Constructive Discrepancy Minimization for Convex Sets

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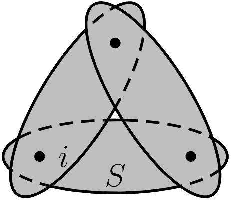
Discrepancy theory

- Set system $\mathcal{S} = \{S_1, \dots, S_m\}, S_i \subseteq [n]$



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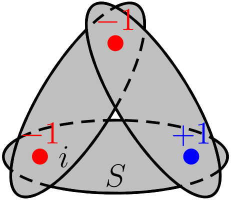

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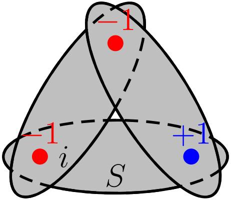

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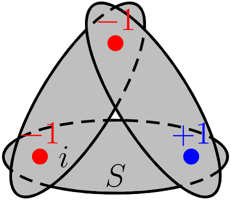

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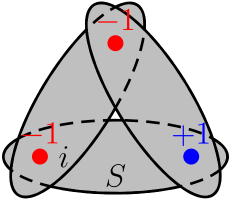

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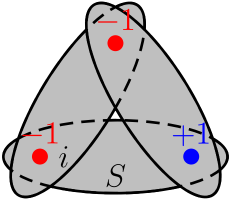

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Main method: Find a **partial coloring** $\chi : [n] \to \{0, \pm 1\}$

- low discrepancy $\max_{S\in\mathcal{S}} |\chi(S)|$
- $|\mathrm{supp}(\chi)| \geq \Omega(n)$

Discrepancy theory (2)

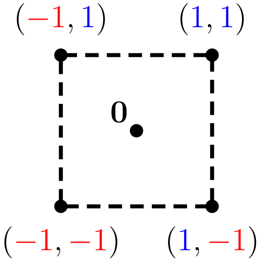
Lemma (Spencer)

For m set on $n \le m$ elements there is a **partial coloring** of discrepancy $O(\sqrt{n \log \frac{2m}{n}})$.

- Run argument $\log n$ times
- Total discrepancy is

$$\lesssim \sqrt{n} + \sqrt{n/2} + \sqrt{n/2^2} + \ldots + 1 = O(\sqrt{n})$$

Thm of Spencer-Gluskin-Giannopolous



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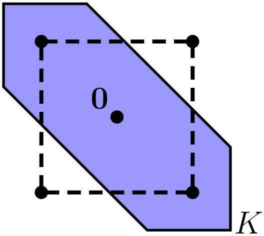

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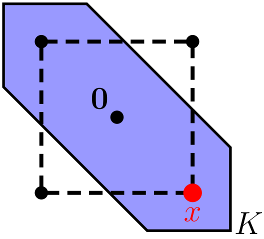

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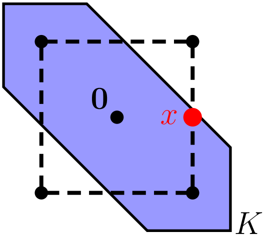

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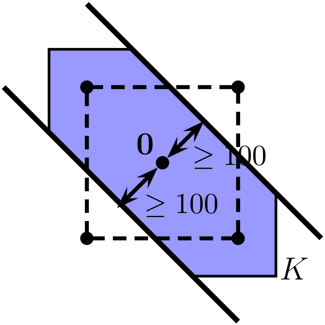

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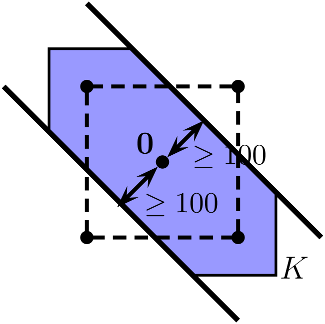

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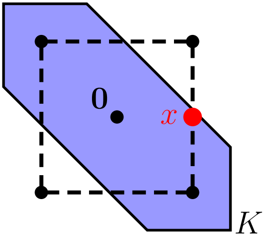

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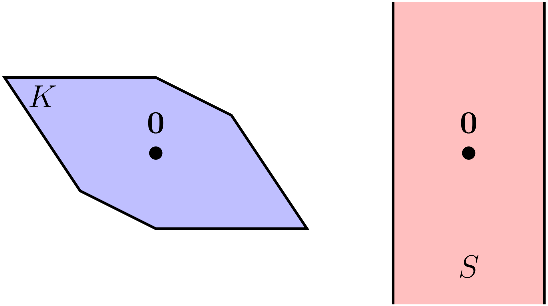
- Counting argument: any such K admits partial coloring

Gaussian measure

Lemma (Sidak-Kathri '67)

For K convex and symmetric and strip S,

$$\gamma_n(K \cap S) \geq \gamma_n(K) \cdot \gamma_n(S)$$



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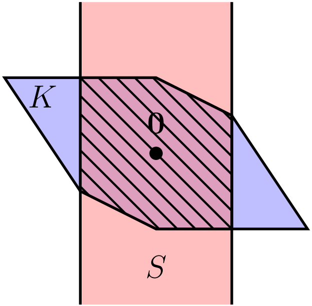

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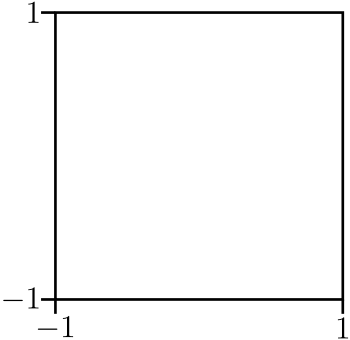
- **Lovett-Meka '12:**
  - (+) poly-time
  - (+) simple and elegant
  - (+/−) Works for any $K = \{x : |\langle x, v_i\rangle| \leq \lambda_i \forall i \in [m]\}$ with $\sum_{i=1}^{m} e^{-\lambda_i^2/16} \leq \frac{n}{16}$

Constructive Partial Coloring Lemma

Lemma [Lovett-Meka '12]



