Neostability theory

Bradd Hart (MacMaster University),
Ehud Hrushovski (Hebrew University at Jerusalem),
Alf Onshuus (Universidad de los Andes),
Anand Pillay (University of Leeds),
Thomas Scanlon (University of Berkeley),
Frank Wagner (Université Lyon I)

12 July 2015 – 17 July 2015

1 Overview of the Field

In recent years many unexpected results have been proven using methods from model theoretic stability theory in unstable contexts and this subject has taken center stage in model theory.

Let us start with a brief chronological review of the subject.

Starting in the late sixties and for all of the seventies and eighties, stability theory played a central role in model theory. Inaugurated with Morley's celebrated proof of his theorem on theories categorical in an uncountable cardinality, the theory reached a high degree of sophistication with Shelah's classification theory and was then developed by Shelah, Lascar, Poizat and others into what has arguably been the deepest and most applicable of the branches of model theory.

Stability theory reached an apex with the geometric stability theory of, especially, Hrushovski and Zilber. In the late 1970s Zilber introduced the group configuration in his work on totally categorical theories and then Hrushovski generalized the group configuration theorem well beyond these logically perfect theories. In so doing, it was revealed that structures of algebraic or algebraic geometric origin explain the complexity of some very general theories in which there is no apparent geometry. This approach of analyzing a stable structure according to the geometry of the types was incredibly fruitful with results such as the trichotomy theorem for Zariski geometries by Hrushovski and Zilber and the latter applications to diophantine number theory (including the Mordell-Lang Conjecture for fields of positive characteristic) by Hrushovski and later by Scanlon.

From the late nineties, especially with the work of Kim and Pillay, people have tried to generalize the results from stability theory to a more general class of structures, notably simple theories, those theories for which the basic properties of forking independence from stable theories continue to hold. Although many results have been obtained, it appears that it was the robustness of the types (automorphism invariant classes) that one has in stable theories which allowed for many of the most striking of the previously mentioned results and this robustness is lost when generalizing from stability to simplicity.

Since the early 2000's, developments turned the attention of many model theorists back into stability, or, really, back to the fundamental ideas and objects of study of stability theory, such as definable types. These developments prompted us to organize the first BIRS meeting on the subject

in 2009, a meeting which directed many of the multiple approaches into defining certain research fields which are now centers of intense research and productivity, something that was reinforced with this last meeting. We will now mention what has happened with the main subfields.

The theory of non forking and the study of definable and coheir extensions outside stable theory has become a very active area of model theory and has manifested itself in disparate areas of mathematics, particularly in dependent and pseudofinite settings. For example, the idea that non forking could be developed relative to any good notion of smallness (in the sense of ideals or measure zero sets) together with a generalization of the generically presented group theorem underlies Hrushovski's recent breakthrough in additive combinatorics which is now known as the "non-abelian Freiman theorem". Towsner then used these techniques to give a model theoretic proof of Szemerédi's theorem on the existence of arithmetic progressions in sufficiently dense sets of integers. On the other hand, deep ideas from combinatorics, such as Szemerédi's graph regularity theorem, have been consciously imported into general model theory.

The foundational studies of dependent theories has continued, and has broadened to the more general setting of NTP_2 theories. NTP_2 theories has now been established as a very adequate generalization of simple and dependent theories, and more and more results show that results which hold in both dependent and simple theories can be generalized to NTP_2 theories. Particular results include the behavior of forking, any result that works in both dependent and simple theories usually works also in NTP_2 theories.

2 Recent Developments

The study of NTP_2 theories has established itself as a very active framework in model theory, and particularly as a setting where stable-inspired notions have a very good behavior. Study of groups in this context has been very active lately.

On the other side of the "simplicity spectrum" (recall that a theory is simple if and only if it is NTP_2 and NTP_1) significant progress has been made towards characterizing and understanding NTP_1 theories. Chernikov and Ramsey have now characterized NSOP_1 structures with the existence of a particular independence relation, which not only helps start the study of definable sets in NSOP_1 theories, but it also might help significantly towards settling the equivalence (or non equivalence) of the concepts of NTP_1 and NSOP_1.

Dependent theories has continue its fruitfulness, with Simon's result that types can be decomposed in terms of stable-like components with ordered-like (distal) quotients. The study of dp-minimal, distal and stable types has also continued, with a characterization of dp-minimal fields by Johnson being a very recent highlight in this setting.

