Strange duality between hypersurface and complete intersection singularities

(joint work with A. Takahashi)

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Oaxaca, 5 October 2015

$f(x_1, \ldots, x_n) \in \mathcal{O}_n = \mathbb{C}\{x_1, \ldots, x_n\}$ convergent power series
$f(0) = 0 \Leftrightarrow f \in \mathfrak{m}_n$
f isolated singularity at 0

$$:\Leftrightarrow \operatorname{grad} f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_x}\right) \text{ has isolated 0 at 0}$$

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Definition

$f, g \in \mathfrak{m}_n$. $f \sim g$ (*right equivalent*) $:\Leftrightarrow \exists$ holomorphic coordinate change $\varphi : \mathbb{C}^n \to \mathbb{C}^n$

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Theorem

$f \in \mathfrak{m}_n$ *isolated singularity at 0* $\Rightarrow f \sim$ *polynomial*

Can assume:
$f(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$, $f(0) = 0$, isolated sing. at 0

Definition

$$\mu := \dim_{\mathbb{C}} \mathcal{O}_n / \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_x} \right) < \infty \textit{ Milnor number}$$

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Definition

- *Unfolding* of f: $F(\underline{x}, \underline{t}) \in \mathbb{C}\{\underline{x}, \underline{t}\}$, $F(\underline{x}, \underline{0}) = f(\underline{x})$
  $\underline{x} = (x_1, \ldots x_n)$, $\underline{t} = (t_1, \ldots, t_k)$
- *Universal unfolding* F of f:
  1. Every unfolding of f equivalent to an unfolding induced from F
  2. k minimal

Definition

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- f simple $\Leftrightarrow$ f 0-modal
- *singularity class* $= \mu$-constant stratum

Definition

K, L singularity classes.
$K \leftarrow L$ (L is *adjacent* to K) $:\Leftrightarrow$ every $f \in L$ can be deformed to a $g \in K$

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Theorem

$K \leftarrow L \Rightarrow$

1. $\mu_K < \mu_L$
2. $m_K \leq m_L$
3. *a Coxeter-Dynkin diagram of* K *is a subgraph of a Coxeter-Dynkin diagram of* L

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Lemma

$K \leftarrow L$, $m_K < m_L \Rightarrow \mu_K \leq \mu_L - 2$

Theorem (Arnold 1975)

Modality $m =$

0: *ADE singularities*

1: *simple elliptic* J_{10}, X_9, P_8;
cusp singularities T_{pqr}, $\Delta(p,q,r) := pqr - pq - pr - qr > 0$;
14 exceptional unimodal:

$$\begin{array}{cccccc} E_{12} & Z_{11} & Q_{10} & W_{12} & S_{11} & U_{12} \\ E_{13} & Z_{12} & Q_{11} & W_{13} & S_{12} & \\ E_{14} & Z_{13} & Q_{12} & & & \end{array}$$

2: *8 infinite series*

$$J_{3,k}, Z_{1,k}, Q_{2,k}, W_{1,k}, S_{1,k}, U_{1,k},\ k \geq 0,\ \textit{and}\ W^{\sharp}_{1,k}, S^{\sharp}_{1,k},\ k \geq 1,$$

14 exceptional bimodal

Name	f	$\alpha_1, \alpha_2, \alpha_3$	$\gamma_1, \gamma_2, \gamma_3$	Dual
E_{12}	$x^2 + y^3 + z^7$	2, 3, 7	2, 3, 7	E_{12}
E_{13}	$x^2 + y^3 + yz^5$	2, 4, 5	2, 3, 8	Z_{11}
E_{14}	$x^3 + y^2 + yz^4$	3, 3, 4	2, 3, 9	Q_{10}
Z_{11}	$x^2 + zy^3 + z^5$	2, 3, 8	2, 4, 5	E_{13}
Z_{12}	$x^2 + zy^3 + yz^4$	2, 4, 6	2, 4, 6	Z_{12}
Z_{13}	$x^2 + xy^3 + yz^3$	3, 3, 5	2, 4, 7	Q_{11}
Q_{10}	$x^3 + zy^2 + z^4$	2, 3, 9	3, 3, 4	E_{14}
Q_{11}	$x^2y + y^3z + z^3$	2, 4, 7	3, 3, 5	Z_{13}
Q_{12}	$x^3 + zy^2 + yz^3$	3, 3, 6	3, 3, 6	Q_{12}
W_{12}	$x^5 + y^2 + yz^2$	2, 5, 5	2, 5, 5	W_{12}
W_{13}	$x^2 + xy^2 + yz^4$	3, 4, 4	2, 5, 6	S_{11}
S_{11}	$x^2y + y^2z + z^4$	2, 5, 6	3, 4, 4	W_{13}
S_{12}	$x^3y + y^2z + z^2x$	3, 4, 5	3, 4, 5	S_{12}
U_{12}	$x^4 + zy^2 + yz^2$	4, 4, 4	4, 4, 4	U_{12}

Definition

$f(x,y,z)$ weighted homogeneous polynomial, $W := (w_1, w_2, w_3; d)$ reduced system of weights attached to f.

$$a_f := d - w_1 - w_2 - w_3 \quad (\epsilon_W = -a_f)$$

Gorenstein parameter of f.

