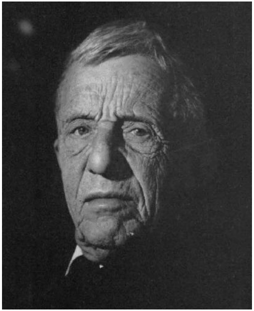

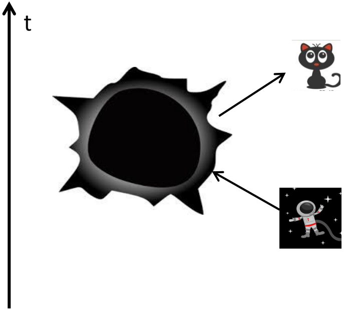

Symmetries of the damped harmonic oscillator and the Bateman system

Marco Cariglia

ICEB, Universidade Federal de Ouro Preto, MG, Brasil

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Black Holes' New Horizons

Oaxaca, 17 May 2016

Main results from arXiv:1605.01932

"Eisenhart lifts and symmetries of time-dependent systems"

In collaboration with

Christian Duval, Marseille
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- Quantization of time-dependent systems
- Special case of damped harmonic oscillator

Damped oscillator and Black Holes

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- Two interacting copies of Schrödinger algebras. What is their physical interpretation?

Summary

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2. Quantizing time dependent systems with the Eisenhart lift
3. Symmetries of the damped oscillator

Ringing frequencies of black holes

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- r_* dependency determined from regularity, t dependency by asking solution does not diverge in time.
- Find countable spectrum of frequencies ω_n. For example Schwarzschild: the $n \to +\infty$ modes for given spin j are independent of angular momentum and satisfy $e^{8\pi M\omega_j} = -(1 + 2\cos \pi j)$. [Motl 2002]

Ringing frequencies of black holes 2

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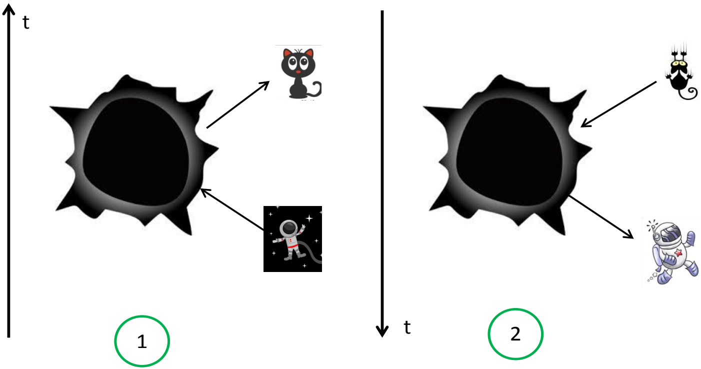

Beginning of quantization of quasinormal modes 2

- Express the Hamiltonian as

$$\hat{H} = \sum_{\lambda,\bar{\lambda}} \left[\omega_{R\lambda} \left(\hat{a}^\dagger_\lambda \hat{a}_\lambda - \hat{a}^\dagger_{\bar{\lambda}} \hat{a}_{\bar{\lambda}} \right) + i\omega_{I\lambda} \left(\hat{a}^\dagger_\lambda \hat{a}_{\bar{\lambda}} - \hat{a}^\dagger_\lambda \hat{a}_{\bar{\lambda}} \right) \right] .$$

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- Describes an oscillator with damping $2\omega_I$ and angular frequency ω_R, $\ddot{x} + 2\omega_I \dot{x} + \left(\omega_R^2 + \omega_I^2 \right) x = 0$ plus a "doppelgänger" with anti-damping $\ddot{y} - 2\omega_I \dot{y} + \left(\omega_R^2 + \omega_I^2 \right) y = 0$.

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- A time dependent canonical transformation transforms it into a double Caldirola-Kanai oscillator

$$H_{CK} = e^{-\gamma t} \frac{p_x'^2}{2m} + \frac{1}{2} m\omega^2 x'^2 e^{\gamma t} - e^{\gamma t} \frac{p_y'^2}{2m} - \frac{1}{2} m\omega^2 y'^2 e^{-\gamma t} \, .$$

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- What is the role of the amplifying modes? Not clear. Pal et Al. suggest an antiparticle. Related to the quantization of unstable 'quantons'.

