \textstyle\sum_{i=1}^{k} r^i s_i + \sum_{j=1}^{\ell} p^j y_j \in S \\
&\Rightarrow \quad \phi_{S,B}(\textstyle\sum_{i=1}^{k} r^i s_i + \sum_{j=1}^{\ell} p^j y_j) \geq 1 \\
&\qquad\qquad \text{[Validity of } \phi_{S,B}\text{]} \\
&\Rightarrow \quad \textstyle\sum_{i=1}^{k} \phi_{S,B}(r^i s_i) + \sum_{j=1}^{\ell} \phi_{S,B}(p^j y_j) \geq 1 \\
&\qquad\qquad \text{[Subadditivity of } \phi_{S,B}\text{]} \\
&\Rightarrow \quad \textstyle\sum_{i=1}^{k} \phi_{S,B}(r^i) s_i + \sum_{j=1}^{\ell} \phi_{S,B}(p^j) y_j \geq 1 \\
&\qquad\qquad \text{[Positive Hom. of } \phi_{S,B}\text{]} \\
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Claim: $\{r \in \mathbb{R}^n : \psi'_S(r) \le 1\}$ is an S-free convex set.

If $\bar{r} \in S$ and $\psi'_S(\bar{r}) < 1$, then consider $X_S(R, P)$ with $R = [\bar{r}]$ and any P. $(s_{\bar{r}} = 1, y = 0)$ is a valid solution but

$$\begin{aligned} & \textstyle\sum_{i=1}^{k} \psi_S(r^i)s_i + \sum_{j=1}^{\ell} \pi_S(p^j)y_j \\ = \; & \psi'_S(\bar{r})s_{\bar{r}} \\ = \; & \psi'_S(\bar{r}) < 1 \end{aligned}$$

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This implies $B = \{r \in \mathbb{R}^n : \psi'_S(r) \leq 1\}$

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Polars

Polar of a convex set C is

$$C^* = \{y \in \mathbb{R}^n \ : \ y \cdot x \le 1 \ \ \forall x \in C\}$$

Convex analysis fact: $D := \{d \in \mathbb{R}^n : r \cdot d \le \psi'_S(r) \ \ \forall r \in \mathbb{R}^n\}$.

Then $D^* = \{r \in \mathbb{R}^n : \psi'_S(r) \le 1\}$

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   Basu, Cornuéjols, Zambelli 2011
   Conforti, Cornuéjols, Daniilidis, Lemaréchal, Malick 2015

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4. $r \cdot a^i \leq \psi'_S(r)$ for every $i \in I, r \in \mathbb{R}^n$.

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$$\psi_S(r) = \max_{i \in I} a^i \cdot r \leq \psi'_S(r).$$

Computing with Cut Generating Functions

$$X_S(R,P) := \{(s,y) \in \mathbb{R}^k_+ \times \mathbb{Z}^\ell_+ : Rs + Py \in S\}$$

Focus on $S = (b + \mathbb{Z}^n) \cap Q$ where $b \notin \mathbb{Z}^n$, Q rational polyhedron.

Let $B \in \mathbb{R}^n$ be a maximal S-free convex set with $0 \in \mathbf{int}(B)$:

$$B = \{x \in \mathbb{R}^n : a_i \cdot x \leq 1,\ i \in I\}.$$

Define the function

$$\phi_{S,B}(r) = \max_{i \in I} a_i \cdot r, \quad \forall r \in \mathbb{R}^n.$$

THEOREM: $\psi_S = \pi_S = \phi_{S,B}$ is a partially minimal valid pair, i.e.,

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Towards a fully minimal pair

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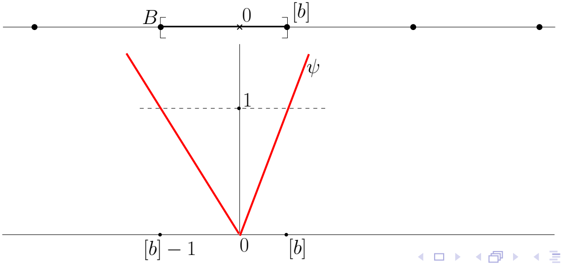

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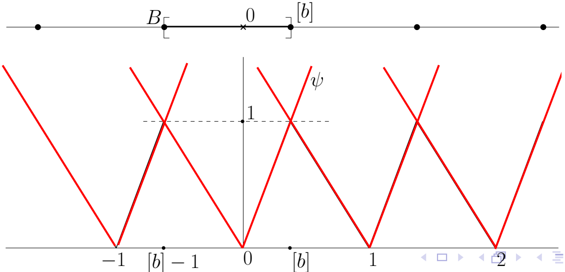

