PREPROJECTIVE ALGEBRAS INTERACTING WITH SINGULARITIES, COHEN-MACAULAY MODULES AND WEIGHTED PROJECTIVE SPACES

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1 The topic of the Workshop

Classically, singularity theory happens in the range of analysis and differential topology, where a major invariant is the *Milnor lattice*, an invariant obtained by resolutions of singularities. During the last ten years, the socalled *algebraic analysis of singularities*, which works with quite different methods, has become more and more important. One reason for this is that, typically, the algebraic analysis of singularities yields a *triangulated category* that serves as a categorification of the Milnor lattice, and thus provides the information about the Milnor lattice itself and, additionally, new and powerful invariants, thus giving additional insight.

To describe the procedure let's start with an isolated singularity R of Gorenstein type; a typical example being a hypersurface singularity given by a (graded) polynomial $f(x_1, \cdots, x_n)$, where $R = k[x_1, \ldots, x_n]/(f)$. There are two triangulated categories, associated to R, one $\mathcal{T}_1$ is the singularity category consisting of the stable category of (graded) Cohen-Macaulay modules over R, the other one $\mathcal{T}_2$ is the bounded derived category of the category of coherent sheaves on the projective spectrum $\mathbf{X}$, obtained from R as the Serre quotient of finitely generated (graded) R-modules modulo the Serre subcategory of finite dimensional ones. By design, $\mathcal{T}_1$ has an algebraic flavour, and will be studied by methods of Commutative Algebra with all available tools from Cohen-Macaulay theory, while $\mathcal{T}_2$, and more specifically $\mathbf{X}$, have a geometric flavour; typically $\mathbf{X}$ is an object of non-commmutative algebraic geometry, and in many situations more specifically, even a *weighted projective space*. In classical singularity theory, $\mathbf{X}$, and therefore $\mathcal{T}_2$ have not been studied.

In the above context it is important to realize, that due to Orlov [26] the two triangulated categories $\mathcal{T}_1$ and $\mathcal{T}_2$ are related to each other by means of a semi-orthogonal decomposition. In particular, and depending on the Gorenstein parameter g of the singularity R, the category $\mathcal{T}_1$ is contained in $\mathcal{T}_2$ if g is positive, the two categories are equivalent if and only if g is zero, and $\mathcal{T}_2$ is contained in $\mathcal{T}_1$ if g is negative.

This setup provides a strong link to finite dimensional algebras and their representation theory: In many cases $\mathcal{T}_2$ and/or $\mathcal{T}_2$ have a *tilting object* T, yielding an associated finite dimensional algebra A, the endomorphism ring of T, with the property that the triangulated category in question is equivalent to the bounded derived category of finite dimensional A-modules.

To sum up: The analysis of (Gorenstein) singularities offers a set-up where many mathematical theories interact: classical singularity theory, Cohen-Macaulay modules over commutative and non-commutative singularities, non-commutative algebraic geometry, in particular weighted projective spaces, triangulated categories and tilting theory, therefore the representation theory of finite dimensional algebras.

A key role in this set-up is played by the preprojective algebras, in particular the higher preprojective algebras of Iyama and collaborators. These arise to be associated to a finite dimensional algebra A of finite global dimension d. They are infinite dimensional as vector spaces and non-commutative; they may be noetherian or not. A quick way to define them is as tensor algebras (over A) of the (A, A)-bimodule $\mathrm{Ext}^d(DA, A)$, where DA denotes the vector space dual of A. This bimodule is a natural invariant of the category of all A-modules, since it describes the extension behavior between all injective and projective A-modules. In the case $d = 1$, preprojective algebras and their associated singularity and sheaf categories are well understood. For instance, they contain the simple (or Kleinian) singularities, labelled by the Dynkin diagrams of type A-D-E. For $d > 1$ the corresponding relationship is the matter of active current research.

2 The format of the Workshop

The workshop activities run from Monday to Friday; Wednesday afternoon was free and used for an excursion to Montealban. There were no formal activities on Friday afternoon.

For the lectures on Monday and Tuesday morning, the organizers have asked selected participants to prepare introductory lectures on preassigned topics. The remaining lectures were selected during the Workshop on the basis of abstracts submitted to the organizers. Almost all talks had a duration of 50 minutes.

The introductory topics covered the following subjects:

1. The foundations and highlights of the theory of (maximal) *Cohen-Macaulay modules* were presented by Graham Leuschke [24, 23] in two talks, starting with the background and history of the subject, explaining the link to representation theory, for instance McKay theory, and then proceeding to a subject of current interest, the cluster tilting theory for Cohen-Macaulay modules.