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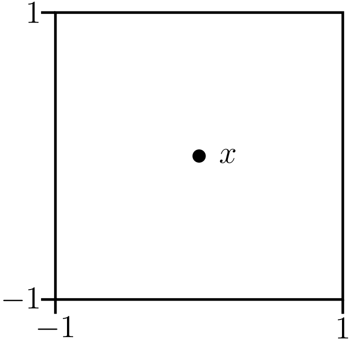

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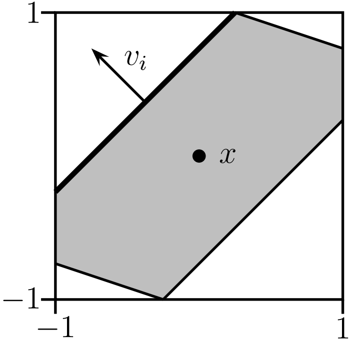

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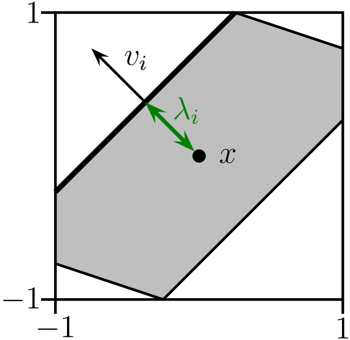

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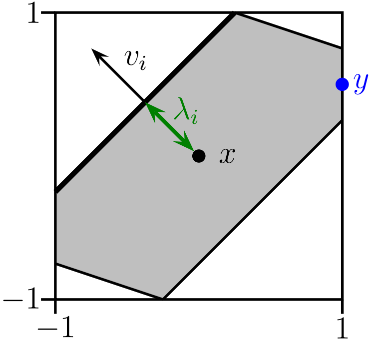

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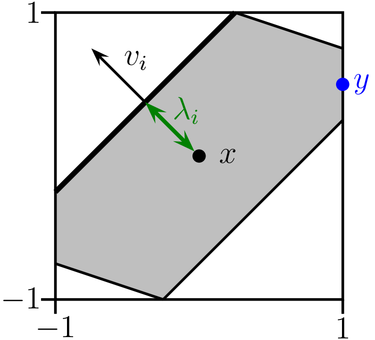

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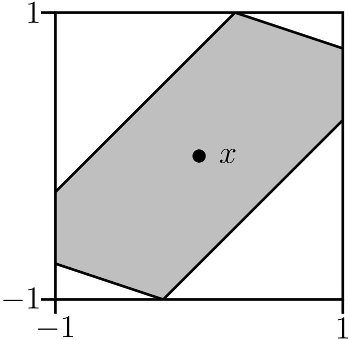

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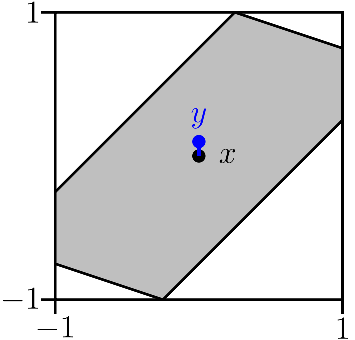

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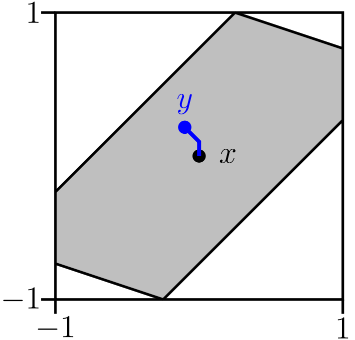

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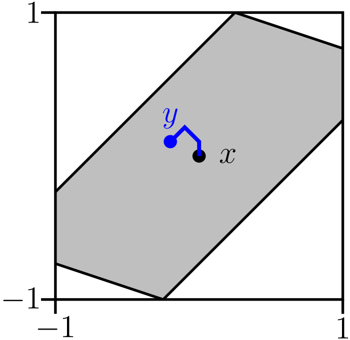

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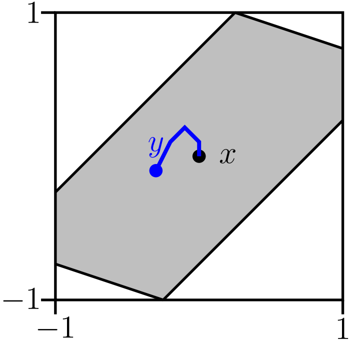

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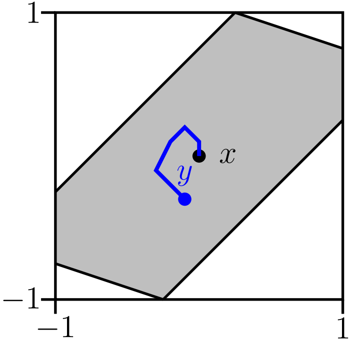

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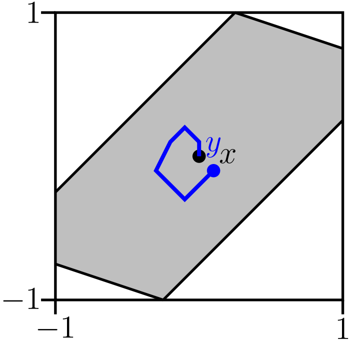

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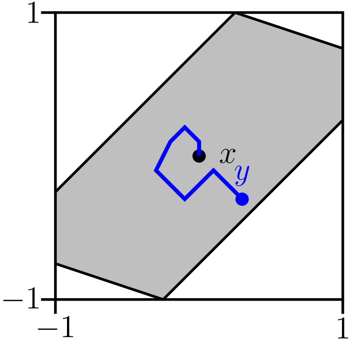

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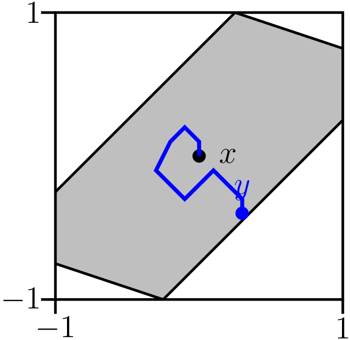