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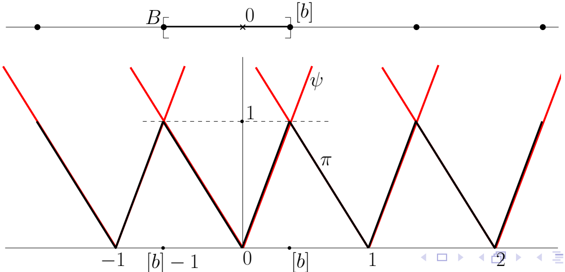

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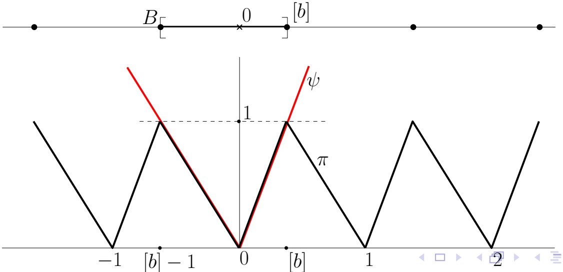

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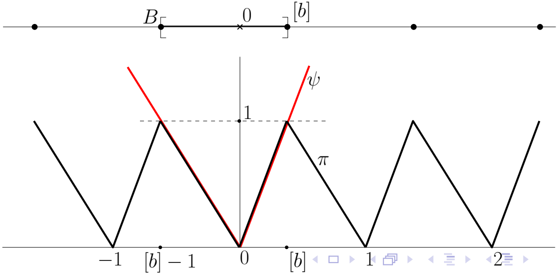
This is the Gomory Mixed-Integer (GMI) Cut!



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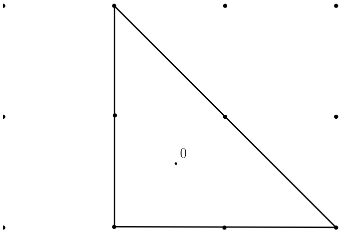
QUESTION: When is this really minimal?

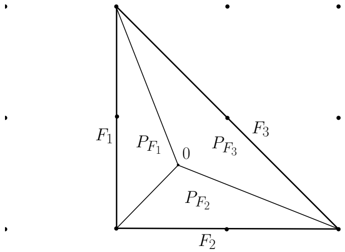
The Lifting Region

1. Given
$S = (b + \mathbb{Z}^n) \cap Q,\ B$



The Lifting Region

1. Given $S = (b + \mathbb{Z}^n) \cap Q$, B.
2. For every facet F, $P_F := \{r \in \mathbb{R}^n : \arg\max_i a_i r \text{ indexes } F\}$.



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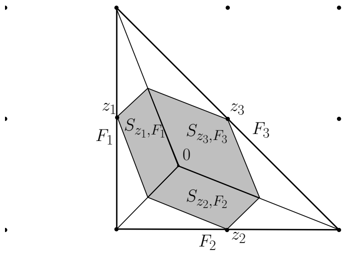

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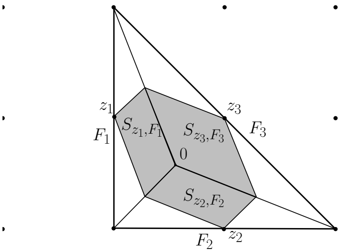

$$R(S,B) = \bigcup_{\text{Facets } F} \bigcup_{z \in S \cap F} S_{z,F}$$

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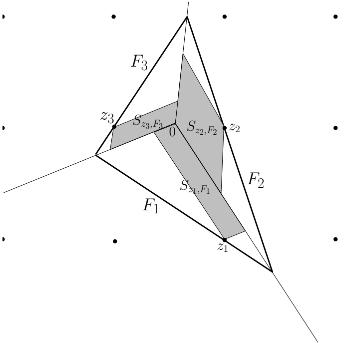

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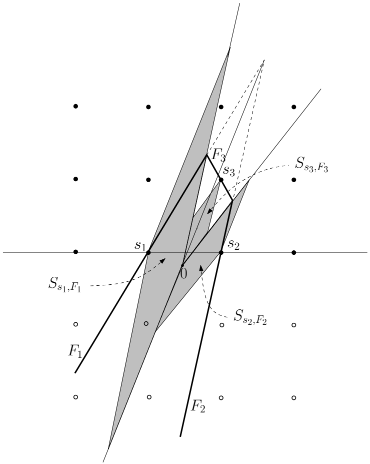

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The Lifting Region and the Covering Property

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THEOREM Basu, Campelo, Conforti, Cornuéjols, Zambelli 2011
$\psi_S(r) = \max_{i \in I} a_i r$ and $\pi_S(r) = \min_{w \in W_S} \psi_S(r+w)$ form a minimal pair if (and only if) $R(S,B) + W_S = \mathbb{R}^n$.

