Affine Geometric Analysis

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Over the last 25 years the affine geometry of convex bodies has changed dramatically. Milestones were the extension of affine area to general convex bodies and the development of the theory of affine invariant valuations. The resulting body of work has proved to be an invaluable tool in fields like stochastic geometry and PDEs.

Even more recently, the new field of Affine Geometric Analysis started to emerge. A central question is to establish functional versions of results and problems from the affine geometry of convex bodies. Moreover, $\mathrm{SL}(n)$ invariants and linearly associated norms and tensors of functions and their associated inequalities are studied systematically. The resulting analytic inequalities are almost invariably stronger than their Euclidean counterparts. Methods and results from the affine geometry of convex bodies and from affine differential geometry, in particular, affine flows are applied. It is impossible to describe all the recent developments, but we hope that four examples will give some idea of the range and potential of this emerging field.

1 Affine Sobolev Inequalities

A fundamental analytic inequality with geometric content is the sharp L^1 Sobolev inequality established by Federer & Fleming and Maz′ya in 1960. For $f : \mathbb{R}^n \to \mathbb{R}$ in the Sobolev space $W^{1,1}(\mathbb{R}^n)$, it states that

$$\frac{1}{n}\int_{\mathbb{R}^n} |\nabla f(x)|\, dx \geq n\, v_n^{\frac{1}{n}}\, \|f\|_{\frac{n}{n-1}}, \tag{1}$$

where v_n is the n-dimensional volume of the n-dimensional unit ball, on the left side is the Euclidean norm of the gradient and

$$\|f\|_p = \Big(\int_{\mathbb{R}^n} |f(x)|^p dx\Big)^{1/p}.$$

Its geometric equivalent is the Euclidean isoperimetric inequality. Gaoyong Zhang [52] established the *affine Zhang-Sobolev inequality* for $f \in W^{1,1}(\mathbb{R}^n)$,

$$\Big(\frac{1}{n}\int_{\mathbb{S}^{n-1}} \|D_u f(x)\|^{-n}\, du\Big)^{-\frac{1}{n}} \geq c_n\, \|f\|_{\frac{n}{n-1}}, \tag{2}$$

where $D_u f$ is the partial derivate of f in direction u and $c_n = 2v_{n-1}/v_n$. The affine Sobolev inequality is significantly stronger and implies (by Hölder's inequality) the sharp L^1 Sobolev inequality. Moreover, the Zhang-Sobolev inequality is *affine*, which means that both sides of the inequality are not changed by applying a special linear transformation in the domain of the functions. The corresponding geometric inequality is the generalized Petty projection inequality, which is sharp precisely for ellipsoids (whereas the isoperimetric inequality is sharp precisely for balls). The affine Sobolev inequality has become a cornerstone of Affine Geometric Analysis. Extensions and analogues were obtained by Lutwak, Yang & Zhang [35], Cianchi, Lutwak, Yang & Zhang [17], Haberl & Schuster [23], and Tuo Wang [50].

Many of the affine inequalities for convex bodies center around the generalization of the Minkowski problem introduced by Lutwak [32]: Given a real number p, what are the necessary and sufficient conditions on a Borel measure on the unit sphere $\mathbb{S}^{n-1}$ to be the L^p-surface area measure of a convex body (that is, the classical surface area measure multiplied with the support function raised to the $(1-p)$-th power). The case $p = 1$ is the classical Minkowski problem with ground-breaking contributions to the regularity question by Cheng & Yau, Nirenberg and Pogorelov. For $p < 1$, many cases are still open and intensely studied by researchers working within convex geometry and on Monge-Ampère equations. First results on the case $p = 0$ were obtained by Stancu [45, 46] and very recently, the case $p = 0$ has been solved under assumption of symmetry by Böröczky, Lutwak, Yang & Zhang [10].