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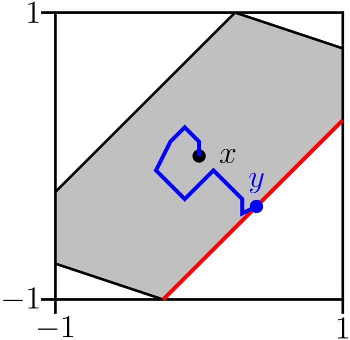

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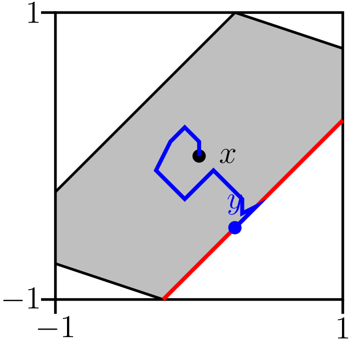

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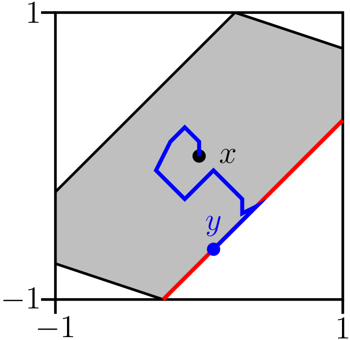

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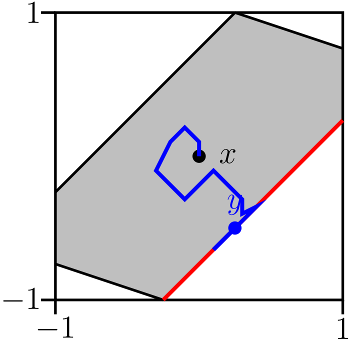

Constructive Partial Coloring Lemma

Lemma [Lovett-Meka '12]

Given $x \in [-1,1]^m$, unit vectors v_i , parameters $\lambda_i \geq 0$ s.t. $\sum_i e^{-\lambda_i^2/16} \leq \frac{m}{16}$. Then one can find $y \in [-1,1]^m$ with

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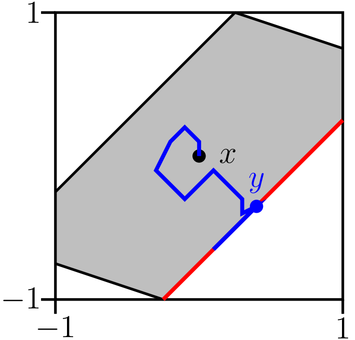

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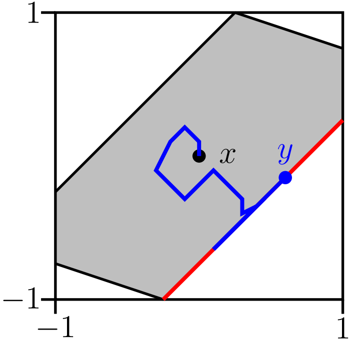

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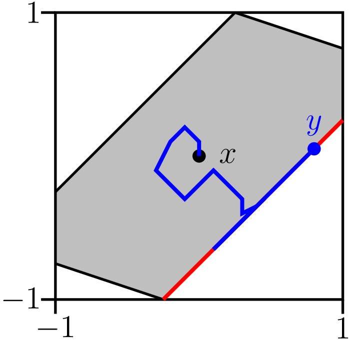

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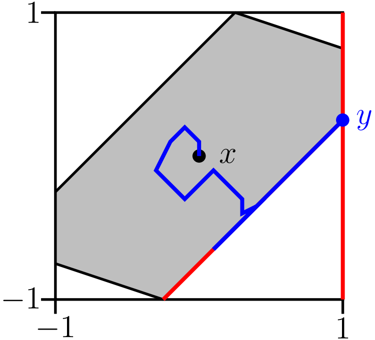

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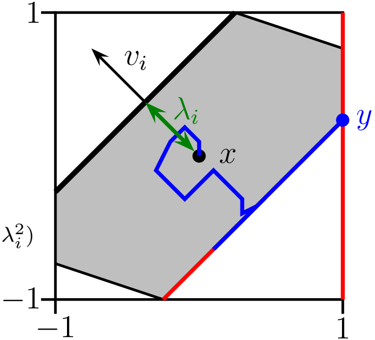

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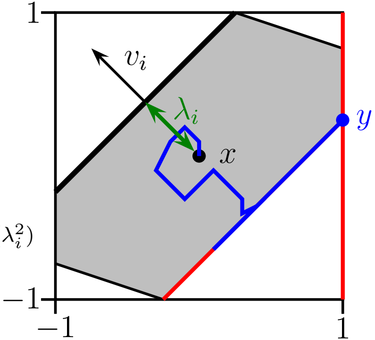
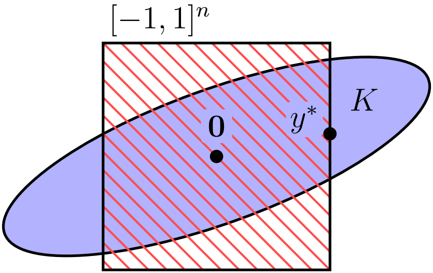


Our main result

Theorem (R. 2014)

For a **convex symmetric set** $K \subseteq \mathbb{R}^n$ with $\gamma_n(K) \geq e^{-\Theta(n)}$, one can find a $y \in K \cap [-1,1]^n$ with $|\{i : y_i = \pm 1\}| \geq \Theta(n)$ in **poly-time**.