Combinatorial aspects of definable graphs and bounds has had very important results lately. The continuous use of (p,q)-theorem and results by Matouscek to understand combinatorial behaviour of definable sets in dependent theories has provided a very important connection between combinatorics and dependent theories. Some of this results are now being expanded by Chernikov to NTP_2 theories.

Finally, the connection between topological dynamics and different definable compactifications in dependent theories are providing a very interesting a fruitful way of studying definable groups in dependent theories.

3 Testimonials

In order to understand the scientific progress made and the presentation highlights, we asked the participants about their experience, the scientific progress made, and which were the talks they found as particular highlights of the meeting. This were their replies.

- **Frank Wagner:** "my favourite talks were Will Johnson, Artem Chernikov, Isaac Goldbring, Nick Ramsey, Silvain Rideau and Sergei Starchenko. And I am planning to delve deeper into the papers of Artem, Isaac and Sergei (although I have not done this yet). Connections : I guess I got to know someof the youngsters I had not met before."

- **Assaf Hasson:** "As far as I am concerned, the talk by Will Johnson was certainly the most interesting in the meeting (and closely related to a problem I have been interested in for a few years now). Also the discussions I had with Will following his talk was interesting, and useful.

  Some discussions I had with Alf Dolich helped me focus a project a couple of my students worked on (around the topic I talked about – weakly o-minimal non-valuational structures), and so were a few conversations I had with Chirs Laskowski about the same topic (and his paper with Shaw on related questions). "

- **Dierdre Haskell:** "I made excellent use of the meeting to talk through a lot of different ideas. Specifically 1) I talked with Dugald Macpherson, who I have collaborated with a lot, and made plans for our next project 2) I talked with Assaf Hasson, who helped me to clarify some ideas related to a project that I have been working on. Based on this, I developed some ideas which are part of my current research grant proposal."

- **Artem Chernikov:** "I could mention that Sergei and I made some progress on the improved Ramsey bounds in various NIP structures, especially we have obtained an analogue of a theorem of Bukh and Matousek over p-adics: given a sequence of singletons, and a finite set of formulas Delta in k variables, there is some constant $c = c(k)$ such that if the sequence is of length at least $2^{2^c} n$, then it contains a Delta-indiscernible subsequence of length n. We continue investigating this and related question in other classes of NIP theories."

- **Anand Pillay:** "The meeting showed the vitality and richness of the current state of "pure" or "purish" model theory, with many high level contributions from "young" people. Highlights include: Constructing a valuation in dp-minimal fields (Johnson), combinatorics and NIP (Chernikov, although I forget details), chain conditions on groups definable in "generalized" stable structures (e.g. NTP_2) (Hempel Onshuus), Characterizations of NSOP_1 (Ramsay, Chernikov). "

- **Byunghan Kim:** "This was a very-well organized meeting. By listening to talks of experts and conversing with colleagues I have learned a lot and have corrected/expanded my points of view toward the subjects. Being supplied with enough time for discussions with collaborators made the meeting more attractive."

- **Dugald Macpherson:** " I think the highlight was Artem Chernikov's talk which suggested a question I mean to return to, but I very much enjoyed many other talks, several of which I am likely to look back at. During the meeting I had discussions particularly with Charlie Steinhorn and Dario Garcia about pseudofinite structures, and with Deirdre Haskell on imaginaries in valued fields with analytic structure, pushing along long-term collaborations. I also talked to Gwyneth Harrison-Shermoen, who is now in Leeds on a 1-year postdoc, a possibility we discussed in Oaxaca."

- **Pierre Simon:** "I found essentially all the talks of this workshop very interesting and especially varied in topics. It was very nice to see developments in areas of pure model theory outside of the big three stable/simple/NIP: Nick Ramsey's talk on NSOP, Nadja Hampel's talk on NTP_2 come to mind. I also liked very much Will Johnson's talk on dp-minimal fields. The meeting gave me a chance to talk with Erik Walsberg and we started working on a project related to the talk I gave."

4 Presentation Highlights

Some of the talks that were highlighted by the people who attended the workshop were.

- **Will Johnson** Classified all dp-minimal fields, showing that every dp-minimal field possibly considered in an expansion of the language of rings, is either a strongly minimal expansion of an algebraically closed field, is real closed, or possesses a definable valuation with respect to which it is henselian. He even provided a complete classification of which theories of valued fields are

dp-minimal for the Henselian case. The definition of a valuation with just understanding the behaviour of the ring structure was particularly impressive.