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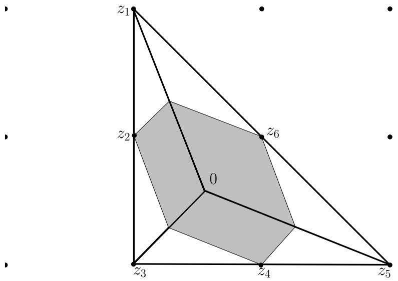
Main Credit for sparking this line of research:
Santanu Dey and Laurence Wolsey 2009.



z_1
z_2
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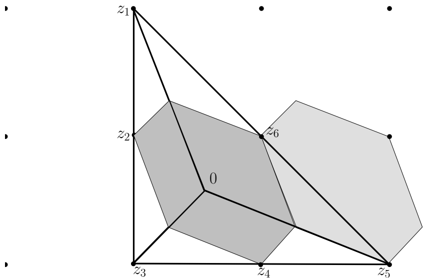

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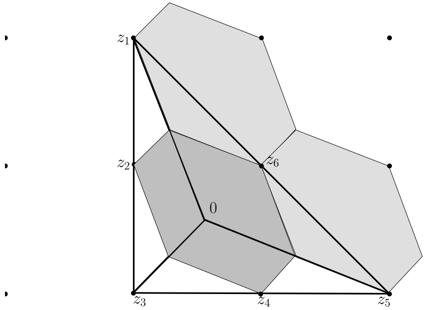
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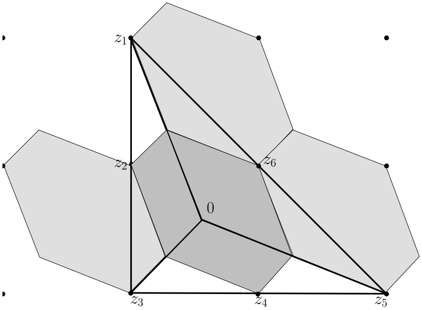
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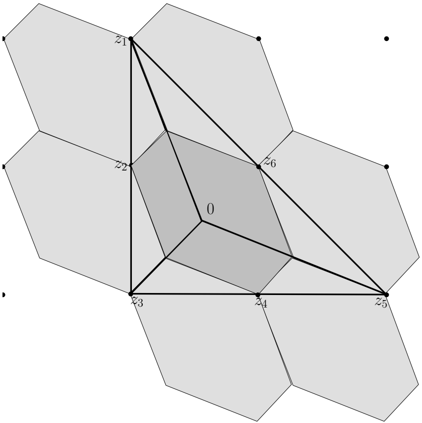
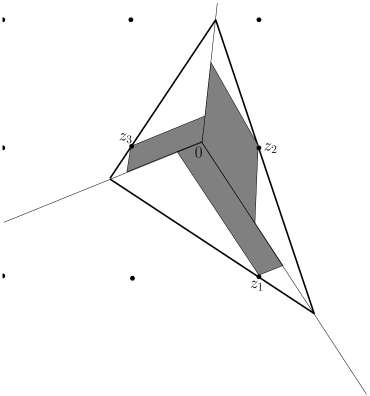
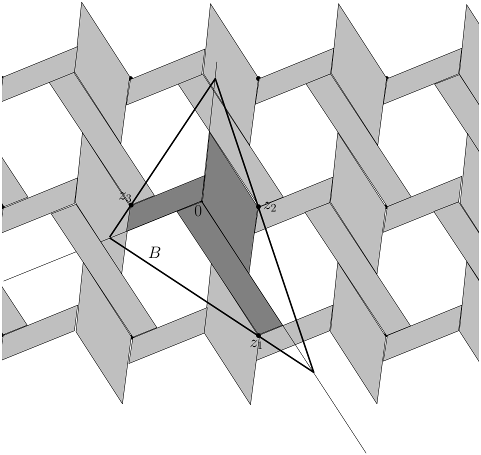
z_5

$R(S, B) + W_S = \mathbb{R}^n$




z_3
0
z_2
z_1

$R(S, B) + W_S \neq \mathbb{R}^n$



MORAL :

We want maximal S-free sets B such that $R(S, B) + W_S = \mathbb{R}^n$.
This gives us closed form formulas for cut generating pairs.