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Theorem

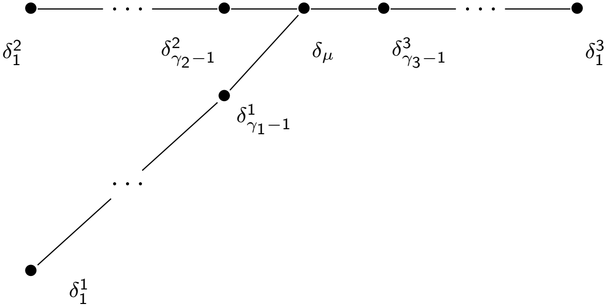
a_f

< 0: *ADE singularities*

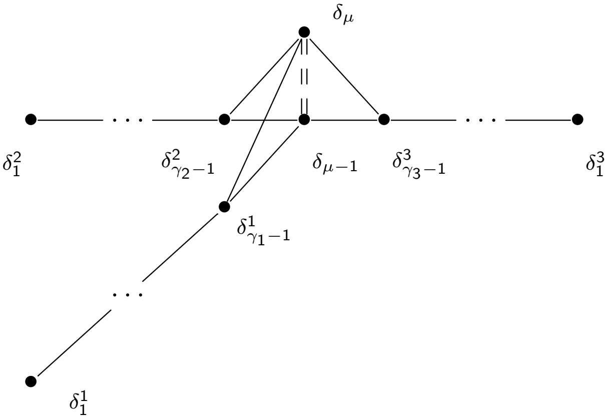
$=0$: *simple elliptic singularities* P_8, X_9, J_{10}

$=1$: *14 exceptional unimodal,* $J_{3,0}, Z_{1,0}, Q_{2,0}, W_{1,0}, S_{1,0}, U_{1,0}$
+ 11 others

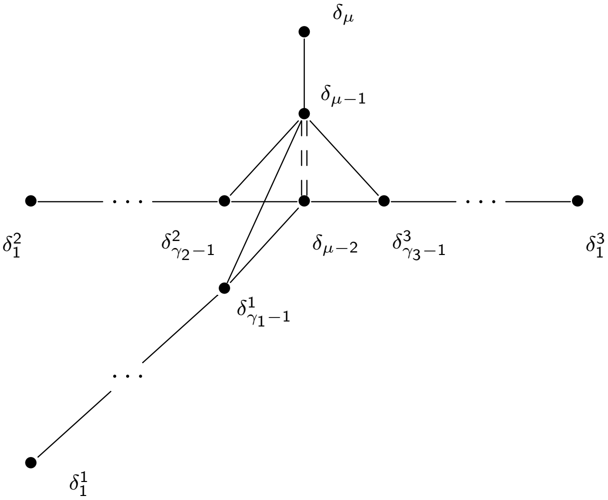
Coxeter-Dynkin diagrams of the ADE singularities:



The graph $T_{\gamma_1,\gamma_2,\gamma_3}$



The graph $S_{\gamma_1,\gamma_2,\gamma_3}$:



Theorem

Coxeter-Dynkin diagram of type

- *tree* $\Leftrightarrow$ *ADE*
- $T_{\gamma_1,\gamma_2,\gamma_3} \Leftrightarrow$ *simple elliptic* $T_{236}, T_{244}, T_{333}$ *(*$\Delta(\gamma_1,\gamma_2,\gamma_3) = 0$*); cusp singularity (*$\Delta(\gamma_1,\gamma_2,\gamma_3) > 0$*)*
- $S_{\gamma_1,\gamma_2,\gamma_3} \Leftrightarrow$ *14 exceptional unimodal*

$$\gamma_1\gamma_2\gamma_3 = \begin{matrix} 237 & 245 & 334 & 255 & 344 & 444 \\ 238 & 246 & 335 & 256 & 345 & \\ 239 & 247 & 336 & & & \end{matrix}$$

- ADE diagram $\simeq$ path algebra of ADE-quiver

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- $J_{3,0} : x^2 + y^3 + z^9 \to A_2 \otimes A_8 \simeq \underline{\mathrm{vect}}\text{-}\mathbb{P}^1_{2,3,9}$
- $W_{1,0} : x^2 + y^4 + z^6 \to A_3 \otimes A_5 \simeq \underline{\mathrm{vect}}\text{-}\mathbb{P}^1_{2,4,6}$

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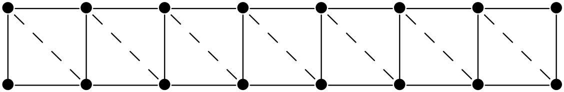
Two distinguished bases of vanishing cycles are related by sequence of *Gabrielov transformations*