- **Nick Ramsey** Reported on a joint work with Artem Chernikov on SOP_1 and k-TP_1. These notions which were introduced by Džamonja and Shelah, and by Kim and Kim (respectively) to understand better the behavior of of simple theories, and possible generalizations.

  In this talk it was shown that SOP_1 can be characterized by a version of independent amalgamation of types which gives rise to a very useful criterion for establishing that a structure is NSOP_1. Leveraging work of Granger and Chatzidakis, this then was used to show that the two-sorted theory of vector spaces over an algebraically closed field with a generic bilinear form and ω-free PAC fields of characteristic zero are NSOP_1.

  This promises to be a very important step towards starting the study of NSOP_1 theories, and towards making some progress towards settling whether or not NSOP_1 and NTP_1 theories are equivalent.

- **Artem Chernikov** spoke about the Zarankiewicz problem for (hyper-)graphs and counting types. The Zarankiewicz problem comes from in extremal graph theory and asks for the largest possible number of edges in a bipartite graph that has a given number of vertices but has no complete bipartite subgraphs of a given size. Chernikov discussed how bounds for the hypergraph version of this problem proved by Erdős can be used to count (partial) types in n-dependent and NTP_2 theories, and how improved bounds may be obtained for graphs definable in various NIP structures.

- **Nadja Hempel** spoke about a joint work with Alf Onshuus, about finding definable groups around groups with a certain property. Poizat showed that in the stable context, for any nilpotent/solvable subgroup of a stable group there is a definable nilpotent/solvable group of the same class which contains the given group. Appropriate generalizations of this results were given for dependent theories (Shelah, de Aldama) and for simple theories (Milliet). Hempel and Onshuus proved that these results could be proved to abelian/normal nilpotent/normal soluble subgroups of groups definable in an NTP_2 theory.

- **Isaac Goldbring** reported on joint work with Nasso, Jin, Leth, Lubini and Mahlburg, on progress on a famous conjecture of Erdős, that a set of natural numbers of positive lower density contains the sum of two infinite sets.

- **Silvain Rideau**'s talk centered around a joint work with Pierre Simon on definable and invariant types in enrichments of NIP theories. Using the theory of honest definitions, they showed how under certain weak hypotheses invariant types in a reduct may be obtained from definable types in an expansion.

- **Pierre Simon** proved a remarkable result for types in any NIP theory. He proved that in an dependent theory into a stable part and a distal "quotient". This, together with the ongoing study and knowledge on distal types makes this a very important result in dependent theories and in particular already Simon reported using this for a stronger form of honest definitions.

- **Sergei Starchenko** continued to show results about the combinatorics of definable sets in o-minimal theories. He showed that the following theorem from [1]

  Theorem. *Let $\varphi(x_1, \ldots, x_r)$, with all x_i singletons, be a semi-algebraic formula over the reals. Then there is a constant $C = C_\varphi$ such that for every $n \in \mathbb{N}$ setting $N = 2^{2^{Cn}}$ every sequence of real numbers of length N contains a φ-indiscernible subsquence of length n.*

  can be generalized to any polynomially bounded o-minimal expansion of the real field.

5 Scientific Progress Made

Probably the main scientific progress made was to socialize and consolidate the different efforts and results in the area. The community building that has been achieved during the BIRS meetings is clear and significant when one sees the evolution of stability-theoretic methods in non stable contexts such as dependent, NTP_2, rosy, and SOP_1 theories.

More specifically many people reported progress made in their research during the meeting. Amador Martin-Pizarro reported progress in his research in equationality of pairs of algebraically closed fields of characteristic 0. Dierdre Haskell and Dugald Machpherson reported progress on their research about the characterization and study of imaginaries in valued fields. Atem Chernikov reported that together with Sergei Starchenko they obtained an analogue of the Buch-Matousek result mentioned in the highlights, this time in the p-adic context. Dugald Macpherson, Dario Garcia and Charles Steinhorn reported progress in their study of pseudofinite structures.

6 Open Problems

Model theory, and even more stable-like methods outside stable theories-, is a very young field, and although we have come a long way towards understanding and setting the foundations, there are many open problems and research areas that need to be explored. We list only a few of this.

6.1 Stable forking conjecture and related problems

For the last decade or so, no progress has been made towards settling some of the open problems in simplicity theory. The stable forking conjecture (even in the supersimple case), elimination of hyperimaginaries in simple theories, the equivalence between forking and thorn forking are all questions that seem as far away from being solved as we were ten years ago.