Connects with a lot of research on coverings and tilings by star-shaped bodies, extensively studied in Geometry of Numbers.

Operations that preserve the covering property

THEOREM Basu, Paat 2014
Let B be a maximal S-free polytope in $\mathbb{R}^n (n \geq 2)$ let $t \in \mathbb{R}^n$ such that $B + t$ still contains the origin. Then $R(S, B) + W_S = \mathbb{R}^n$ if and only if $R(S + t, B + t) + W_{S+t} = \mathbb{R}^n$

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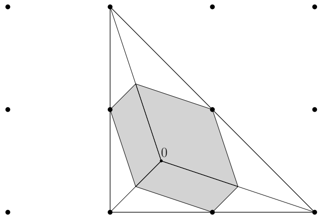

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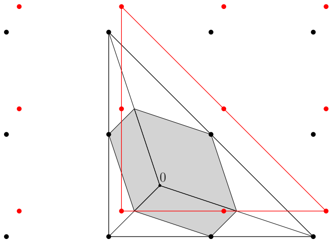

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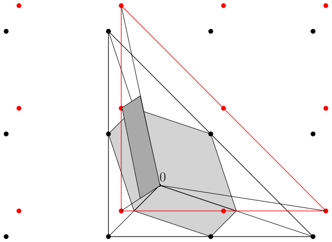

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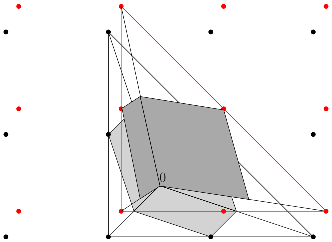

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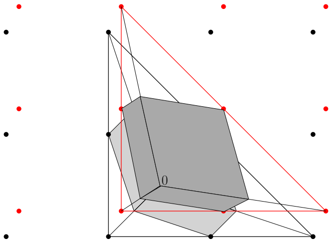

Operations that preserve the covering property

Coproduct Construction. Let $B_1 \subseteq \mathbb{R}^{n_1}$ and $B_2 \subseteq \mathbb{R}^{n_2}$. Let $0_i \in \textbf{int}(B_i)$, $i = 1, 2$. For any $0 < \mu < 1$, define the coproduct as a polytope in $\mathbb{R}^{n_1+n_2}$:

$$B_1 \diamond B_2 := \mathrm{conv}((\{\frac{B_1}{1-\mu} \times \{0_2\}) \cup (\{0_1\} \times \frac{B_2}{\mu})).$$

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Let $B_i \subseteq \mathbb{R}^{n_i}$ be maximal S_i-free polytopes and let $0 < \mu < 1$. Then $B_1 \diamond B_2 \subseteq \mathbb{R}^{n_1+n_2}$ is a maximal $S_1 \times S_2$-free polytope. Moreover, if B_1, B_2 both have the covering property, then so does $B_1 \diamond B_2$.

Extended to general unbounded B_1, B_2 by Basu, Paat 2014

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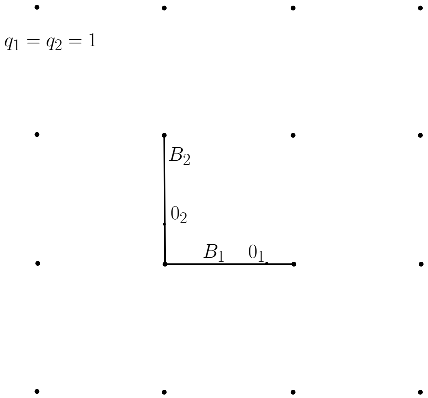

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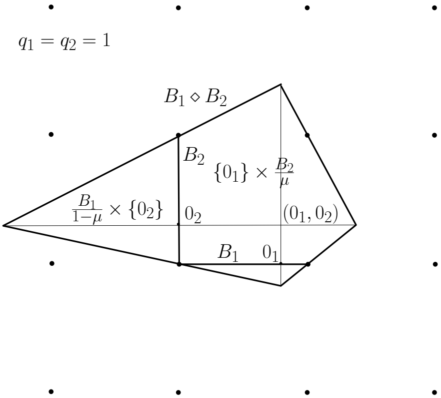

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Extended to general unbounded B_1, B_2 by Basu, Paat 2014

Operations that preserve the covering property

THEOREM Basu, Paat 2014

Let B_t be a sequence of maximal S-free polytopes that "converge" to a maximal S-free polytope B. If every polytope in the sequence has the covering property, then the "limit" polytope B has the covering property.