2. Atsushi Takahashi and Wolfgang Ebeling gave a coordinated sequence of two highlights of *singularity theory*, see [13, 12, 11]: (generalizations of) Arnold's strange duality and the treatment of mirror symmetry by means of invertible polynomials, invoking the Berglund-Hübsch transform. The presentation included an overview on the hierarchy of singularities and their connections to weighted projective lines.

3. *Weighted projective lines* and *GL-spaces* (Geigle-Lenzing spaces) and associated algebras formed another block of foundational material. Here, a weighted projective line can be thought of as the usual projective line equipped with a finite number of marked points endowed with positive integral multiplicity. The basics of this 1-dimensional theory were presented by Hagen Meltzer, outlining the two major categories associated to a weighted projective line: This uses that its projective coordinate algebra is (graded) complete intersection, isolated singularity. Thus the algebraic analysis of singularities, mentioned before, applies and yields a category of coherent sheaves with Serre duality and a category of (graded) Cohen-Macaulay modules yielding a stable category, the singularity category, that is triangulated. In this one-dimensional situation the interplay between these categories is well understood, because the category of (graded) Cohen-Macaulay modules is equivalent to the category of vector bundles. In particular, the simplest weighted projective lines (of socalled domestic representation type) yield — in a natural way — the simple surface singularities of A-D-E type together with their categories of Cohen-Macaulay modules (as categories and Auslander-Reiten quivers). Further the representation theory of a weighted projective line is completely understood, because of tame type, if the Gorenstein parameter of the coordinate algebra is greater or equal than zero, see [14, 22, 21, 20]. Moreover, each weighted projective line has a tilting object, in fact, a tilting bundle T composed of line bundles, whose endomorphism ring is a canonical algebra in the sense of Ringel [27].

These facts in dimension one form the basis and a challenge for an extension to higher dimensions. The Workshop program contained three talks (by Steffen Oppermann, Osamu Iyama and Martin Herschend) giving a survey on a comprehensive study of higher dimensional weighted projective spaces (Geigle-Lenzing spaces [GL-spaces]) from Herschend-Iyama-Minamoto-Oppermann, see [16], see also [4], [5] for a related treatment from a different perspective. Such a GL-space $\mathbf{X}$ of dimension d is an ordinary projective space $\mathbf{P}^d$, where positive integral weights are placed on a finite number of hyperplanes (these, typically, are assumed to be in general position). Such a space $\mathbf{X}$ comes with a (graded) projective coordinate algebra S which is complete intersection, just as for the case of a weighted projective line. Moreover, the category of coherent sheaves on $\mathbf{X}$ admits a tilting bundle T, composed of line bundles, whose endomorphism ring is by definition a d-canonical algebra (for $d = 1$ this specializes to the usual concept of a canonical algebra in Ringel's sense.) This is a new class of algebras with a promising future. Moreover, the (graded) Cohen-Macaulay theory of S is, again, well understood: the singularity category, that is, the stable category of Cohen-Macaulay modules turns out to have a tensor product decomposition into blocks that are categories of finite dimensional representations of finite linear quivers (of type A). These categories of Cohen-Macaulay modules further provide an important link to cluster theory. Also, since d-canonical algebras have finite global dimension, it is possible to form the associated preprojective algebra and study their properties in relationship to the GL-space. A good understanding of this preprojective algebra affords additional restrictions on $\mathbf{X}$ and is the topic of current investigation.

3 Further workshop topics

3.1 Weighted projective spaces and stacks

We should first remark that the class of GL-spaces does by no means exhaust the class of weighted projective spaces, since positive integral weights may also be inserted in subspaces (even further in subvarieties) of smaller dimension, not just in hyperplanes. One thus arrives to a hierarchy of very different kinds of weighted projective spaces. This problem does not occur in dimension one, where one has just a single notion of weighted projective line. Returning to higher dimension, the GL-spaces by Herschend-Iyama-Minamoto-Oppermann [16] should in that context be considered to be the most basic version of a higher dimensional weighted projective space, because of the well-formed and complete theory.

But even for GL-spaces, not to mention more complicated weighted spaces, a problem remains: There is no convincing concept of morphisms between these objects or of an embracing category. Though this has sofar been established only in a few cases,see [2, 3], one may morally think of weighted projective spaces as (Deligne-Mumford) stacks, thus solving the morphism problem. It is thus not surprising that a number of Workshop lectures were focussing on stacks: Daniel Chan attacked the problem of an application of stacks to the representation theory of finite dimensional algebras by addressing the question how to construct (if it exists) a projective stack that is derived equivalent to a given finite dimensional algebra, and showed that the stack, corresponding to a canonical algebra, is a weighted projective line, compare [8]. In a related, but different direction, Kazushi Ueda related deformations of an algebraic stack (with a tilting object T) by the moduli of relations of the endomorphism algebra of T, see [1].