6.2 Minimal fields

Another famous open problem in the field is whether or not a minimal field (one where every definable set is finite or cofinite) is algebraically closed. This is known ([14], [15]) that the results holds if the characteristic is positive or if no partial order is defined (this was partially solved at a previous Neostability meeting). Solving this conjecture (by Podewski) is an important goal for our field.

6.3 Equivalence of NSOP_1 and NTP_1

. As reported previously, both the concepts of NTP_1 and NSOP_1 complement NTP_2 as the "other side" of generalizing simple theories (a theory is simple if and only if it has NTP_2 and either NTP_1 or NSOP_1). However, this branch of neostability has not been as active as NTP_2, to the extent where we do not know yet whether any NTP_1 has NSOP_1. This equivalence (or non equivalence) together with an understanding of definable sets in such theories, is an area which should be explored.

6.4 Describing different compactifications of definable groups

. It is known that in a group G definable in any dependent theory, the quotient G/G^{00} is compact. In o-minimal expansions of the real field, if G is definably compact, we "recover" a real algebraic group (over the actual real numbers). Another way of finding a compactification was recently discovered by finding the Ellis group of the action of G on the set of types. It is known that in definably amenable groups this two compactifications coincide. But a lot more needs to be studied: How different are this two structures in non amenable groups definable in o-minimal structures? Is there a nice Lie-type description for such compactifications in groups definable in, say, the p-adic numbers? Pillay's talk about compactifications in Pressburger arithmetic seems to be a very good starting point to explore this field.

7 Outcome of the Meeting

This meeting offered a very rich array of results, and proved that the research area started in 2009 is consolidating into a very exciting and fruitful area. Young researchers are working in the area and achieving substantial results. Contexts such as NTP_2 which started as a possible generalization of simple and dependent theories is now a solid area with plenty of results in the study of groups and types which appear in NTP_2 theories, solidifying its role as a major fault line and classification theory of dependent theories is taking on the robust character of stable classification theory.

This meeting had the effect of solidifying the body of work in neostability theory as a coherent project to discern robust divisions within the class of all theories and to develop stability theoretic methods in an appropriate level of generality.

References

[1] B. Bukh, Boris and J. Matousek, "Erdős-Szekeres-type statements: Ramsey function and decidability in dimension 1", *Duke Math. J.*, 2014.

[2] A. Chernikov, I. Kaplan, and S. Shelah, On non-forking spectra, `arXiv:1205.3101`, preprint, 2012.

[3] M. Malliaris and S. Shelah, Constructing regular ultrafilters from a model theoretic point of view, `arXiv:1204.1481`, preprint, 2012.

[4] J. Goodrick, A monotonicity theorem for DP-minimal densely ordered groups. *J. Symbolic Logic* **75** (2010), no. 1, 221–238.

[5] P. Simon, On dp-minimal ordered structures. *J. Symbolic Logic* **76** (2011), no. 2, 448–460.

[6] A. Chernikov and P. Simon, Externally definable sets and dependent pairs II (arXiv 1202.2650).

[7] A. Chernikov and I. Kaplan, Forking and dividing in NTP2 theories, *Journal of Symbolic Logic* **77**, Issue 1 (2012), 1-20.

[8] V. Guingona, On uniform definability of types over finite sets, *Journal of Symbolic Logic* **77**, Issue 2 (2012), 499-514.

[9] M. Aschenbrenner, A. Dolich, D. Haskell, D. Macpherson and S. Starchenko, Vapnik-Chervonenkis density in some theories without the independence property, II (arXiv 1109.5437).

[10] M. Aschenbrenner, A. Dolich, D. Haskell, D. Macpherson and S. Starchenko, Vapnik-Chervonenkis density in some theories without the independence property, I (arXiv 1109.5438).

[11] M. Bays, M. Gavrilovich and M. Hils, Some definability results in abstract Kummer theory (arXiv:1112.6012).

[12] I. Kaplan, A. Onshuus and A. Usvyatsov, Additivity of the dp-rank (arXiv:1109.1601).

[13] F. Wagner, Plus ultra (arXiv:1201.1114).

[14] Krupinski, Krzysztof ; Tanovic, Predrag ; Wagner, Frank O. Around Podewski's conjecture. *Fund. Math.* 222 (2013), no. 2, 175–193.

[15] Wagner, Frank O. Minimal fields. *J. Symbolic Logic* 65 (2000), no. 4, 1833–1835.