In this direction, a completely different approach to the affine L^p Sobolev inequality uses the so-called LYZ operator (Lutwak, Yang & Zhang [37]) to formulate and solve the functional L^p Minkowski problem. The solution of the even *functional* L^p Minkowski problem associates with a function (from a suitable Sobolev space) an origin-symmetric convex body, which is (as shown by Lutwak, Yang & Zhang [37]) the unit ball of the *optimal* Sobolev norm. A general version of the L^1 Sobolev inequality is

$$\frac{1}{n}\int_{\mathbb{R}^n} \|\nabla f(x)\|_{K^*}\,dx \geq n\,v_n^{1/n}\Big(\int_{\mathbb{R}^n} |f(x)|^{n/(n-1)}dx\Big)^{(n-1)/n}, \tag{3}$$

where $\|\cdot\|_L$ is the norm with unit ball L and $K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}$ is the polar body of the convex body K. Here the origin-symmetric convex body has volume v_n (as in (2)). The right side of (3) does not depend on K and hence it makes sense to ask for the optimal Sobolev norm which is the norm that for a given function $f \in W^{1,1}(\mathbb{R}^n)$ minimizes the left side. The functional L^p Minkowski problem has thus become a central notion within Affine Geometric Analysis. Whereas for $p \geq 1$ (and, in particular, $p = 1$) fundamental results have been established, many questions remain open for $p < 1$ and in the general (not even) setting.

In his talk (based on [25]), Carlos Hugo Jiménez presented a new approach to some sharp affine functional inequalities, including log-Sobolev, Sobolev and Gagliardo-Nirenberg inequalities, using the L_p Busemann-Petty centroid inequality along with some classical results for general norms. These results are not using the solution to the L_p Minkowski problem and are in this sense more elementary than the original approach by Zhang [52] and Lutwak, Yang & Zhang [35].

Andrea Cianchi presented new results (based on [16]) on Sobolev trace inequalities and their connections to relative isoperimetric inequalities. These results show that indicator functions of Euclidean balls minimize the so-called trace constants in the space of functions of bounded variations. They are thus not affine invariant. Hence it is a natural question to find the corresponding results within Affine Geometric Analysis.

Closely connected to affine Sobolev inequalities is the new notion of variational affine capacity that was presented by Jie Xiao (based on [51]). While capacities play a major role in geometric analysis, within the affine context these questions just have been started to be studied.

2 Functional Versions of Classical Inequalities

There is a general approach to extend invariants and inequalities of convex bodies to corresponding invariants and inequalities for functions. The important connection between the Euclidean isoperimetric inequality and the sharp Sobolev inequality was already mentioned. Another important connection is that between the classical Brunn-Minkowski inequality and the Prékopa-Leindler inequality. For Borel sets K and L in $\mathbb{R}^n$ and given $\lambda \in (0,1)$, the Brunn-Minkowski inequality in its multiplicative form states that the n-Lebesgue measure V_n of the Minkowski linear combination $\lambda K + (1-\lambda)L = \{(1-\lambda)x + \lambda y \in \mathbb{R}^n : x \in K, y \in L\}$ is bounded from below in the following way

$$V_n(\lambda K + (1-\lambda)L) \geq V_n(K)^{1-\lambda}\,V_n(L)^{\lambda}. \tag{4}$$

The Prékopa-Leindler inequality states that, for given $\lambda \in (0,1)$ and measurable functions $f, g, h : \mathbb{R}^n \to [0,\infty)$ if, for any $x, y \in \mathbb{R}^n$,

$$h\big((1-\lambda)x + \lambda y\big) \geq f(x)^{1-\lambda} g(y)^{\lambda},$$

then

$$\int_{\mathbb{R}^n} h\,dx \geq \left(\int_{\mathbb{R}^n} f\,dx\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g\,dx\right)^{\lambda}. \tag{5}$$

It is easy to derive the geometric inequality (4) from the functional inequality (5) and vice versa.

Among the most important affine geometric inequalities is the Blaschke-Santaló inequality, which states that the so-called volume product, that is, the product of the volume of an origin-symmetric convex set and its polar, is maximized by centered ellipsoids:

$$V_n(K)\,V_n(K^*) \leq v_n^2 \tag{6}$$

for K an origin-symmetric convex set in $\mathbb{R}^n$. The corresponding functional Blaschke-Santaló inequality for log concave functions, which involves the Legendre transform, was established by Keith Ball in his thesis (University of Cambridge 1987) and has inspired many results in the field.

The Blaschke-Santaló inequality was proved by Blaschke with the help of the so-called affine isoperimetric inequality (established by Blaschke, Deicke, and Santaló for convex sets with smooth boundary). The affine isoperimetric inequality gives a sharp upper bound of affine area of a convex body in terms of its volume. The equivalent of this inequality for log concave functions is a *reverse log Sobolev inequality* for entropy. It was established recently by Artstein-Avidan, Klartag, Schütt & Werner [4]. This was the starting point to introduce the concept of *f-divergence* for log concave and s-concave functions [13, 14]. Such divergences and their related inequalities are important tools in information theory, statistics, probability theory and machine learning. These recent developments are yet other instances of the rapidly developing, fascinating connections between Affine Geometric Analysis and information theory. Further examples can be found in e.g., [15, 33, 34, 36, 42].