An interesting case of weighted projective spaces that are not GL-spaces was given by Lutz Hille (presenting joint work with Ragnar Buchweitz). Hille starts with a weighted projective space X, given by a positive $\mathbf{Z}$-grading of the polynomial algebra, and shows that in the Fano case the crepant resolution of the resulting Fano stack X and the associated GL-space P, constructed by the same weight system, have derived equivalent categories of coherent sheaves, by establishing a full, strongly exceptional sequence of line bundles on the crepant resolution.

Ryo Kanda succeeded to show that the atom spectrum, a generalization of the Gabriel spectrum, of a GL-space $\mathbf{X}$ is obtained from the atom spectrum of the underlying projective space by adding multiplicities for the defining hyperplanes. This allows to classify all Serre subcategories of the category of coherent sheaves on a GL-space, see [18].

3.2 Higher dimensional aspects

For a one-dimensional smooth projective variety, that is a smooth projective curve, the complexity of classifying indecomposable objects is well understood: it is tame for genus at most one and wild otherwise. For varieties of higher dimension, this decomposition complexity is always wild. In order to distinguish higher dimensional varieties one thus needs a different measure, decomposition complexity does not work. A promising approach to higher dimension was outlined by the lecture of Rosa-Maria Miró-Roig defining the complexity of projective varieties by means of their full subcategories of arithmetically Cohen-Macaulay (ACM) sheaves, that is, sheaves without intermediate cohomology. This complexity is studied in terms of the dimension and number of families of indecomposable ACM sheaves, compare [9]. In particular, varieties of finite representation type are completely determined.

Another important aspect, more close to representation theory, was presented by Izuru Mori demonstrating how to apply noncommutative algebraic geometry to representation theory; in particular he studied the Beilinson algebra R of a Artin-Shelter regular algebra A, and showed, that under proper restrictions the preprojective algebra of R is Morita-equivalent to A. This allows to study n-regular modules in the sense of higher Auslander-Reiten theory, compare [25].

Eleonore Faber reported on joint work with Colin Ingalls and Ragnar-Olaf Buchweitz. Starting from classical McKay theory that relates the (resolutions of) Kleinian (or simple) surface singularities of type A-D-E to the representation theory of binary polyhedral groups, that is, the finite subgroups of $\mathrm{SL}(2, \mathbf{C})$. Faber was outlining a higher dimensional McKay theory, relating the representation theory of a finite subgroup G of $GL(n, k)$, k a field, to the skew group algebra $G * S$, where S is the polynomial algebra in n variables over k. This process works in good characteristic if G is generated by reflections and yields noncommutative resolutions of the singularities defined by the discriminant of the group action.

3.3 Singularity theory

Matthew Ballard, discussing the question "Where do derived equivalences come from?", exhibited a pattern that semi-orthogonal decompositions, like in the fundamental theorem of Orlov, relating the singularity category and the sheaf category associated to a Gorenstein singularity, appear during wall-crossing in moduli with stability [6]. This provides a coherent explanation for a couple of phenomena discussed before.

David Favero, pointing in a similar direction, in joint work with C. Doran and T. Kelly [10], completed the introductory talks by Ebeling and Takahashi, presenting surprising examples of derived equivalences and semi-orthogonal decompositions arising from the symmetry of invertible polynomials. In linking the Berglund-Hübsch-Krawitz mirror construction to the Homological Mirror Conjecture (HMC) of Kontsevich, Favero was able to confirm further instances of the HMC.

3.4 Higher Auslander-Reiten theory and preprojective algebras

Christof Geiss reported on joint work with B. Leclerc and J. Schröer [15]. They introduce a vast generalization of classical preprojective algebras of quivers by first introducing a class of Gorenstein algebras defined via quivers with relations associated with symmetrizable Cartan matrices. In their treatment the usually required fields are replaced by truncated polynomial algebras. These Gorenstein algebras generalize the path algebras of quivers associated with symmetric Cartan matrices and then allow to define generalized preprojective algebras having homological properties resembling closely the known properties of classical preprojective algebras. In particular, one obtains generalizations of classical results of Gabriel, Dlab-Ringel, and Gelfand-Ponomarev.

Louis-Philippe Thibault discussed the problem when the skew group algebra $G * R$ has the structure of a higher preprojective algebra, when G is a finite subgroup of $\mathrm{SL}(n, k)$ and R is the polynomial algebra of n variables over k. Classically, that is for $n = 2$, a result from I. Reiten and M. Van den Bergh shows that $G * R$ is Morita-equivalent to the preprojective algebra of the extended Dynkin quiver corresponding to G by McKay theory. Thibault addressed the question wether a corresponding result holds for $n \geq 3$. Based on former results from C. Amiot, O. Iyama and I. Reiten, Thibault shows [28] that this holds if and only if G is a finite cyclic subgroup which does not embed in a product of $\mathrm{SL}(m, k)$'s.