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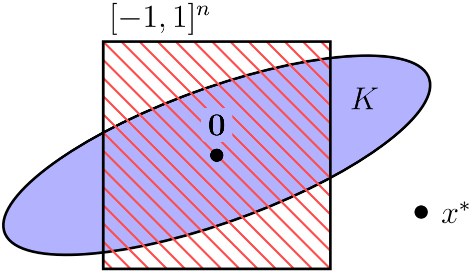

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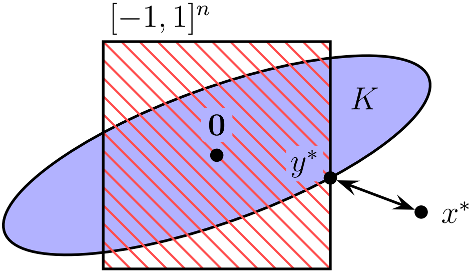
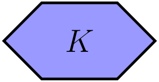


Isoperimetric inequality

- For set K



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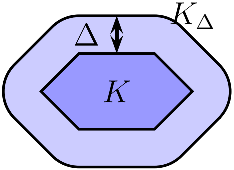

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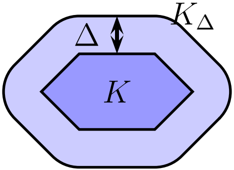

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$\gamma_n(K) \geq e^{-\delta n} \implies \gamma_n(K_{3\sqrt{\delta n}}) \geq 1 - e^{-\delta n}$

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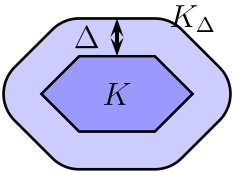

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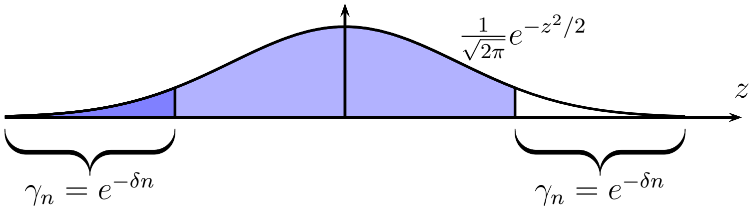

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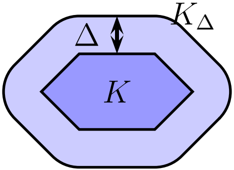

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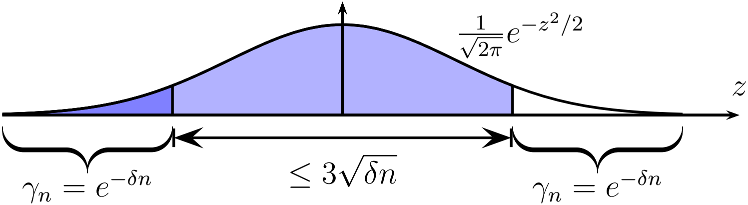

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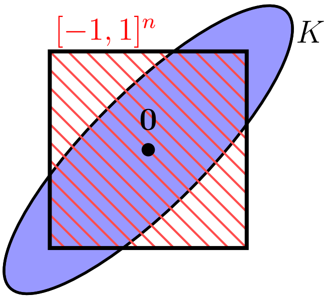

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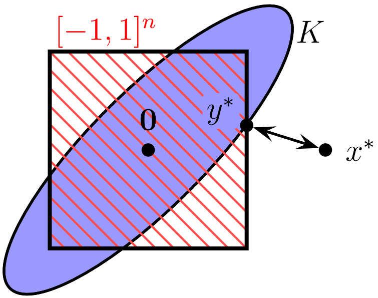

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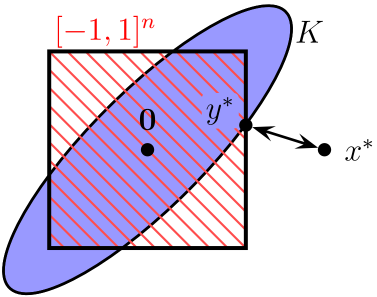

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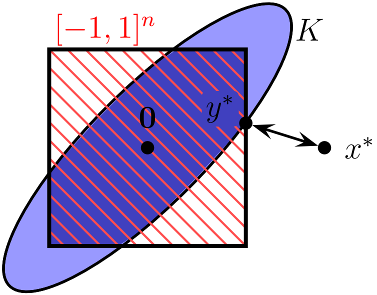

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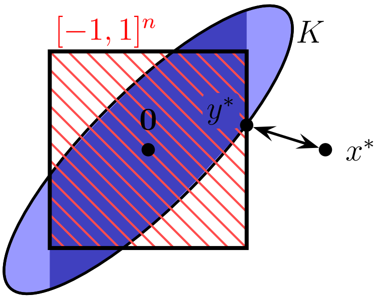

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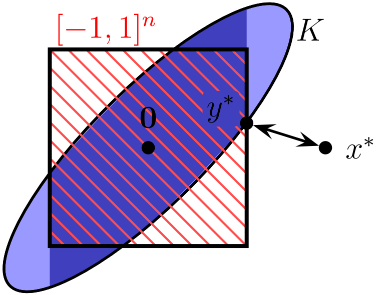

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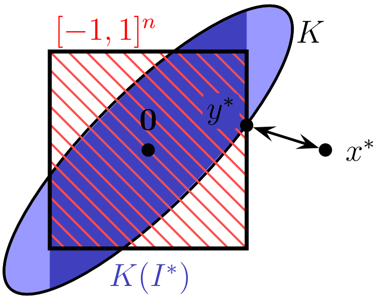

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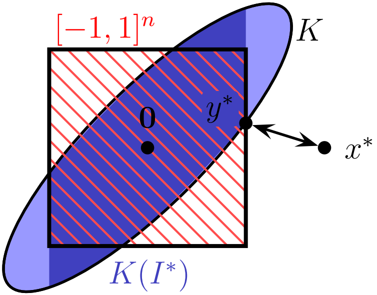

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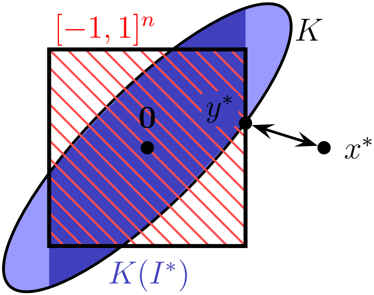