Among the classical fundamental inequalities in the affine geometry of convex bodies are the Petty projection inequality, the Busemann-Petty centroid inequality and the Busemann intersection inequality. As mentioned in the last section, the Petty projection inequality is the core of the affine Sobolev-Zhang inequality, while the Busemann-Petty centroid was used in [25] in the proof of affine Sobolev inequalities.

In his talk, Jesús Yepes Nicolás (based on [18]) presented a linear refinement of the Prékopa-Leindler inequality (5). If f and g have a common projection onto a hyperplane (which is the analytic counterpart of the projection of a set onto a hyperplane), the Prékopa-Leindler inequality admits a linear refinement. That is, under such an assumption for the functions f and g, the right-hand side of (5) may be exchanged by the convex combination of the integrals, which yields a stronger inequality. Moreover, the same inequality can be obtained when assuming that both projections (not necessarily equal as functions) have the same integral.

Alexander Segal presented functional inequalities involving the geometric inf-convolution, which corresponds to the Minkowski addition of level sets. He also presented a further geometric analogue of the Prékopa-Leindler inequality as well as of the Borell-Brascamp-Lieb inequalities.

Besides the already mentioned Brunn-Minkowski inequality and Petty projection inequality, also the Busemann intersection inequality is of central importance in the affine geometry of convex bodies. It bounds the volume of the *intersection body* of a convex body by the volume of the body itself. In her talk, Susanna Dann (based on [19]) presented, among other things, a functional version of the Busemann intersection inequality. She also obtained functional versions of inequalities for affine quermassintegrals. These new inequalities bound marginals of probability densities.

3 Affine Area and its Applications

It is Wilhelm Blaschke who introduced in dimension three the celebrated affine area and showed the intrinsic connection of affine geometry to convexity. While at Blaschke's time affine geometry referred to the study of geometric invariants of convex hypersurfaces in $\mathbb{R}^n$ with respect to the equi-affine group of transformations (that is, translations combined with the special linear group, $\mathrm{SL}(n)$) of the Euclidean space, soon afterwards it included the study of the local and global invariants of convex hypersurfaces under the general linear group and special linear group of transformations of the Euclidean space.

In the theory of partial differential equations, Neil Trudinger & Xujia Wang's solution [49] of the affine Plateau problem and their work on the affine Bernstein problem [48] centers around the affine area functional. Asymptotic behavior of affine flows studied by Ben Andrews [1], and in non-compact setting by Loftin & Tsui [26], employs its properties as well and relies essentially on the classical affine isoperimetric inequality relating the affine area of a convex body to its volume. In 2011, a class of centro-affine curvature flows were introduced by Stancu [47] such that each flow in the class corresponds, in a certain way, to a L_p-affine surface area, for a fixed $p \neq -n,\ -\infty < p < +\infty$. Given the SL$(n)$-invariance of each flow, any centered ellipsoid, of arbitrary eccentricity, is a homothetic solution to these flows. The monotonicity of the evolution of the L_p isoperimetric ratio under these flows, which remained constant for centered ellipsoids, strongly suggested that any convex body will shrink under these flows to a point while its shape approaches that of an ellipsoid. For many of these flows, the results on the long term existence and asymptotic behavior of the flow confirmed that any initial convex body, sufficiently smooth, will flow to an ellipsoid of the same volume. Mohammad Najafi Ivaki [24] proved, and successfully employed, the asymptotic behavior of such a flow to obtain a stability result to the planar Busemann-Petty centroid inequality. In his talk, he showed how such methods can be potentially used for other stability results of equiaffine invariant geometric inequalities.