Gabrielov transformations $\leftrightarrow$ mutations of the category

Example: $J_{3,0}$



$A_2 \otimes A_8$

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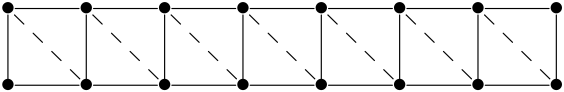

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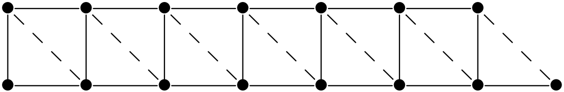
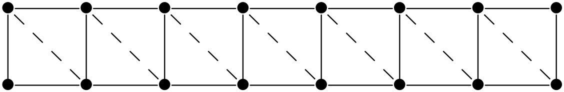


$S_{23,10}$

Example: $J_{3,0}$



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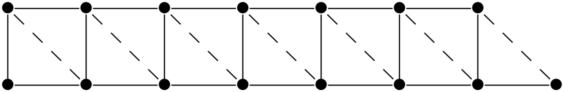

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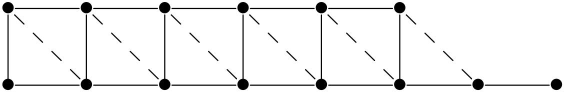
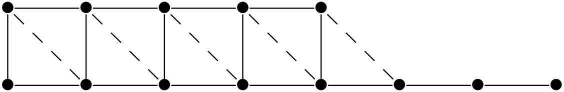
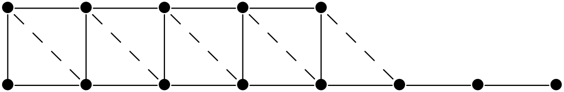


S_{239}



T_{239}

$$T_{239} \subset S_{239} \subset S_{23,10} \subset A_2 \otimes A_8$$



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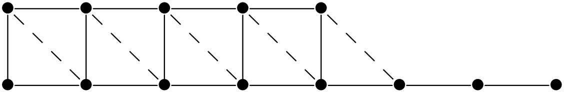
Corresponding categories:

$$\text{coh-}\mathbb{P}^1_{2,3,9} \subset S_{239} \subset S_{23,10} \subset \underline{\text{vect}}\text{-}\mathbb{P}^1_{2,3,9}$$



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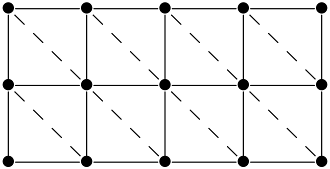
Corresponding singularities:

$$\underbrace{T_{239} \leftarrow E_{14}}_{\text{unimodal}} \leftarrow \; ??? \; \leftarrow \underbrace{J_{3,0}}_{\text{bimodal}}$$

Example: $W_{1,0}$



$$T_{246} \subset S_{246} \subset S_{256} \subset \begin{matrix} S_{266} \\ S_{257} \end{matrix} \subset A_3 \otimes A_5$$

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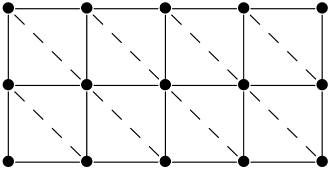

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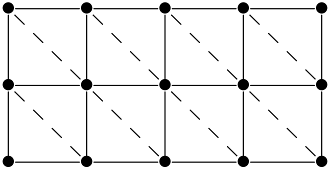

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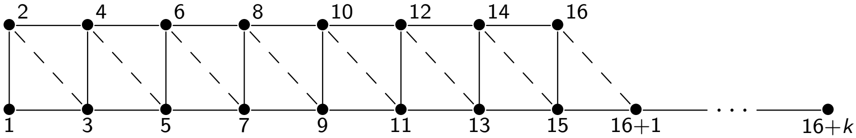
$$\underbrace{T_{246} \leftarrow Z_{12} \leftarrow W_{13}}_{\text{unimodal}} \begin{matrix} \swarrow \\ \nwarrow \end{matrix} \begin{matrix} ??? \\ ??? \end{matrix} \begin{matrix} \nwarrow \\ \swarrow \end{matrix} \underbrace{W_{1,0}}_{\text{bimodal}}$$

E., Wall 1985:

Example: bimodal series $J_{3,k}$, $k = 0, 1, 2, \ldots$



$\rightarrow$ $J_{3,-1}$ *virtual singularity*

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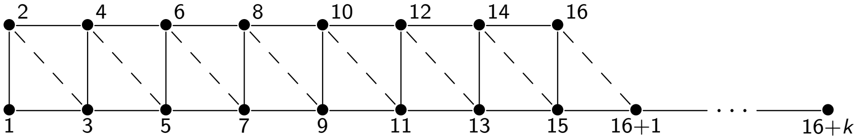