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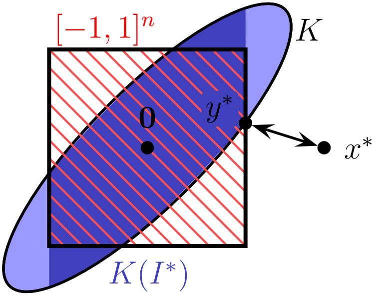

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Application

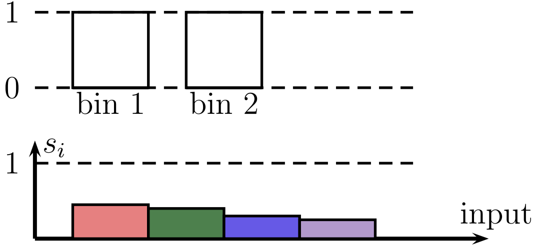
An additive $O(\log OPT)$-apx for Bin Packing

Joint work with Rebecca Hoberg

Bin Packing

Input: Items with sizes $s_1, \ldots, s_n \in [0, 1]$

Goal: Pack items into minimum number of **bins** of size 1.



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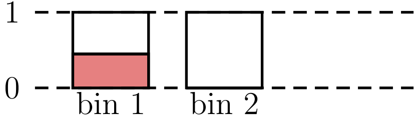

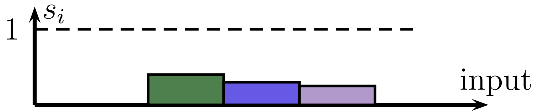

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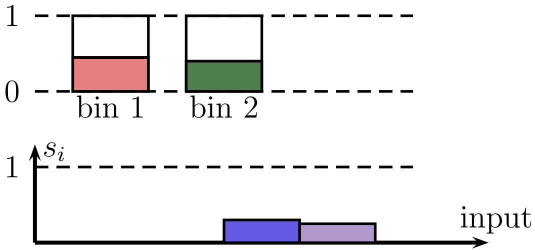

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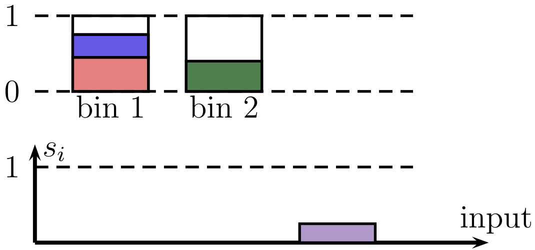

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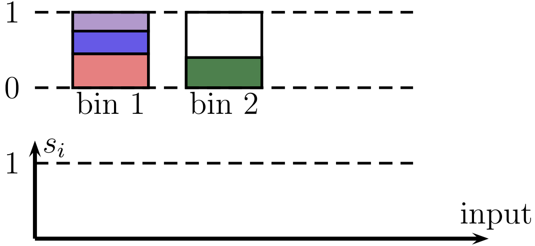

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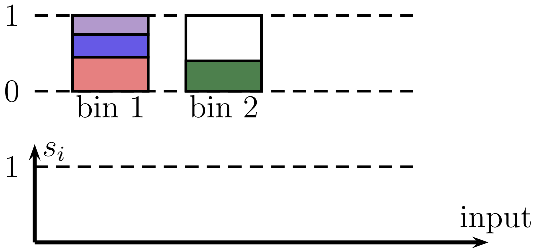

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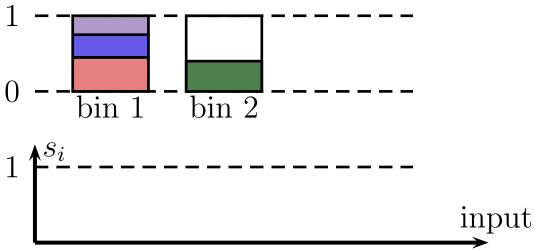

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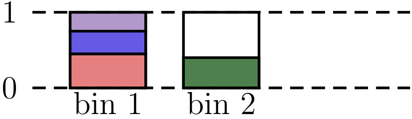

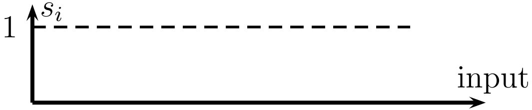

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- First Fit Decreasing [Johnson '73]: $APX \leq \frac{11}{9} OPT + 4$
- [de la Vega & Lücker '81] :
  $APX \leq (1 + \varepsilon) OPT + O(1/\varepsilon^2)$ in time $O(n) \cdot f(\varepsilon)$

The Gilmore Gomory LP relaxation

- $b_i = \#\text{items with size } s_i$
- **Feasible patterns**:

$$\mathcal{P} = \left\{ p \in \mathbb{Z}^n_{\geq 0} \mid \sum_{i=1}^{n} s_i p_i \leq 1 \right\}$$

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- **Gilmore Gomory LP relaxation:**

$$\begin{aligned} \min \sum_{p \in \mathcal{P}} x_p & & & \\ \sum_{p \in \mathcal{P}} p_i \cdot x_p & \geq & b_i & \quad \forall i \in [n] \\ x_p & \geq & 0 & \quad \forall p \in \mathcal{P} \end{aligned}$$

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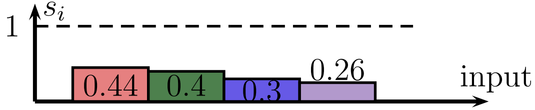
- Can find x with $\mathbf{1}^T x \leq OPT_f + \delta$ in time $\text{poly}(\|b\|_1, \frac{1}{\delta})$

The Gilmore Gomory LP - Example



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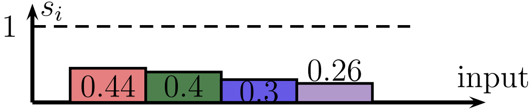

$$\min \mathbf{1}^T x$$

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 3 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} x \geq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$x \geq \mathbf{0}$$