This brings us to the area of (equi-)affine isoperimetric inequalities which are not only central to affine convex geometry, but have also many applications in, for example, quantum information theory. These inequalities aim to find the best possible upper and/or lower bounds, in terms of volume, for SL(n)- or affine invariant functionals on space of convex bodies. Deping Ye talked about recent progress on affine isoperimetric inequalities for geominimal surface area. This notion shares many properties with the affine surface area, but it is different in other respects. For example, geominimal surface area is continuous, while the classical affine surface area is only upper semi-continuous, on the set of all convex bodies equipped with the Hausdorff metric. Ye explained, in particular, how to define the L_p geominimal surface areas for all $-n \neq p < 1$ and its Orlicz extension, which generalize the L_p geominimal surface areas for $p \geq 1$ (defined by Petty for $p = 1$ and by Lutwak for $p > 1$). One should add that equiaffine invariants are now at core of the rapidly developing L_p-Brunn-Minkowski theory which has known several major results during the past few years (cf. [10, 17, 23, 32, 33, 34, 35, 42]).

Intertwined to the notion of affine surface area of a convex body K in $\mathbb{R}^n$ is the *convex floating body* K_δ. For $\delta > 0$ small enough, K_δ is the intersection of all half spaces whose defining hyperplanes cut of a set of volume δ from K [43]. The convex floating body has played an important role in extending the affine surface area from smooth bodies to all convex bodies. This extension is motivated by the fact that affine surface area is a classical and powerful tool in the (equi-)affine geometry of convex bodies and appears in applications ranging from PDEs to affine analytical isoperimetric inequalities, and to the approximation of convex bodies by polytopes. Several talks were devoted to this topic. There are several equivalent representations of affine surface area. Yiming Zhao, a PhD student from New York University, Polytechnic School of Engineering, discussed a new representation of affine surface area based on curvature measures. This new representation fills a *missing* piece from the already known definitions. The new representation [53], which is also equivalent to the existing ones [43], is *polar* to that of Lutwak [31] and and dual to that of Schütt & Werner [43]. Because of its importance, it is desirable to have the affine surface area defined not only in the Euclidean setting. Extensions to spherical space, and to hyperbolic space, were recently achieved by Florian Besau and Elisabeth Werner. Florian Besau, a recent PhD student, gave a talk on this research. The extensions, called floating areas, to spherical and hyperbolic space were achieved also via a notion of spherical and hyperbolic floating bodies [8, 9]. As in the Euclidean setting, differentiation of the volume difference of the body and the floating body gives rise to the floating area.

4 Valuations and Characterization Theorems

Valuations or additive functions are classical concepts in geometry. They were the critical ingredient in Dehn's solution of Hilbert's Third Problem in 1900. A milestone was Hadwiger's classification of rigid motion invariant valuations on convex bodies, which Gian-Carlo Rota would often describe as one of the most important results of twentieth century mathematics. Within the affine geometry of convex bodies, Haberl & Parapatits [22] were able to achieve a breakthrough last year by classifying all $\mathrm{SL}(n)$ invariant valuations on convex bodies containing the origin. Combined with a recent result of Ludwig & Reitzner [30], their theorem establishes a centro-affine Hadwiger theorem and a complete characterization of general affine areas. For classical affine area such a characterization as an $\mathrm{SL}(n)$ and translation invariant, upper semicontinuous valuation was obtained in [29]. Together with the recent results on functional affine area, this should lead to a complete picture for affine areas within Affine Geometric Analysis at large.

More generally, a systematic study of affine valuations on function spaces was recently started by Ludwig [28, 27]. The LYZ operator that is critical for affine Sobolev inequalities turned out to be the unique affine convex-body-valued valuations on the Sobolev space $W^{1,1}(\mathbb{R}^n)$ and the Fisher information matrix the unique affine matrix-valued valuations on $W^{1,2}(\mathbb{R}^n)$. Many important questions are still under investigation.

Within convex geometry, Böröczky & Schneider [11] obtained a simple characterization of polarity using the property that polarity interchanges pairwise intersections and convex hulls of unions. Gardner, Hug & Weil [20] obtained a complete classification of additions of convex sets. The functional version of the Böröczky & Schneider theorem was established by Artstein-Avidan & Milman [5]. They showed the classical Legendre transform plays for convex functions a role similar to polarity for convex sets. Artstein-Avidan & Milman [7] described a new duality for functions and, in [6], an analogue of the support map for functions. The functional version of the Gardner, Hug & Weil theorem is still an open problem, but these results and questions will certainly have important impact on Affine Geometric Analysis.