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$$\begin{cases} L = \dfrac{m}{2\alpha(t)} g_{ij}(x^k)\dot{x}^i\dot{x}^j - \beta(t)V(x^i,t), \\ H = \dfrac{\alpha(t)}{2m} g^{ij}(x^k)p_ip_j + \beta(t)V(x^i,t), \end{cases}$$

m is mass, $g_{ij}(x^k)$ curved metric on configuration space Q with local coordinates x^i, $i = 1,\ldots,n$. $V(x^i,t)$ potential.

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Equations of motion

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} - \frac{\dot{\alpha}}{\alpha}\frac{dx^i}{dt} = -\frac{\alpha\beta}{m}g^{ij}\partial_j V\,,$$

Γ^i_{jk} Christoffel symbols of metric connection. When explicitly time-dependent $\rightarrow$ energy is not conserved.

Generalised Caldirola-Kanai systems 2

Velocity dependent term can be eliminated by introducing a new time-parameter $\tau = \tau(t)$ defined by $d\tau = \alpha\, dt$.

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For $V = \frac{1}{2}m\omega^2 x^2$, $\alpha = \beta^{-1} = e^{-\gamma t}$, get damped harmonic oscillator

$$L = \frac{m}{2}e^{\gamma t}\left(\left|\frac{d\vec{x}}{dt}\right|^2 - \omega^2\vec{x}^2\right), \qquad \frac{d^2\vec{x}}{dt^2} + \gamma\frac{d\vec{x}}{dt} = -\omega^2\,\vec{x}.$$

[Bateman 1931, Caldirola 1941, Kanai 1948, Dekker 1981, Um Yeon George 2002, Aldaya Cossío Guerrero López-Ruiz 2011-2012]

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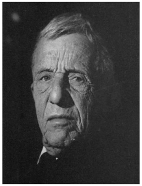

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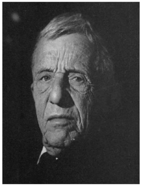

Solutions $t \mapsto x^i(t)$ of $H = \frac{\alpha(t)}{2m} g^{ij} p_i p_j + \beta(t) V(x, t)$ in correspondence with *null geodesics* of the Lorentzian metric in $(n+2)$-dimensions

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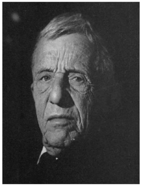

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where $(x^a) = (x^i, t, s)$.

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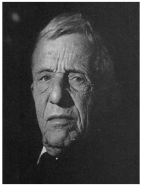

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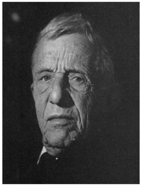

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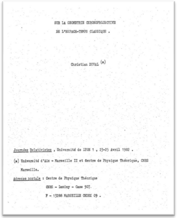
Time reparameterization associated to a conformal rescaling

$$\tilde{g} = \alpha(t) g = g_{ij}(x^k) dx^i dx^j + 2 d\tau ds - \frac{2\beta}{m\alpha} V d\tau^2 .$$

Modern point of view



SUR LA GEOMETRIE CHRONOPROJECTIVE
DE L'ESPACE-TEMPS CLASSIQUE .

Christian DUVAL (*)

Journées Relativistes , Université de LYON 1 , 23-25 Avril 1982 .

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CNRS - Luminy - Case 907.
F - 13288 MARSEILLE CEDEX 09 .

- Christian Duval rediscovers the lift and applies it to non-relativistic mechanics

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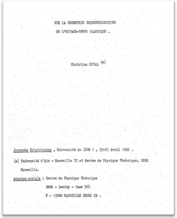

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F - 13288 MARSEILLE CEDEX 09 .

- Christian Duval rediscovers the lift and applies it to non-relativistic mechanics
- Today: Schrödinger equation, non-relativistic electrodynamics [Duval, Gibbons, Horváthy 1991], pp-waves, non-relativistic holography [Balasubramanian, McGreevy 2008; Son 2008; Duval, Hassaïne, Horváthy 2009; Bekaert, Morand 2013], Lorentzian distance and least action [Minguzzi 2007], . . .