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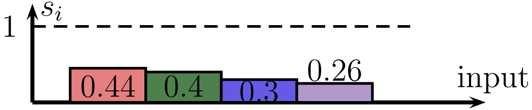

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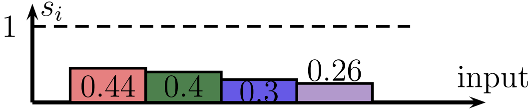

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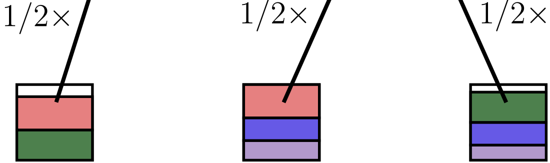

Main result

- [Karmarkar & Karp '82]: $APX \leq OPT + O(\log^2 OPT)$
  Technique: Basic LP solutions

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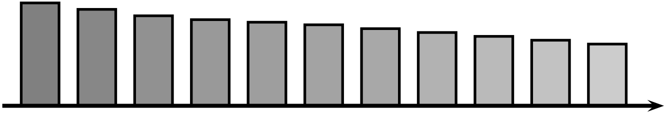
Theorem (Hoberg & R. '15)

There is an $OPT + O(\log OPT)$ algorithm for Bin Packing.

- **Technique:** Discrepancy theory

Karmarkar-Karp's Grouping

input:



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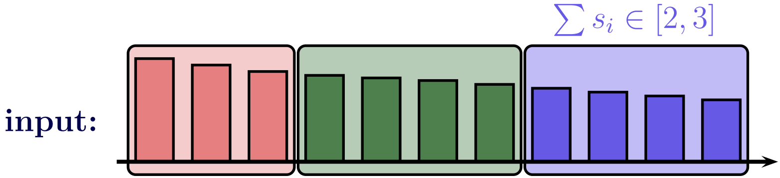

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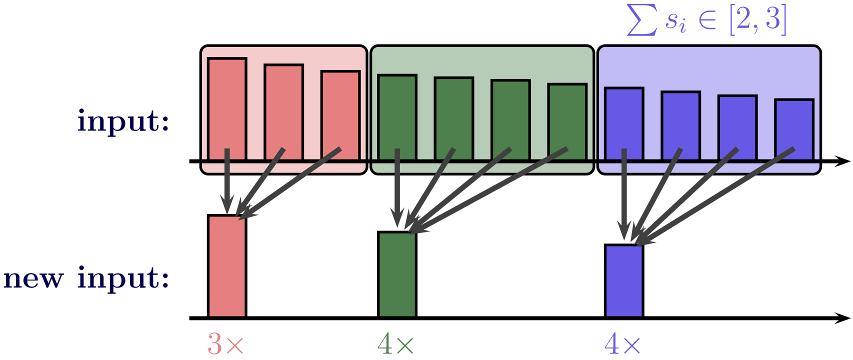

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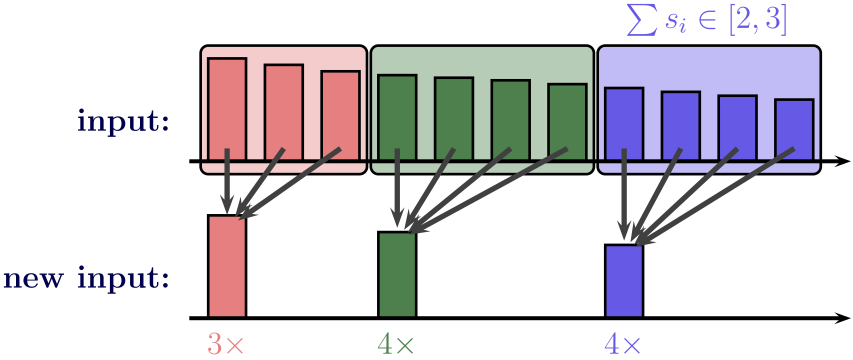

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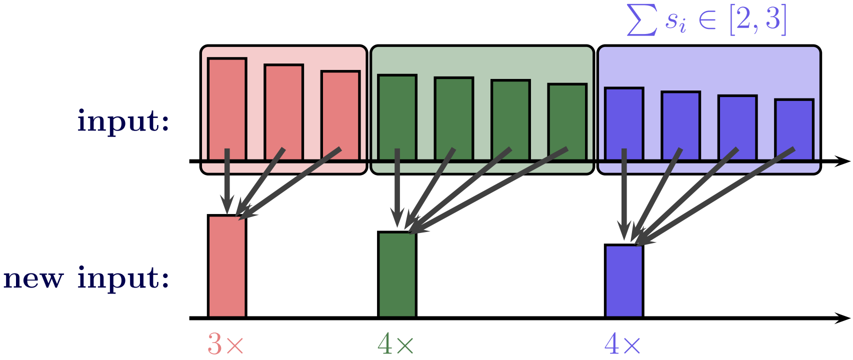

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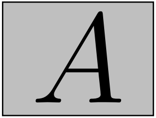

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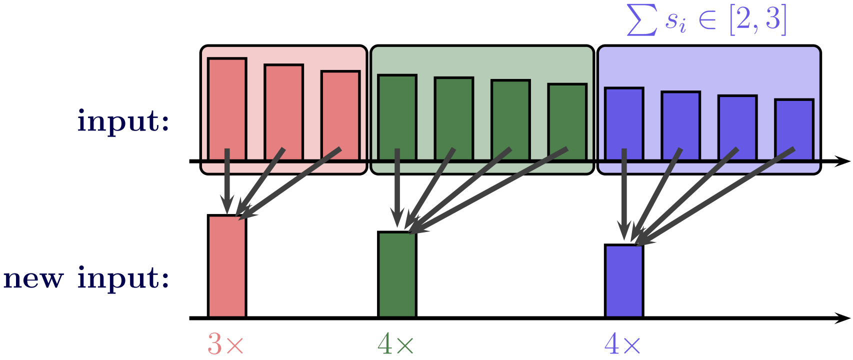