Gustavo Jasso dealt with a subject of higher Auslander-Reiten theory. From joint work with Julian Külshammer [17] Jasso presented higher analogues of Nakayama algebras. More precisely, for each Nakayama algebra A and each positive integer d, there is a finite dimensional algebra $A(d)$ and a d-cluster-tilting $A(d)$-module $M(d)$ whose endomorphism algebra is a higher analogue of the Auslander algebra of A. As an application Jasso constructed d-abelian categories which are analogues of tubes from classical representation theory.

The classical correspondence between representation-finite hereditary algebras and representation-finite selfinjective algebras goes back to Riedtmann, Hughes and Waschbüsch. Erik Darpö dealt with the challenge to extend such a correspondence to higher Auslander-Reiten theory. For a large class of selfinjective algebras he was able to show that they have an n-cluster-tilting module by generalizing Riedtmann's construction. These results further can be used to reprove a former result of Iyama-Oppermann that higher preprojective algebras have n-cluster-tilting-modules.

3.5 Noncommutative algebraic geometry

Colin Ingalls reported on a joint coordinated effort of a large group of researchers, succeeding in extending previous results of Chan and Ingalls for orders on surfaces to all dimensions, and thus obtaining substantial progress in their *non-commutative Mori program*.

Hiroyuki Minamoto presented joint work with Iyama by treating *tilting bundles on Fano algebras*. Since, in general, the class of Fano algebras is not closed under derived equivalence it is reasonable to impose suitable restrictions on derived equivalences such that the Fano property is preserved. One natural restriction is to consider derived equivalences induced by tilting bundles, where the wanted result follows from the fact that endomorphism rings of tilting bundles over Fano algebras are again Fano. And, indeed, the derived equivalence in such a case even preserves the attached non-commutative projective scheme.

3.6 Further results

Henning Krause dealt with highest weight categories and recollements [19]. In particular, he was able to show that full exceptional sequences in abelian categories are basically equivalent (up to derived equivalence) to the concept of a sequence of standard objects in a highest weight category.

David Pauksztello, reporting on joint work with Nathan Broomhead and David Ploog, dealt with an averaging process of t-structures [7]. In particular, he presented a method to construct new t-structures from old by taking the extension closure of aisles. Sofar, the main application of this technique is to the bounded derived category of a tame hereditary algebra.

3.7 Last day of the meeting

On Friday morning there were two lectures on recent results in Cohen-Macaulay representation theory given by Hailong Dao and Ryo Takahashi. (Let us remark that all but four participants of the Workshop were still present.) Dao was attacking representation theory of Cohen-Macaulay modules from the point of view of a commutative algebraist, discussing for instance, how singularities influence this representation theory. He based his talk on joint work with O. Iyama, R. Takahashi and I. Shipman.

In the second talk, Ryo Takahashi presented joint work with Srikanth Iyengar, focussing on the concept of dimension of a triangulated category as introduced by Bondal, Van den Bergh and Rouquier. He applied this concept to commutative rings, relating it to annihilation properties in Cohen-Macaulay representation theory.

The Workshop finished with an open discussion on matrix factorizations.

4 Outcome of the Meeting

The meeting had 42 participants, including the 4 organizers and a fairly large group of 8 Mexican participants (including one of the organizers). The Workshop assembled mathematicians with quite diverse expertise, which was one of the reasons to start the Workshop with a sequence of introductory lectures. The meeting itself provided a faithful picture of a wide range of present activities as shown by the attached bibliography. Throughout, the Workshop showed a stimulating atmosphere with a lot of interchange between participant. This exchange was further facilitated by the Oaxaca site, providing blackboards at many places, forming crystallization points of discussions outside the formal meeting and continuing until late at night.

In the opinion of the organizers, a major outcome of the meeting lies in the information exchange across the boundaries of different mathematical fields, the identification of open problems, and in the corresponding stimulus to expand the personal point of view. In this respect, the Oaxaca meeting was definitely a success.

5 Recommendations, specific to Oaxaca

Generally speaking, we found the site very convenient, and the staff friendly and helpful, thus caring for a productive working atmosphere. Nevertheless, there are a few minor issues, where the Oaxaca setup can be improved:

The organizers strongly recommend to encourage a stay until Saturday. This would add substantial value to the stay in Oaxaca, since for most participants travel to Oaxaca takes a quite long time.

It would further be important to improve the lighting of the blackboard(s). In the setup of the meeting, blackboard presentations were difficult to follow for people not sitting in central position.

A number of participants found the taxi information on the Oaxaca web site confusing.

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