Quantization using Eisenhart. Step 1: Schrödinger equation

Massless minimally coupled scalar wave equation in Eisenhart spacetime

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$$i\partial_t\Psi = \hat{H}\Psi\,, \qquad \hat{H} = -\frac{\alpha}{2m}\nabla^2 + \beta V\,.$$

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Find $\partial_t\big(\bar{\Psi}\Psi\big) = i\frac{\alpha}{2m}\nabla^j\big(\bar{\Psi}\partial_j\Psi - \Psi\partial_j\bar{\Psi}\big)$, where ∇^i is the Levi-Civita covariant derivative of the metric g_{ij}. Then the conserved probability is

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Notice that

$$\Psi(t+t') \neq e^{-i\hat{H}t}\Psi(t')\,.$$

Quantization using Eisenhart. Step 2: second quantised QFT

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- Choose Cauchy surface as $t = t_0 \rightarrow$

$$(\phi', \phi) = 2i\int ds \int \sqrt{-g}d^n x\big(\bar{\phi}'\partial_s\phi - \phi\partial_s\bar{\phi}'\big)\,.$$

  This generates the *superselection rule* $m = m'$.

Summary

1. Invitation/Possible application: Black holes' quasinormal modes

2. Quantizing time dependent systems with the Eisenhart lift

3. Symmetries of the damped oscillator

Reminder of main formulas

Hamiltonian of the Caldirola-Kanai oscillator

$$H = \frac{1}{2m} e^{-\gamma t} p^2 + \frac{1}{2} e^{\gamma t} m\omega^2 x^2 \,.$$

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Eisenhart lift

$$g_{ab} dx^a dx^b = e^{\gamma t} dx^2 + 2dtds - e^{\gamma t} \omega^2 x^2 dt^2 \,.$$

Arnold Transformation

- The most general linear second-order differential equation in one dimension

$$\ddot{x} + \dot{f}(t)\dot{x} + \omega^2(t)x = F(t) \,,$$

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- $$x(t) = u_p(t) + a u_1(t) + b u_2(t)\,.$$

Dividing times $u_1(t)$ when allowed:

$$\xi(\tau) = a\tau + b\,, \qquad \xi = \frac{x - u_p}{u_2}\,, \qquad \tau = \frac{u_1}{u_2}\,.$$

Arnold Transformation 2

- One finds *the Eisenhart metric is conformally flat*:

$$g_{ab}dx^a dx^b = \underbrace{e^f}_{\alpha^{-1}} dx^2 + 2dtds - 2e^f \underbrace{\left(\frac{1}{2}\omega^2 x^2 - F(t)x\right)}_{\frac{V}{m}} dt^2$$

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- For the damped harmonic oscillator $e^f = e^{\gamma t}$, $F(t) = 0$,

$$u_1 = e^{-\gamma t/2} \frac{\sin \Omega t}{\Omega} , \quad u_2 = e^{-\gamma t/2} \big(\cos \Omega t + \frac{\gamma}{2\Omega} \sin \Omega t \big) , \quad \Omega^2 = \omega^2 - \gamma^2/4 .$$

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- For $\gamma = 0$ reduces to 'Niederer's trick' [Niederer 1973] lifted to higher dimension.

Symmetries of the damped oscillator

Conformally related Hamiltonians share identical symmetries generated by conformal Killing vectors: import symmetries from those of flat space!

Symmetries of the damped oscillator 2

Extended Schrödinger algebra

$$\begin{aligned} T &= p_\xi \\ B &= -p_\sigma \xi + p_\xi \tau \\ E &= -p_\tau \\ m &= p_\sigma \\ D &= -2\tau p_\tau - \xi p_\xi \end{aligned}$$

$$K = \tau^2 p_\tau + \tau \xi p_\xi - \frac{1}{2}\xi^2 p_s$$

Symmetries of the damped oscillator 2

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m &= p_\sigma = p_s\,.\\
D &= -2\tau p_\tau - \xi p_\xi\\
&= -2e^{\gamma t}u_1 u_2 p_t - \left(1 + 2e^{\gamma t}u_1 \dot u_2\right)x\,p_x + \frac{e^{\gamma t}}{u_2}\left[\dot u_2 - e^{\gamma t}u_1\left(\omega^2 u_2^2 - \dot u_2^2\right)\right]x^2 p_s\,,\\
K &= \tau^2 p_\tau + \tau\xi p_\xi - \frac{1}{2}\xi^2 p_s\\
&= e^{\gamma t}u_1^2 p_t + \frac{u_1}{u_2}\left(1 + e^{\gamma t}u_1\dot u_2\right)x\,p_x\\
&\quad + \frac{1}{2u_2^2}\left[-1 - 2e^{\gamma t}u_1\dot u_2 + e^{2\gamma t}u_1^2\left(\omega^2 u_2^2 - \dot u_2^2\right)\right]x^2 p_s\,.
\end{aligned}
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Symmetries of the Bateman oscillator