$\rightarrow$ $J_{3,-1}$ *virtual singularity*

$$\underbrace{T_{239} \leftarrow E_{14}}_{\text{unimodal}} \leftarrow J_{3,-1} \leftarrow \underbrace{J_{3,0}}_{\text{bimodal}}$$

$$\underbrace{T_{246} \leftarrow Z_{12} \leftarrow W_{13}}_{\text{unimodal}} \quad \begin{matrix} \swarrow & W_{1,-1} & \nwarrow & \\ & & & \underbrace{W_{1,0}}_{\text{bimodal}} \\ \nwarrow & W^{\sharp}_{1,-1} & \swarrow & \end{matrix}$$

Theorem (E., Wall 1985)

Name	Dol	Gab	Dol	Gab	*Dual*
$J_{3,-1}$	2, 2, 2, 3	2, 3, 10	2, 3, 10	2, 2, 2, 3	J'_9
$Z_{1,-1}$	2, 2, 2, 4	2, 4, 8	2, 4, 8	2, 2, 2, 4	J'_{10}
$Q_{2,-1}$	2, 2, 2, 5	3, 3, 7	3, 3, 7	2, 2, 2, 5	J'_{11}
$W_{1,-1}$	2, 2, 3, 3	2, 6, 6	2, 6, 6	2, 2, 3, 3	K'_{10}
$W^\sharp_{1,-1}$	2, 3, 2, 3	2, 5, 7	2, 5, 7	2, 3, 2, 3	L_{10}
$S_{1,-1}$	2, 2, 3, 4	3, 5, 5	3, 5, 5	2, 2, 3, 4	K'_{11}
$S^\sharp_{1,-1}$	2, 3, 2, 4	3, 4, 6	3, 4, 6	2, 3, 2, 4	L_{11}
$U_{1,-1}$	2, 3, 3, 3	4, 4, 5	4, 4, 5	2, 3, 3, 3	M_{11}

14 exceptional unimodal + 8 bimodal $S_{\gamma_1,\gamma_2,\gamma_3}$
$\leftrightarrow$ extended canonical algebras with $t = 3$

Virtual singularities exist:

Series	Arnold's equation
$J_{3,-1}$	$x^3 + x^2y^3 + z^2 + y^8$
$Z_{1,-1}$	$x^3y + x^2y^3 + z^2 + y^6$
$Q_{2,-1}$	$x^3 + x^2y^2 + yz^2 + y^5$
$W_{1,-1}$	$x^3z^2 + y^2 + z^4 + x^5$
$W^\sharp_{1,-1}$	$(x^3 + z^2)^2 + y^2 + x^4z$
$S_{1,-1}$	$xy^2 + x^2z^2 + yz^2 + x^4$
$S^\sharp_{1,-1}$	$x^3y + xy^2 + yz^2 + x^3z$
$U_{1,-1}$	$x^2y + y^3 + yz^3 + x^2z$

Isolated singularity at the origin, but additional A_1 singularities.
Considered as global polynomial, Coxeter-Dynkin diagram as above

Recall:
$f(x_1, \dots, x_n)$ invertible polynomial, $q_1, \dots, q_n$ rational weights

Definition

maximal group of diagonal symmetries

$$G_f = \{(\lambda_1, \dots, \lambda_n) \in (\mathbb{C}^*)^n : f(\lambda_1 x_1, \dots, \lambda_n x_n) = f(x_1, \dots, x_n)\}$$

- G_f always contains the exponential grading operator

$$g_0 := (\mathbf{e}[q_1], \dots, \mathbf{e}[q_n]), \quad \mathbf{e}[\bullet] := \exp(2\pi\sqrt{-1} \cdot \bullet).$$

Denote by G_0 the subgroup of G_f generated by g_0.

Heads of the bimodal series ($k=0$):
Given by invertible polynomials with $[G_f : G_0] = 2$

Name	F	$A_{(f,G_0)}$
$J_{3,0}$	$x^3+xy^6+z^2+ax^2y^3,\ a \neq \pm 2$	$2,2,2,3$
$Z_{1,0}$	$x^5y+xy^3+z^2+ax^2y^3,\ a \neq \pm 2$	$2,2,2,4$
$Q_{2,0}$	$x^3+xy^4+yz^2+ax^2y^2,\ a \neq \pm 2$	$2,2,2,5$
$W_{1,0}$	$x^6+y^2+yz^2+ax^3y,\ a \neq \pm 2$	$2,2,3,3$
$S_{1,0}$	$x^5+xy^2+yz^2+ax^3y,\ a \neq \pm 2$	$2,2,3,4$
$U_{1,0}$	$x^3+xy^2+yz^3+ax^2y,\ a \neq \pm 2$	$2,3,3,3$

Correspond to weighted projective lines with 4 isotropic points of orders $A_{(f,G_0)} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ (a modulus)
Exponent matrix: 4×3-matrix $E = (E_{ij})_{j=1,2,3}^{i=1,2,3,4}$

Definition

$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i \prod_{j=1}^{n} x_j^{E_{ij}}$ invertible polynomial