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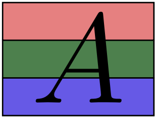

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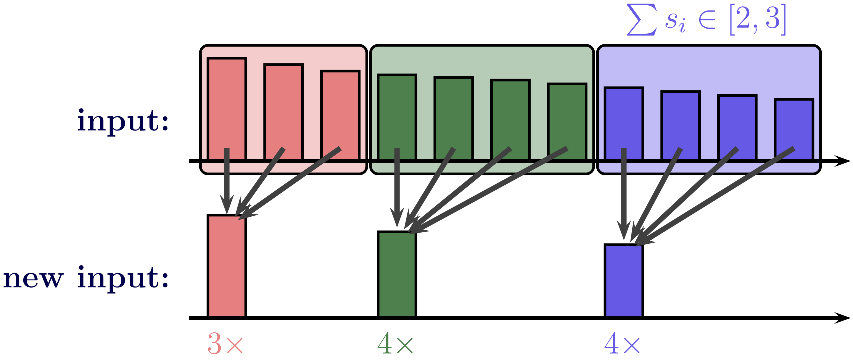

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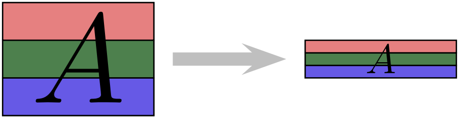

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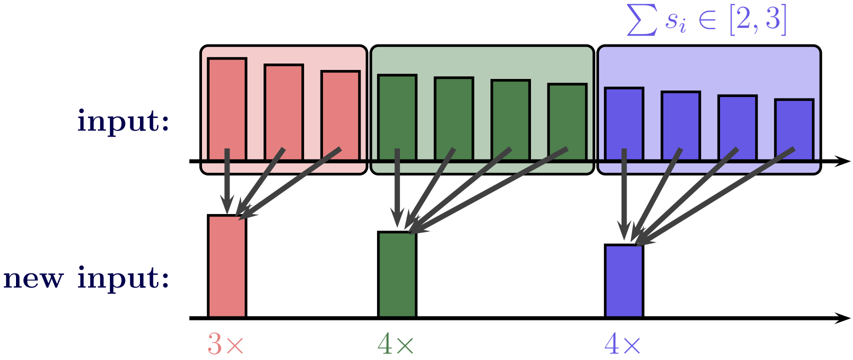

- increases OPT by $O(\log n)$
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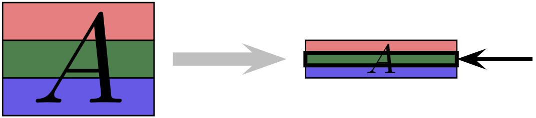

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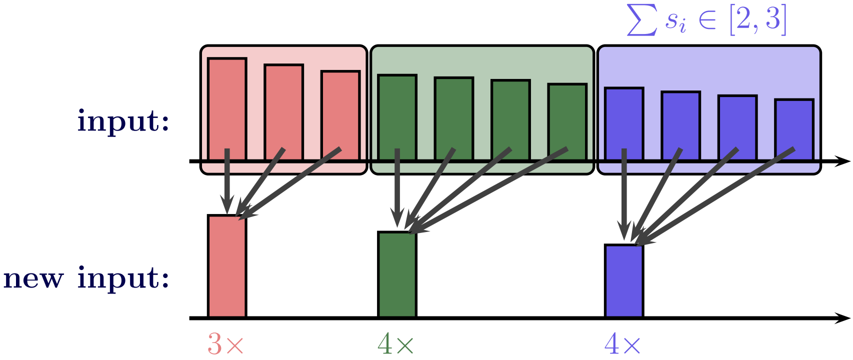

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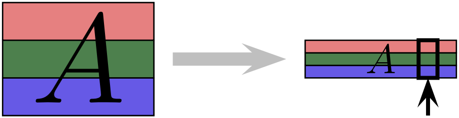

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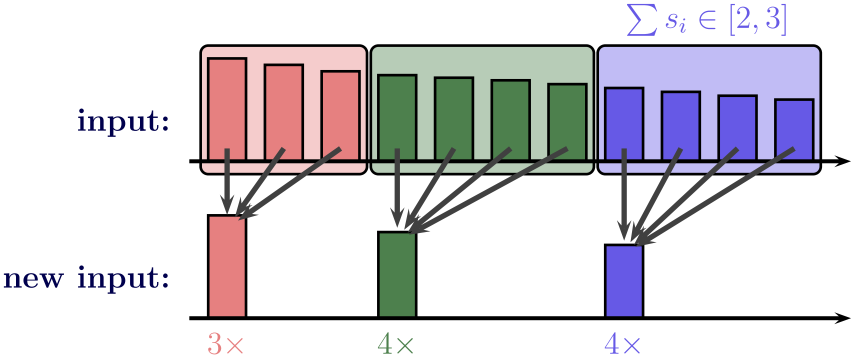

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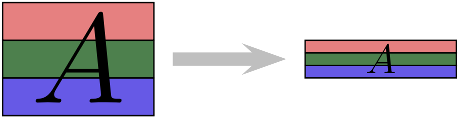
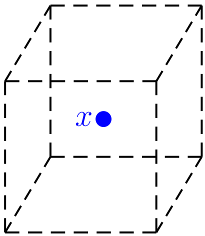
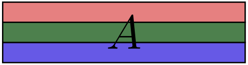


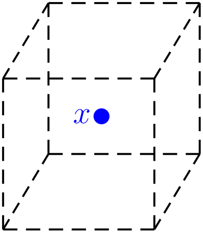
Karmarkar-Karp algo (2)



Karmarkar-Karp algo (2)

- After grouping: # constraints $\leq \frac{1}{2}|\mathrm{supp}(x)|$





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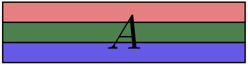