- Recall the Bateman-Feshbach-Tikochinksy oscillator:

$$\hat{H}_B = p_x p_y + \frac{\gamma}{2}(y p_y - x p_x) + m\Omega^2 xy\,, \qquad \Omega = \sqrt{\omega^2 - \frac{\gamma^2}{4}}\,.$$

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- Describes a damped particle and an anti-damped double, or doppelgänger:

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- A time dependent canonical transformation transforms it into a double Caldirola-Kanai oscillator

$$H_{CK} = e^{-\gamma t}\frac{p_x'^2}{2m} + \frac{1}{2}m\omega^2 x'^2 e^{\gamma t} - e^{\gamma t}\frac{p_y'^2}{2m} - \frac{1}{2}m\omega^2 y'^2 e^{-\gamma t}\,.$$

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$$p_x' = \frac{\partial F_2}{\partial x'}\,, \quad p_y' = \frac{\partial F_2}{\partial y'}\,, \quad x = \frac{\partial F_2}{\partial p_x}\,, \quad y = \frac{\partial F_2}{\partial p_y}\,, \quad H_B = H_{CK} + \frac{\partial F_2}{\partial t}\,,$$

$$F_2 = \frac{1}{\sqrt{2}}\left(e^{\gamma t}x' + y'\right)p_y + \frac{1}{\sqrt{2}}\left(x' - e^{-\gamma t}y'\right)p_x + \frac{\gamma}{4m\Omega^2}e^{-\gamma t}p_x^2 - \frac{m\gamma}{8}\left(e^{\frac{\gamma t}{2}}x' - e^{-\frac{\gamma t}{2}}y'\right)^2$$

Symmetries of the Bateman oscillator

- Eisenhart lift of the double Caldirola-Kanai

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- Two interacting copies of the Schrödinger algebras. Define $T_1 = T$, $B_1 = B$ and $E_1 = E$ for the first copy. Then $v_{1,2}(t)$ obtained from $u_{1,2}(t)$ using $\gamma \to -\gamma$,

$$\begin{aligned}
T_2 &= v_2\, p'_y - e^{-\gamma t}\, \dot{v}_2\, y'\, p_s \,, \\
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- Heisenberg subalgebras are mutually commuting:

$$\begin{aligned}
\{T_1, T_2\} = 0 = \{T_1, B_2\} \,, \\
\{B_1, T_2\} = 0 = \{B_1, B_2\} \,.
\end{aligned}$$

Symmetries of the Bateman oscillator 2

However there exist two independent definitions of 'conformally flat time', and each Heisenberg subalgebra is time dependent with respect to the other time: infinite copies of mutually commuting Heisenberg subalgebras. For $i \neq j$, $i, j = 1, 2$

$$\{E_i, T_j\} := \tau_j^{(1)} \qquad \{E_i, \tau_j^{(n)}\} := \tau_j^{(n+1)} ,$$

$$\{E_i, B_j\} := \beta_j^{(1)} \qquad \{E_i, \beta_j^{(n)}\} := \beta_j^{(n+1)}$$

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$$\{\tau_i^{(n)}, \beta_j^{(n)}\} = \delta_{ij}\mu_i^{(n)} ,$$

For concreteness, specialising to the first $n = 1$ level we find the following generators:

$$\tau_2^{(1)} = e^{\gamma t}\, u_2^2\, \dot{v}_2\, p_y' + \omega^2\, u_2^2\, v_2\, y'\, p_s ,$$
$$\tau_1^{(1)} = e^{-\gamma t}\, v_2^2\, \dot{u}_2\, p_x' + \omega^2\, v_2^2\, u_2\, x'\, p_s ,$$
$$\beta_2^{(1)} = e^{\gamma t}\, u_2^2\, \dot{v}_1\, p_y' + \omega^2\, u_2^2\, v_1\, y'\, p_s ,$$
$$\beta_1^{(2)} = e^{-\gamma t}\, v_2^2\, \dot{u}_1\, p_x' + \omega^2\, v_2^2\, u_1\, x'\, p_s ,$$
$$\mu_2^{(1)} = e^{2\gamma t}\omega^2 u_2^4\, p_s ,$$
$$\mu_1^{(2)} = e^{-2\gamma t}\omega^2 v_2^4\, p_s .$$

(there are also cross commutators)

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Future perspectives:

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Future perspectives:

- Study in detail a concrete example of quasinormal modes to get insight.