- *maximal grading* of the invertible polynomial f:

$$L_f := \bigoplus_{i=1}^{n} \mathbb{Z}\vec{x}_i \oplus \mathbb{Z}\vec{f} \,/ I_f \ ,$$

  I_f is the subgroup generated by

$$\vec{f} - \sum_{j=1}^{n} E_{ij}\vec{x}_j, \quad i = 1, \ldots, n.$$

- $\widehat{G}_f := \mathrm{Spec}(\mathbb{C}L_f)$

Definition

- $\widehat{G}_0$ subgroup of $\widehat{G}_f$ defined by

$$\begin{array}{ccccccccc}
\{1\} & \longrightarrow & G_0 & \longrightarrow & \widehat{G}_0 & \longrightarrow & \mathbb{C}^* & \longrightarrow & \{1\} \\
 & & \big\downarrow & & \big\downarrow & & \| & & \\
\{1\} & \longrightarrow & G_f & \longrightarrow & \widehat{G}_f & \longrightarrow & \mathbb{C}^* & \longrightarrow & \{1\}
\end{array}$$

- L_0 quotient of L_f corresponding to $\widehat{G}_0$

Classification of non-degenerate invertible polynomials $f(x,y,z)$ with $[G_f : G_0] = 2$:

Proposition

There are the following non-degenerate invertible polynomials $f(x,y,z)$ with $[G_f : G_0] = 2$:

I: $f(x,y,z) = x^{p_1} + y^{p_2} + z^{p_3}$; p_1, p_2 *even,*

IIA: $f(x,y,z) = x^{p_2} + xy^{p_3/p_2} + z^{p_1}$; p_2 *odd,* p_3/p_2 *even,*

IIB: $f(x,y,z) = x^{p_1} + y^{p_2} + yz^{p_3/p_2}$; p_1, p_2 *even,*

III: $f(x,y,z) = x^{q_2+1}y + xy^{q_3+1} + z^{p_1}$; q_2, q_3 *even,*

IV: $f(x,y,z) = x^{p_1} + xy^{\frac{p_2}{p_1}} + yz^{\frac{p_3}{p_2}}$; p_2/p_1 *even,* p_1 *odd.*

(Coordinates are chosen so that the action of $\widetilde{G}_0 = \mathbb{Z}/2\mathbb{Z}$ on $\widetilde{f}$ is given by $(x,y,z) \mapsto (-x,-y,z)$.)

We shall now classify 4×3-matrices $E = (E_{ij})_{j=1,2,3}^{i=1,2,3,4}$ such that

$$\mathbb{Z}\vec{x} \oplus \mathbb{Z}\vec{y} \oplus \mathbb{Z}\vec{z} \oplus \mathbb{Z}\vec{f} / \langle E_{i1}\vec{x} + E_{i2}\vec{y} + E_{i3}\vec{z} = \vec{f}, i = 1, \ldots, 4 \rangle \cong L_0$$

and $\mathcal{C}_{(F,G_0)} := [(F^{-1}(0) \setminus \{0\})/\widehat{G}_0]$, where $F := \sum_{i=1}^4 a_i x^{E_{i1}} y^{E_{i2}} z^{E_{i3}}$, is a smooth projective line with 4 isotropic points whose orders are $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, where $A_{(f,G_0)} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ are the Dolgachev numbers of the pair (f, G_0), for general a_1, a_2, a_3, a_4.

Proposition

The possible matrices E are classified into the following types up to a permutation of the rows:

I: $a_1x^{p_1} + a_2y^{p_2} + a_3z^{p_3} + a_4x^{\frac{p_1}{2}}y^{\frac{p_2}{2}}$

IIA: $a_1x^{p_2} + a_2xy^{\frac{p_3}{p_2}} + a_3z^{p_1} + a_4x^{\frac{p_2+1}{2}}y^{\frac{p_3}{2p_2}}$

IIB: $a_1x^{p_1} + a_2y^{p_2} + a_3yz^{\frac{p_3}{p_2}} + a_4x^{\frac{p_1}{2}}y^{\frac{p_2}{2}}$

IIB$^\sharp$: $(p_2 = 2)\ a_1x^{\frac{p_1}{2}}z^{\frac{p_3}{2}} + a_2y^2 + a_3yz^{\frac{p_3}{2}} + a_4x^{\frac{p_1}{2}}y$

III: $a_1x^{q_2+1}y + a_2xy^{q_3+1} + a_3z^{p_1} + a_4x^{\frac{q_2}{2}+1}y^{\frac{q_3}{2}+1}$

IV: $a_1x^{p_1} + a_2xy^{\frac{p_2}{p_1}} + a_3yz^{\frac{p_3}{p_2}} + a_4x^{\frac{p_1+1}{2}}y^{\frac{p_2}{2p_1}}$

IV$^\sharp$: $(\frac{p_2}{p_1} = 2)\ a_1x^{\frac{p_1-1}{2}}z^{\frac{p_3}{p_2}} + a_2xy^2 + a_3yz^{\frac{p_3}{p_2}} + a_4x^{\frac{p_1+1}{2}}y$

(The matrices are described by the corresponding polynomials F.)