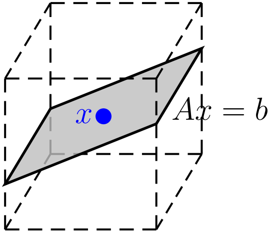

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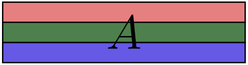

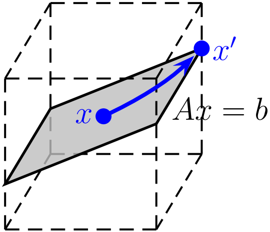

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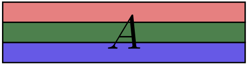

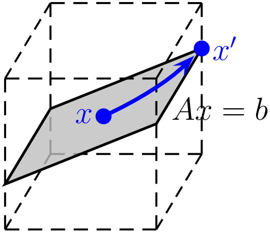

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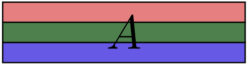
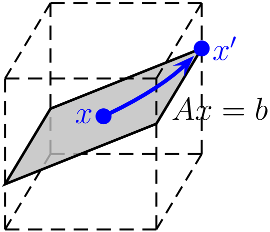
- Repeat $O(\log n)$ times $\rightarrow \boldsymbol{O(\log^2 n)}$

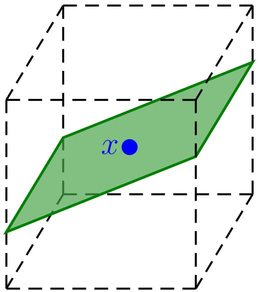




Applying the Constr. Part. Col. Lem.

Comparison:

- **Basic solutions:** $\Theta(m)$ constraints with $\lambda_i = 0$



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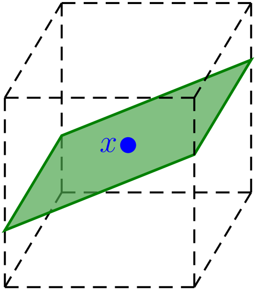

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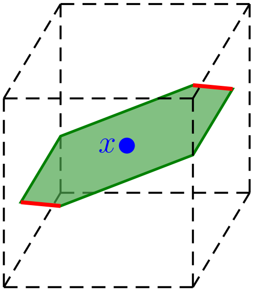

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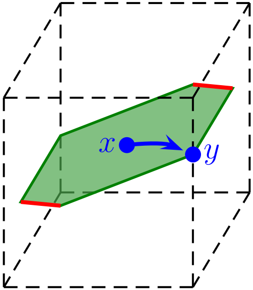

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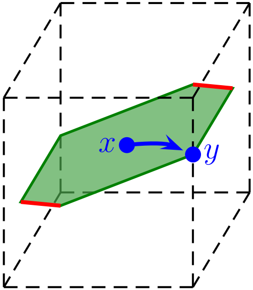


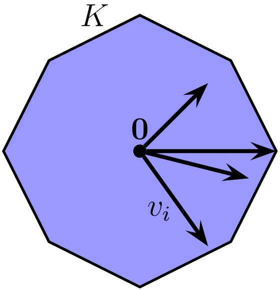
- **Complication:** Need to rebuild incidence matrix for improvement

Open problems

Open problems (1)

Theorem (Banaszczyk '98)

Given a convex body K with $\gamma_n(K) \geq \frac{1}{2}$, vectors $v_1, \ldots, v_n$ with $\|v_i\|_2 \leq \frac{1}{5}$. There is a **coloring** $x \in \{\pm 1\}^n$: $\sum_{i=1}^n x_i v_i \in K$



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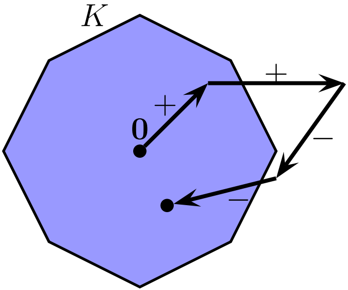

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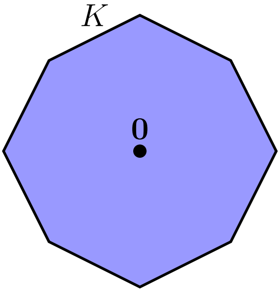

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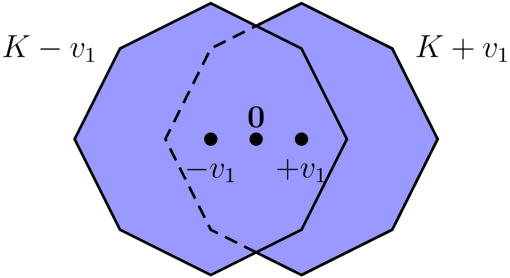

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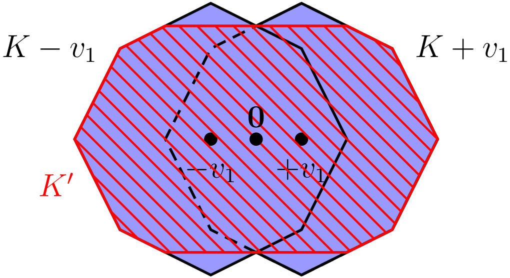

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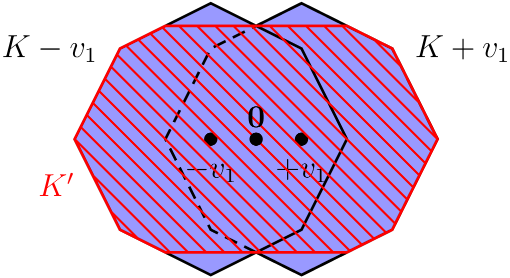

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- Prove $\gamma_n(K') \geq \frac{1}{2}$ $\rightarrow$ **Non-constructive**

Open problem (2)

Setting: Given a set system $S_1, \ldots, S_m \subseteq [n]$ where each element is in at most t sets.

Beck-Fiala Conjecture: disc $\leq O(\sqrt{t})$

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Known bounds:

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Thanks for your attention