Liran Rotem, a PhD student from Tel Aviv University, presented the new definition of the *geometric mean* of convex bodies and also of convex functions. The definition is motivated by letting the polar body of convex body and the Legendre transform of a convex function play the role of inversion for real numbers and uses a suitable limit to obtain the geometric mean. The construction fits nicely within the affine geometry of convex bodies and it suggests many interesting questions.

A key structural property of convex bodies is that of symmetry which is relevant in many problems. We only mention the still open Mahler conjecture about the the minimal volume product of polar reciprocal convex bodies. The affine structure of convex bodies is closely related to the symmetry structure of the bodies. A systematic study of symmetry was initiated by Grünbaum in his seminal paper [21]. A crucial notion in his work is that of *affine invariant point*. It allows to analyze the symmetry situation. In a nutshell: the more affine invariant points, the fewer symmetries. Let $\mathcal{K}_n$ be the set of all convex bodies in $\mathbb{R}^n$ (i.e., compact convex subsets of $\mathbb{R}^n$ with nonempty interior) equipped with the Hausdorff distance. A map $p : \mathcal{K}_n \to \mathbb{R}^n$ is called an affine invariant point, if p is continuous and if for every nonsingular affine map $T : \mathbb{R}^n \to \mathbb{R}^n$ one has,

$$p(T(K)) = T(p(K)).$$

An important example of an affine invariant point is the centroid g. Several talks addressed issues related to affine invariant points. For instance, Grünbaum conjectured that for every convex body K we have

$$\mathcal{P}_n(K) := \{p(K) : p \text{ affine invariant point}\}$$

equals to

$$\mathcal{F}_n(K) = \{x : Tx = x \text{ for every } T \text{ affine linear with } T(K) = K\}.$$

An answer in the case that the set of affine invariant points has codimension 1 was given in [38]. The general case was recently solved by Olaf Mordhorst [40], a PhD student from University of Kiel, who presented his proof at the meeting. Carsten Schütt, also from University of Kiel, introduced the new notion of dual affine point q of an affine invariant point p. Motivated by the duality of the centroid of a convex body and the Santaló point of a convex body, the dual affine invariant point q of an affine invariant point p [39] is given by the formula $q(K^{p(K)}) = p(K)$ for every convex body K, where $K^{p(K)}$ denotes the polar of K with respect to $p(K)$.

A recent trend in the latest development of convex geometry is to find what characterizes *operations* on convex bodies. In this direction, Gabriele Bianchi reported on joint work with Richard Gardner and Paolo Gronchi on i-symmetrization. Providing a convenient framework for most of the familiar symmetrization processes on convex sets, the i-symmetrization, which, in particular, includes the Steiner and, respectively, the Minkowski symmetrizations, are characterized in terms of some of their natural properties. Bianchi then introduced several new symmetrizations and discussed the relations between different properties of i-symmetrizations.

5 Further Developments

Over the last ten years, subtle geometric properties of symmetrization have become much better understood through work of Bianchi, Burchard, Cianchi, Fusco, Gronchi, Klain, Klartag, Lutwak, V. Milman, Volcic, D. Yang, G. Zhang and others. Symmetrization was used before in geometry and analysis in search of solutions to extremal problems, but lately the emphasis was on the question of convergence of sequences of symmetrizations or the minimum number of symmetrizations needed to achieve a certain property. For example, regarding infinite sequences of Steiner symmetrizations, it is now known that any sequence that uses only a finite set of directions converges to a body that has at least partial symmetry. On the other hand, convergence can fail even for Steiner symmetrizations of a convex body along a dense set of directions, and, for any given sequence, symmetrizations along a dense set of directions can be made to converge or diverge by simply reordering the sequence.

In two of the talks, these type of questions were addressed for a rearrangement known as two-point symmetrization which is particularly useful for proving geometric inequalities on spheres. In a first talk, Almut Burchard [12] talked about recovering full rotational symmetry from partial information. She considered a family of piecewise isometries that fold a hemisphere of $\mathbb{S}^{n-1}$ across a hyperplane onto the complementary hemisphere with the aim of answering the question of when can a dense subset of points in $\mathbb{S}^{n-1}$ be reached from an arbitrary point by applying a given set of folding maps. This question relates to the characterization of sequences of random two-point symmetrizations that converge to the symmetric decreasing rearrangement. Burchard then explained that two conditions are needed, one geometric and one algebraic. Surprinsingly, this could be used to show that the random walk generated by randomly alternating these maps is uniquely ergodic.