1. Choose for each of the matrices E special values a_1, a_2, a_3, a_4 such that the corresponding polynomial F has a non-isolated singularity. Denote this polynomial by $\mathbf{f}$.

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2. In all cases except $\mathrm{IIB}^\sharp$ and $\mathrm{IV}^\sharp$, choose conditions such that

$$
\begin{aligned}
\mathbf{f}(x,y,z) &= u(x,y,z) + v(x,y,z)(x-y^e)^2 \text{ or} \\
\mathbf{f}(x,y,z) &= u(x,y,z) + v(x,y,z)(y-x^e)^2
\end{aligned}
$$

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3. Consider the cusp singularity $\mathbf{f}(x,y,z) - xyz$ and perform a coordinate change $x \mapsto x + y^e$ or $y \mapsto y + x^e$:

$$\mathbf{f}(x,y,z) - xyz \to \mathbf{h}(x,y,z) - xyz$$

h has 3 or 4 monomials. Consider only those with 4 monomials: *virtual singularities*

Virtual singularities:

Type	Conditions	$\mathbf{h}(x,y,z)$
IIA	$p_2 = 3$	$-y^{\frac{p_3}{6}+1}z + z^{p_1} + x^3 + x^2y^{\frac{p_3}{6}}$
IIB	$p_2 = 2$	$-x^{\frac{p_1}{2}+1}z + y^2 + yz^{\frac{p_3}{2}} + x^{\frac{p_1}{2}}z^{\frac{p_3}{2}}$
$\mathrm{IIB}^\sharp$	$p_2 = 2$	$-x^{\frac{p_1}{2}+1}z + y^2 + yz^{\frac{p_3}{2}} + x^{\frac{p_1}{2}}y$
III	$q_2 = 2$	$-y^{\frac{q_3}{2}+1}z + z^{p_1} + x^3y + x^2y^{\frac{q_3}{2}+1}$
IV_1	$p_1 = 3$	$-y^{\frac{p_2}{6}+1}z + x^3 + yz^{\frac{p_3}{p_2}} + x^2y^{\frac{p_2}{6}}$
IV_2	$\frac{p_2}{p_1} = 2$	$-x^{\frac{p_1+1}{2}}z + xy^2 + yz^{\frac{p_3}{p_2}} + x^{\frac{p_1-1}{2}}z^{\frac{p_3}{p_2}}$
$\mathrm{IV}_2^\sharp$	$\frac{p_2}{p_1} = 2$	$-x^{\frac{p_1+1}{2}}z + xy^2 + yz^{\frac{p_3}{p_2}} + x^{\frac{p_1+1}{2}}y$

$$\mathbf{h}(x,y,z) = \sum_{i=1}^{4} a_i x^{A_{i1}} y^{A_{i2}} z^{A_{i3}}$$

$\mathrm{Supp}(\mathbf{h}) = \{(A_{i1}, A_{i2}, A_{i3}) \in \mathbb{Z}^3 \mid i = 1, \ldots, 4\}$
$\Delta_\infty(\mathbf{h}) = \text{conv.clos.}_{\mathbb{R}^n}(\mathrm{Supp}(\mathbf{h}) \cup \{0\})$ has two faces Σ_1 and Σ_2 which do not contain the origin.
$I_k := \{i \in \{1, \ldots, 4\} \mid (A_{i1}, A_{i2}, A_{i3}) \in \Sigma_k\}$, $k = 1, 2$

$$\mathbf{h}_k = \sum_{i \in I_k} a_i x^{A_{i1}} y^{A_{i2}} z^{A_{i3}}.$$

$\mathbf{h}_k$ invertible polynomial with a non-isolated singularity at the origin:

Type		$\mathbf{h}_1(x,y,z)$	$\mathbf{h}_2(x,y,z)$
IIA	$p_2 = 3$	$-y^{\frac{p_3}{6}+1}z + z^{p_1} + x^2y^{\frac{p_3}{6}}$	$z^{p_1} + x^3 + x^2y^{\frac{p_3}{6}}$
IIB	$p_2 = 2$	$-x^{\frac{p_1}{2}+1}z + y^2 + x^{\frac{p_1}{2}}z^{\frac{p_3}{2}}$	$y^2 + yz^{\frac{p_3}{2}} + x^{\frac{p_1}{2}}z^{\frac{p_3}{2}}$
$\mathrm{IIB}^\sharp$	$p_2 = 2$	$-x^{\frac{p_1}{2}+1}z + yz^{\frac{p_3}{2}} + x^{\frac{p_1}{2}}y$	$y^2 + yz^{\frac{p_3}{2}} + x^{\frac{p_1}{2}}y$
III	$q_2 = 2$	$-y^{\frac{q_3}{2}+1}z + z^{p_1} + x^2y^{\frac{q_3}{2}+1}$	$z^{p_1} + x^3y + x^2y^{\frac{q_3}{2}+1}$
IV_1	$p_1 = 3$	$-y^{\frac{p_2}{6}+1}z + yz^{\frac{p_3}{p_2}} + x^2y^{\frac{p_2}{6}}$	$x^3 + yz^{\frac{p_3}{p_2}} + x^2y^{\frac{p_2}{6}}$
IV_2	$\frac{p_2}{p_1} = 2$	$-x^{\frac{p_1+1}{2}}z + xy^2 + x^{\frac{p_1-1}{2}}z^{\frac{p_3}{p_2}}$	$xy^2 + yz^{\frac{p_3}{p_2}} + x^{\frac{p_1-1}{2}}z^{\frac{p_3}{p_2}}$
$\mathrm{IV}_2^\sharp$	$\frac{p_2}{p_1} = 2$	$-x^{\frac{p_1+1}{2}}z + yz^{\frac{p_3}{p_2}} + x^{\frac{p_1+1}{2}}y$	$xy^2 + yz^{\frac{p_3}{p_2}} + x^{\frac{p_1+1}{2}}y$