Covering Property of Pyramids

THEOREM Averkov, Basu (IPCO 2014)
Let P be a maximal S-free pyramid in $\mathbb{R}^n$ such that every facet of P contains exactly one integer point in its relative interior. P has the covering property if and only if P is an affine unimodular transformation of $\mathrm{conv}\{0, ne^1, \ldots, ne^n\}$.

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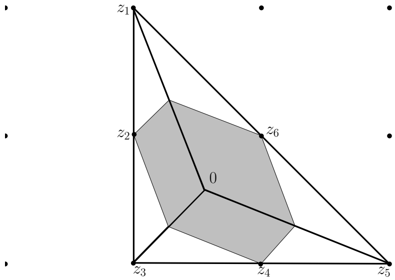

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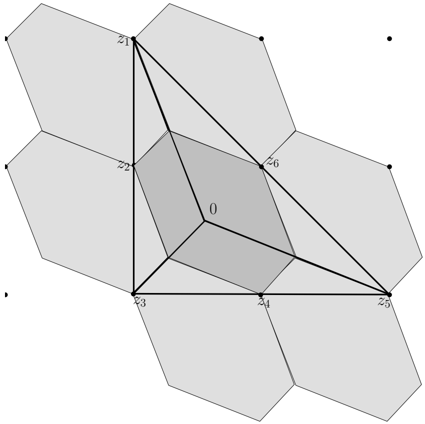

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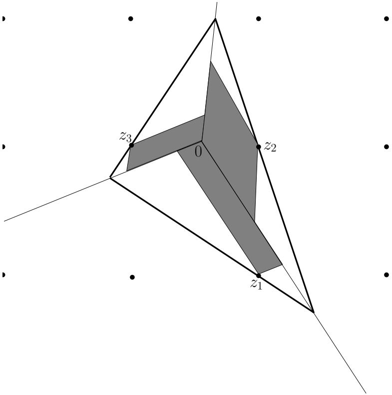

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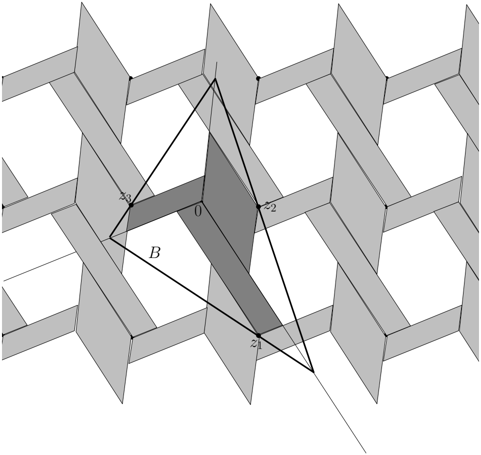

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PROOF:
1. Let 0 be the apex of P and consider $S_{z,F}$, where F is the base.
2. Venkov-Alexandrov-McMullen theorem $\Rightarrow$ $S_{z,F}$ is centrally symmetric with centrally symmetric facets. $S_{z,F}$ spindle $\Rightarrow$ every belt is of length 4 $\Rightarrow$ $n-2$ face is centrally symmetric.
3. McMullen's characterization of zonotopes $\Rightarrow$ $S_{z,F}$ is a zonotope whose every belt is of length 4.
4. Combinatorial geometry of zonotopes $\Rightarrow$ $S_{z,F}$ is a parallelotope. This implies P is a simplex.
5. Minkowski-Hajós theorem $\Rightarrow$ P is an affine unimodular transformation of $\mathrm{conv}\{0, ne^1, \dots, ne^n\}$.

$$X_S(R,P) := \{(s,y) \in \mathbb{R}^k_+ \times \mathbb{Z}^\ell_+ : \sum_{i=1}^{k} r^i s_i + \sum_{j=1}^{\ell} p^j y_j \in S\}$$

Want minimal valid pair (ψ_S, π_S) such that we have efficiently computable formulas.

Approach: Start with maximal S-free set
$B = \{x \in \mathbb{R}^n : a_i \cdot x \leq 1,\ i \in I\}$ with the covering property.

$$\psi_S(r) = \max_{i \in I} a_i \cdot r, \qquad \pi_S(r) = \min_{w \in W_S} \psi_S(r+w)$$

where $W_S = \mathbb{Z}^n \cap (\mathrm{lin}(\mathrm{conv}(S)))$.

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Another approach: First find "minimal valid" π and then try to find functions ψ that can be appended to π to create a valid pair.

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Balas 1971: Intersection Cuts

Balas, Jeroslow 1980: Monoidal Strengthening

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Gomory, Johnson 1971-74: Group Relaxations

THANK YOU !

Questions/Comments ?