Burchard's student from University of Toronto, Qin Deng, reported on their joint work on the two-point symmetrization. Starting with the known fact that i.i.d. random sequences of two-point symmetrizations almost surely transform every subset of $\mathbb{S}^{n-1}$ into a spherical cap and that the expected distance decreases at least like a power law in the number of iterations, Deng showed that the rate of convergence of random sequences of two-points symmetrization on $\mathbb{S}^1$ exactly obeys the power law. The key to the proof is an analogue of the Riesz rearrangement inequality on $\mathbb{S}^1$ which Burchard and Deng conjectured that extends to higher dimensions.

Several talks reported on recent progress in classical long standing open problems. One was by Dan Florentin from Tel Aviv University reporting on a recent work with Shiri Artstein-Avidan, Keshet Einhorn and Yann Ostrover on Godbersen's conjecture [3] on the reverse Prékopa-Leindler inequality and an application to the Godbersen conjecture. They provide a natural generalization of a geometric conjecture of Fáry and Rédei regarding the volume of the convex hull of a convex $K \subset \mathbb{R}^n$ and its reflection $-K$. They show that it implies Godbersen's conjecture regarding the mixed volumes of the convex bodies K and $-K$. They use the same type of reasoning to produce the currently best known upper bound for the mixed volumes $V(K[j], -K[n-j])$, which is not far from Godbersen's conjectured bound. This conjectured bound was suggested in 1938 by Godbersen and says: For any convex body $K \subset \mathbb{R}^n$ and any $1 \leq j \leq n-1$

$$V(K[j], -K[n-j]) \leq \binom{n}{j} \mathrm{Vol}(K),$$

with equality attained only for simplices. Here $V(K_1, \ldots, K_n)$ denotes the mixed volume of the n convex bodies $K_1, \cdots, K_n$ and $V(K[j], T[n-j])$ denotes the mixed volume of j copies of the convex body K and $n-j$ copies of the convex body T.

Another recent trend in convex geometry concerns the algebrization of the geometric theory of convex bodies. In one direction, this can be pursued through the theory of Newton polytopes which provides a beautiful connection between algebraic geometry and convex geometry. Conversely, this view provides geometric proofs of algebraic geometry results when none others exist. The classical Bezout inequality in algebraic geometry relates the degrees of hypersurfaces X_i to the degree of their intersection and through the theory of Bernstein-Kushnirenko-Khovanskii is equivalent to an inequality of mixed volumes for the Newton polytopes P_i of those hypersurfaces and the standard n-simplex Δ_n which is the Newton polytope of a generic hyperplane, namely:

$$V(P_1,\dots,P_r,\Delta^{n-r})V_n(\Delta)^{r-1} \le \prod_{i=1}^{r} V(P_i,\Delta^{n-1}) \text{ for } 2 \le r \le n.$$

Artem Zvavitch talked on the geometric proof for the above inequality (based on [44]). He then discussed some remarks related to the conjecture that the solution to the problem

$$V(K_1,\dots,K_r,D)V_n(D)^{r-1} \le \prod_{i=1}^{r} V(K_i,D^{n-1}) \text{ for } 2 \le r \le n, \text{ and for any}$$

convex bodies $K_1,...,K_r$ in $\mathbb{R}^n$, implies that D is an n-implex. Zvavitch showed that D must be indecomposable and, in particular, if D is a simple polytope, then D must be an n-simplex. Directly from the indecomposability, it follows that the conjecture is true in dimension $n = 2$. However, in dimension 3, and higher, there exist counterexamples to the conjecture, thus one needs to restate the conjecture possibly by considering an isomorphic version of it.

Felix Dorrek presented joint work with Franz Schuster. Dual to Koldobsky's notion of j-intersection bodies, the class of j-projection bodies is introduced, generalizing Minkowski's notion of projection bodies of convex bodies. A fundamental Fourier-analytic characterization of j-intersection bodies due to Koldobsky initiated further investigations of this class. Here a dual version of this theorem for j-projection bodies will be discussed. It turns out that this characterization is closely related to another - valuation-theoretic - characterization involving the Alesker-Fourier transform.