3×4-matrix $E^T \to$

$$\widetilde{f}(x,y,z,w) := x^{E_{11}} y^{E_{21}} z^{E_{31}} w^{E_{41}} + x^{E_{12}} y^{E_{22}} z^{E_{32}} w^{E_{42}} + x^{E_{13}} y^{E_{23}} z^{E_{33}} w^{E_{43}}$$

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- $\ker E^T$ yields another $\mathbb{C}^*$-action on $\widetilde{f}$, induces $\mathbb{Z}$-grading $R = \bigoplus_{i \in \mathbb{Z}} R_i$ on $R := \mathbb{C}[x,y,z,w]$
- $\widetilde{f} \in R_0$, written in invariant coordinates X, Y, Z, W:

$$\widetilde{\mathbf{f}}_1(X,Y,Z,W) \text{ relation}, \quad \widetilde{\mathbf{f}}_2(X,Y,Z,W) := \widetilde{f}(X,Y,Z,W)$$

yields *complete intersection singularity* $\widetilde{\mathbf{f}}_1 = \widetilde{\mathbf{f}}_2 = 0$

1. $\ker E^T = \langle (1,1,0,-2)^T \rangle$ (I, IIA, IIB, III, IV)

$$\lambda * (x,y,z,w) = (\lambda x, \lambda y, z, \lambda^{-2} w) \quad \text{for } \lambda \in \mathbb{C}^*.$$

$$\widetilde{f} \in R_0 = \mathbb{C}[x^2 w, y^2 w, z, xyw]$$

$$X := x^2 w, \quad Y := y^2 w, \quad Z := z, \quad W := xyw.$$

$$\widetilde{\mathbf{f}}_1(X,Y,Z,W) = XY - W^2, \quad \widetilde{\mathbf{f}}_2(X,Y,Z,W) = \widetilde{f}(X,Y,Z,W).$$

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$$\widetilde{\mathbf{f}}_1(X,Y,Z,W) = XY - W^2, \quad \widetilde{\mathbf{f}}_2(X,Y,Z,W) = \widetilde{f}(X,Y,Z,W).$$

2. $\ker E^T = \langle (1,1,-1,-1)^T \rangle$ ($\mathrm{IIB}^\sharp$, $\mathrm{IV}^\sharp$)

$$\lambda * (x,y,z,w) = (\lambda x, \lambda y, \lambda^{-1} z, \lambda^{-1} w) \quad \text{for } \lambda \in \mathbb{C}^*.$$

$$\widetilde{f} \in R_0 = \mathbb{C}[xw, yz, xz, yw]$$

$$X := xw, \quad Y := yz, \quad Z := xz \quad W := yw.$$

$$\widetilde{\mathbf{f}}_1(X,Y,Z,W) = XY - ZW, \quad \widetilde{\mathbf{f}}_2(X,Y,Z,W) = \widetilde{f}(X,Y,Z,W).$$

Dolgachev and Gabrielov numbers for $(\widetilde{\mathbf{f}}_1, \widetilde{\mathbf{f}}_2)$:

Definition

$\mathcal{C}_{\widetilde{\mathbf{f}}_1, \widetilde{\mathbf{f}}_2} := [(X_{\widetilde{\mathbf{f}}_1, \widetilde{\mathbf{f}}_2} \setminus \{0\})/\mathbb{C}^*]$ smooth projective curve with three isotropic points of orders $\alpha_1, \alpha_2, \alpha_3$: *Dolgachev numbers* of the pair $(\widetilde{\mathbf{f}}_1, \widetilde{\mathbf{f}}_2)$.

