Algebraic stacks in the representation theory of finite-dimensional algebras

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We finally,

- introduce a new moduli stack of "Serre stable representations", which gives a first approximation to answering this question.

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- The *dimension vector* of M is $\vec{\dim} M = (\dim_k M_v)_{v \in Q_0} \in \mathbb{Z}^{Q_0} \simeq K_0(A)$.

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- The diagonal copy of $k^\times$ acts trivially so $PGL(\vec{d}) := GL(\vec{d})/k^\times$ also acts.

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$$\mathrm{RHom}_{\mathbb{P}^1}(\mathcal{T}, -) : D^b(\mathbb{P}^1) \longrightarrow D^b(A), \ - \otimes^L_A \mathcal{T} : D^b(A) \longrightarrow D^b(\mathbb{P}^1)$$

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Think of the isomorphism classes of objects in the category $h(k)$ as the "k-points" & the category now remembers automorphisms.

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Consider case $X = \mathbb{A}^1_x$ & $G = \mu_p = \langle \zeta = \sqrt[p]{1} \rangle$ acts by multn, so action free on $x \neq 0$ but $\mathrm{Stab}_G\, 0 = \mu_p$.

k-**points** are parametrised by $y = x^p$.

- If $y \neq 0$ then $k[x]/(x^p - y)$ is a simple sheaf on $[X/G]$.
- If $y = 0$, then $k[x]/(x^p)$ is non-split extension of p non-isomorphic simples $k[x]/(x)$ with μ_p-action given by the p characters of μ_p.

General Fact

If $\tilde{U} \longrightarrow U$ is a G-torsor, then $[\tilde{U}/G] \simeq U$. Here $[(\mathbb{A}^1_x - 0)/\mu_p] \simeq \mathbb{A}^1_y - 0$.

- $\omega_X = k[x]dx$ & $\omega_{[X/G]} \otimes_{[X/G]} -$ permutes the simples with $x = 0$ cyclically.

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This process is called "stable reduction".

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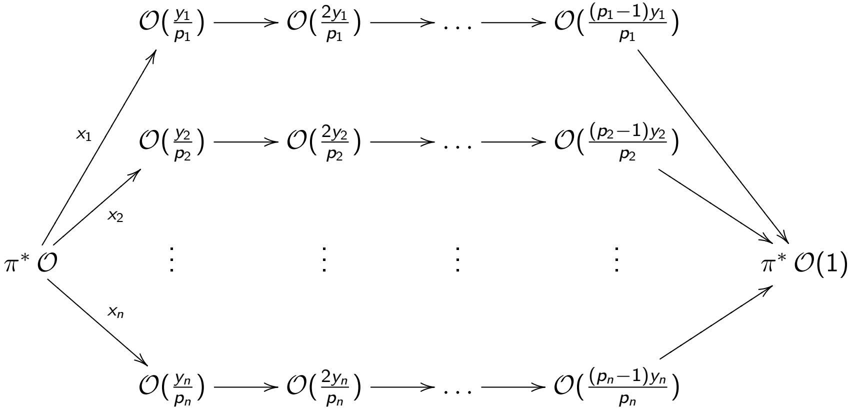
If $f_i \in \mathrm{Hom}_{\mathbb{W}}(\pi^* \mathcal{O}, \pi^* \mathcal{O}(1))$ corresponds to y_i, then

$$\mathrm{coker}\,(f_i : \pi^* \mathcal{O} \longrightarrow \pi^* \mathcal{O}(1))$$

is the non-split extension of p_i non-isomorphic simples on previous slide.