Maria de los Angeles Alfonseca-Cubero, as well as Dmitry Ryabogin investigate classical open problems in convex geometry. Maria de los Angeles Alfonseca-Cubero talked about a joint work with Michelle Cordier, on constructions [2] of examples of two convex bodies K, L in $\mathbb{R}^n$, such that every projection of K onto a $(n-1)$-dimensional subspace can be rotated to be contained in the corresponding projection of L, but K itself cannot be rotated to be contained in L. They also find necessary conditions on K and L to ensure that K can be rotated to be contained in L if all the $(n-1)$-dimensional projections have this property. Dmitry Ryabogin studies the following problem: Let K and L be two convex bodies in $\mathbb{R}^4$ and let $\xi^\perp$ be a three-dimensional subspace orthogonal to the unit vector ξ. Assume that for every ξ, the projections $K|\xi^\perp$, $L|\xi^\perp$ are directly congruent. Does it follow that K and L coincide up to translation and reflection in the origin? We show that if the set of diameters of bodies satisfy an additional condition and certain projections do not have π-symmetries, then the answer is affirmative.

Topology of hyperspaces of compact and closed convex sets has been under investigation for some time with a classical result of Nadler, Quinn and Stavrakas which states that the hyperspace of convex compact subsets of $\mathbb{R}^n$, $n \ge 2$, equipped with the Hausdorff distance topology, is homeomorphic to the punctured Hilbert cube $Q \setminus \{*\}$. Their result has found many applications in convex geometry. In particular, it enabled the proof that the hyperspace of all compact strictly convex bodies is homeomorphic to the separable Hilbert space l_2. Based on joint work with Sergey Antonyan and Bernardo González Merino, Natalia Jonard Pérez presented a talk on the study of the topology of certain subspaces of the hyperspace of convex compact subsets of $\mathbb{R}^n$ obtained by the study of the natural action of Aff(n), the group of all affine transformations of $\mathbb{R}^n$ on convex compact subsets of $\mathbb{R}^n$, $cc(\mathbb{R}^n)$.

In this talk, Pérez addressed the topological structure of the hyperspace $cb(\mathbb{R}^n)$ of all compact convex bodies of $\mathbb{R}^n$. She showed that $cb(\mathbb{R}^n)$ is homeomorphic to the product $Q \times \mathbb{R}^{n(n+3)/2}$. Similarly, the hyperspace $cc_1(\mathbb{R}^n)$ of all compact convex subsets of $\mathbb{R}^n$ of dimension at least 1 is homeomorphic to $Q \times \mathbb{R}^{n+1}$. On the other hand, by studying the topology of the orbit spaces generated by the action of some subgroups of Aff(n) on certain subspaces of $cc(\mathbb{R}^n)$, Pérez showed that the orbit spaces $cb(\mathbb{R}^n)/$Aff(n) and $cc_1(\mathbb{R}^n)/$Sim(n) (where Sim(n) stands for the group of all similarities of $\mathbb{R}^n$) are both homeomorphic to the Banach-Mazur compactum $BM(n)$. Furthermore, if $E(n)$ denotes de Euclidean group, the orbit space $cc(\mathbb{R}^n)/E(n)$ (which corresponds with the Gromov-Hausdorff hyperspace of all compact convex subsets of $\mathbb{R}^n$) is homeomorphic to the open cone over $BM(n)$.

Probabilistic aspects play an important role in convex geometry. This direction was presented in a talk by Petros Valettas, a post doctoral researcher at the University of Missouri, Columbia. His talk was based on joint work with G. Paouris and J. Zinn [41]. Their starting point is the famous Dvoretzky's theorem on

Euclidean sections of convex bodies. More precisely, in a version due to Vitali Milman, it says that for all 0-symmetric convex bodies K in $\mathbb{R}^n$ and for all $0 < \varepsilon < 1$ there is a subspace F of dimension greater than or equal to $c(\varepsilon)\ \log n$ such that $K \cap F$ is $(1+\varepsilon)$ close to the Euclidean ball B_2^F of the same dimension as F,

$$a\, B_2^F \subset K \cap F \subset (1+\varepsilon)\, a\, B_2^F.$$

The subspace F is obtained though a random construction. It is known that $c(\varepsilon) \sim \varepsilon^2$ and that this dependence and the dependence $\log n$ on the dimension cannot be improved in general. In specific situations however, one can do better. Paouris, Valettas and Zinn investigated precicely this question for the unit balls $B_p^n = \{x \in \mathbb{R}^n : \|x\|_p \leq 1\}$ and indeed got better dependence. Their method of proof requires an improvement on a concentration inequality by Talagrand.

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