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Theorem

$$\left\{ \begin{array}{c} \widetilde{\mathbf{f}}_1(W, X, Y, Z) \\ \widetilde{\mathbf{f}}_2(W, X, Y, Z) - ZW \end{array} \right\} \sim \left\{ \begin{array}{c} XY - Z^{\gamma_1} - W^{\gamma_2} \\ X^{\gamma_3} + Y^{\gamma_4} - ZW \end{array} \right\}$$

cusp singularity of type $T^2_{\gamma_1, \gamma_3, \gamma_2, \gamma_4}$

Definition

$(\gamma_1, \gamma_2; \gamma_3, \gamma_4)$ *Gabrielov numbers* of the pair $(\widetilde{\mathbf{f}}_1, \widetilde{\mathbf{f}}_2)$.

Dolgachev and Gabrielov numbers for **h**:

Definition

Gabrielov numbers as for invertible polynomials

Dolgachev and Gabrielov numbers for **h**:

Definition

Gabrielov numbers as for invertible polynomials

$V_i := \{(x,y,z) \in \mathbb{C}^3 \mid \mathbf{h}_i(x,y,z) = 0\}$, $i = 1, 2$, has $\mathbb{C}^*$-action

Definition

exceptional orbit of this action is called *principal* $:\Longleftrightarrow$

(A) V_i contains a coordinate hyperplane: not contained in that hyperplane

(B) V_i does not contain a coordinate hyperplane: not contained in the singular locus

Definition

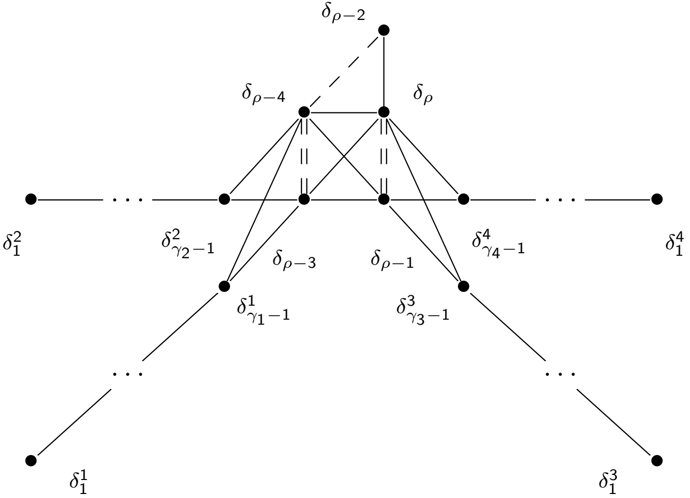
Dolgachev numbers of **h**: $\alpha_1, \alpha_2; \alpha_3, \alpha_4$ orders of isotropy groups of principal exceptional orbits of $\mathbf{h}_1$ and $\mathbf{h}_2$

Theorem (E., Takahashi)

The Gabrielov numbers of the polynomial $\mathbf{h}$ *corresponding to a virtual singularity coincide with the Dolgachev numbers of the dual pair* $(\widetilde{\mathbf{f}}_1, \widetilde{\mathbf{f}}_2)$ *and, vice versa, the Gabrielov numbers of a pair* $(\widetilde{\mathbf{f}}_1, \widetilde{\mathbf{f}}_2)$ *coincide with the Dolgachev numbers of the dual polynomial* $\mathbf{h}$*.*

The $a_f = 1$ case:

Name	$\mathrm{Dol}(\mathbf{h})$	$\mathrm{Gab}(\mathbf{h})$	$\mathrm{Dol}(\widetilde{\mathbf{f}}_1, \widetilde{\mathbf{f}}_2)$	$\mathrm{Gab}(\widetilde{\mathbf{f}}_1, \widetilde{\mathbf{f}}_2)$	Dual
$J_{3,-1}$	$2,2;2,3$	$2,3,10$	$2,3,10$	$2,2;2,3$	J'_9
$Z_{1,-1}$	$2,2;2,4$	$2,4,8$	$2,4,8$	$2,2;2,4$	J'_{10}
$Q_{2,-1}$	$2,2;2,5$	$3,3,7$	$3,3,7$	$2,2;2,5$	J'_{11}
$W_{1,-1}$	$2,2;3,3$	$2,6,6$	$2,6,6$	$2,2;3,3$	K'_{10}
$W^\sharp_{1,-1}$	$2,3;2,3$	$2,5,7$	$2,5,7$	$2,3;2,3$	L_{10}
$S_{1,-1}$	$2,2;3,4$	$3,5,5$	$3,5,5$	$2,2;3,4$	K'_{11}
$S^\sharp_{1,-1}$	$2,3;2,4$	$3,4,6$	$3,4,6$	$2,3;2,4$	L_{11}
$U_{1,-1}$	$2,3;3,3$	$4,4,5$	$4,4,5$	$2,3;3,3$	M_{11}


$\delta_{\rho-2}$
$\delta_{\rho-4}$
δ_{ρ}
δ_1^2
$\delta_{\gamma_2-1}^2$
$\delta_{\rho-3}$
$\delta_{\rho-1}$
$\delta_{\gamma_4-1}^4$
δ_1^4
$\delta_{\gamma_1-1}^1$
$\delta_{\gamma_3-1}^3$
δ_1^1
δ_1^3