Canonical Algebra

Factorising f_i into p_i inclusions gives



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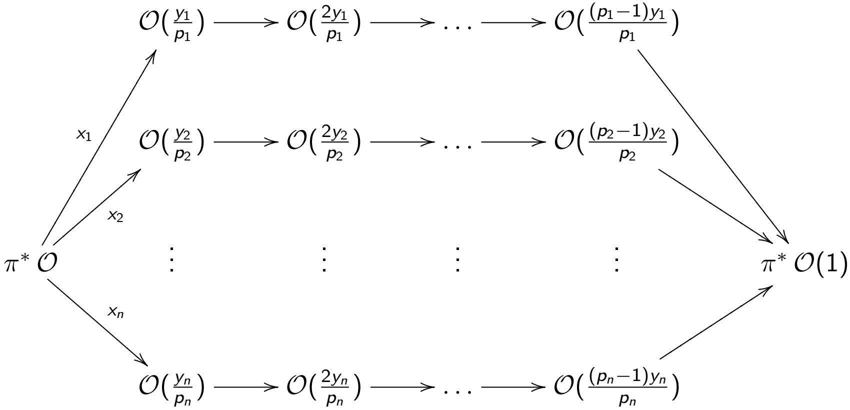

Thm(Geigle-Lenzing) The above is a tilting bundle on $\mathbb{P}^1(\sum p_i y_i)$ with endomorphism ring the corresponding canonical algebra.

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Note These will never be weighted projective lines because all modules have $k^\times$ in their automorphism group!

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We hence obtain a partially defined self-map

$$\nu_d : \text{RigIso}(A, \vec{d})^0 \longrightarrow \text{RigIso}(A, \vec{d})$$

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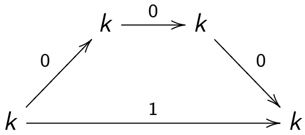

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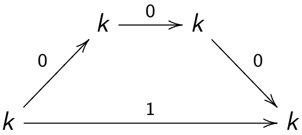

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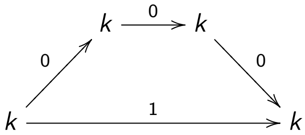

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Why

$$\begin{array}{ccc} M & \longrightarrow & \nu_1 M \\ {\scriptstyle \theta \in (k^\times)^3} \downarrow & & \downarrow {\scriptstyle \nu_d \theta} \\ M & \longrightarrow & \nu_1 M \end{array} \quad , \nu_d \theta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \theta$$

commutes up to scalar $\iff \theta$ is an e-vector of the permutation matrix.

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- The automorphism group in $\mathrm{RigIso}^S(k)$ of any such θ is μ_p where $p =$ no. Wedderburn components of $\operatorname{End}_A M$.

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- Abdelghadir-Ueda have also exhibited weighted projective lines as moduli spaces, but of enriched quiver representations.
- The proof of the derived equivalence is via Bridgeland-King-Reid theory and is independent of Geigle-Lenzing's.

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Remark Serre stability condition makes checking the 2nd condition easy.

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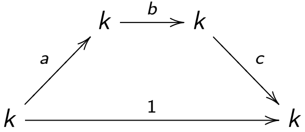

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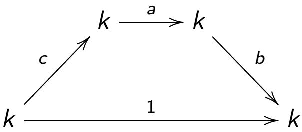

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Step 3 Guess a universal family/moduli space:

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is a μ_3-equivariant family on $\mathbb{A}^1_x$. See $\mathrm{RigIso}^S \simeq \mathbb{P}^1(3y)$.

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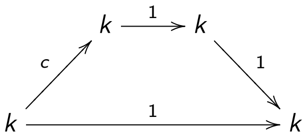

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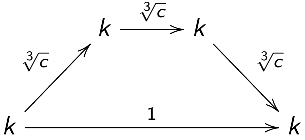

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- For tame hereditary algebras, the preprojective algebra arises naturally in attempting to construct Serre stable objects.

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- Method "works" because Serre stable moduli stack of "skyscraper sheaves" is the tautological moduli problem that recovers many stacks.
- Ideally we can apply Bridgeland-King-Reid theory to obtain independently many derived equivalences. Problem is we don't have many general results about the Serre stable moduli stack e.g. need a stable reduction theorem.
- For tame hereditary algebras, the preprojective algebra arises naturally in attempting to construct Serre stable objects.
- Case where you insert weights on intersecting divisors fails. Perhaps can be fixed by using the cotangent bundle.