Various limits of Kerr-NUT-(A)dS spacetimes

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Outline

- **Off-shell Kerr–NUT–(A)dS metric**
  geometry possessing the principal CCKY tensor
- **On-shell Kerr–NUT–(A)dS metric**
  satisfying the Einstein equations
- **Examples**
  Euclidian instanton, generally rotating black hole
- **Limits of vanishing rotations**
  spherical, deformed and twisted black holes
- **Nutty spacetimes**
  Taub–NUT-like limit

Off-shell Kerr–NUT–(A)dS metric

Kerr–NUT–(A)dS metric

(for simplicity only in even dimensions $D = 2N$)

$$\boldsymbol{g} = \sum_\mu \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \left(\sum_k A_\mu^{(k)} \boldsymbol{d}\psi_k \right)^2 \right]$$

x_μ	radial and latitudinal coordinates ($\mu = 1, \dots, N$)
ψ_k	temporal and longitudinal coordinates ($k = 0, \dots, N{-}1$)
$X_\mu = X_\mu(x_\mu)$	metric functions to be determined by the Einstein equations

$$A^{(k)} = \sum_{\substack{\nu_1,\dots,\nu_k \\ \nu_1<\dots<\nu_k}} x_{\nu_1}^2 \dots x_{\nu_k}^2 \qquad A_\mu^{(k)} = \sum_{\substack{\nu_1,\dots,\nu_k \\ \nu_1<\dots<\nu_k,\ \nu_i\neq\mu}} x_{\nu_1}^2 \dots x_{\nu_k}^2 \qquad U_\mu = \prod_{\substack{\nu \\ \nu\neq\mu}} (x_\nu^2 - x_\mu^2)$$

- Myers R. C., Perry M. J.: *Black Holes in Higher Dimensional Space-Times*, Ann.Phys. 172 (1986) 304
- Gibbons G. W., L H., Page D. N., Pope C. N.: *Rotating Black Holes in Higher Dimensions with a Cosmological Constant*, Phys.Rev.Lett. 93 (2004) 171102
- Chen W., L H., Pope C. N.: *General Kerr-NUT-AdS Metrics in All Dimensions*, Class.Quant.Grav. 23 (2006) 5323

Geometry includes:

- "Kerr" — rotating black holes
- "NUT" — nontrivial NUT behavior
- "(A)dS" — arbitrary cosmological constant
- geometries of various signatures
- maximally symmetric spaces
- Euclidian instantons
- geometry of horizons
- various sectors of geometry of deformed black holes

Properties

- **Uniquely determined by the existence of the principal CCKY tensor**
- **Integrability of geodesic motion**
- **Separability of the Hamilton–Jacobi equations**
- **Commuting scalar symmetry operators**
- **Separability of the Klein–Gordon equations**
- **Commuting Dirac symmetry operators**
- **Separability of the Dirac equations**

Explicit and hidden symmetries

Tower of various Killing objects build form the principal CCKYtensor

Explicit symmetries — N independent **Killing vectors**:

$$\boldsymbol{l}_{(j)} = \boldsymbol{\partial}_{\psi_j} \qquad j = 0, \dots, N-1$$

⇒ observables linear in momenta

Hidden symmetries — N independent **Killing tensors of rank 2**:

$$\boldsymbol{k}_{(j)} = \sum_{\mu} A_{\mu}^{(j)} \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_k A_\mu^{(k)} \boldsymbol{d}\psi_k \Big)^2 \right] \qquad j = 0, \dots, N-1$$

$$\boldsymbol{k}_{(0)} = \boldsymbol{g} \qquad \text{metric for } j = 0$$

⇒ observables quadratic in momenta
(generalization of the Carter constant)

Kerr–NUT–(A)dS metric

— coordinate ranges

$$\boldsymbol{g} = \sum_{\mu} \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \left(\sum_k A_\mu^{(k)} \boldsymbol{d}\psi_k \right)^2 \right]$$

$$X_\mu = X_\mu(x_\mu) \qquad\qquad U_\mu = \prod_{\substack{\nu \\ \nu\neq\mu}} (x_\nu^2 - x_\mu^2)$$

Zeros of U_μ are problematic!

- ranges of x's should avoid zeros of U_μ
- for example, ordering of coordinates $x_1 < x_2 < \cdots < x_N$

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Zeros of X_μ correspond to:

- axes of rotational symmetry
- Killing horizon of temporal symmetry

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Zeros of U_μ are problematic!

- ranges of x's should avoid zeros of U_μ
- for example, ordering of coordinates $x_1 < x_2 < \cdots < x_N$

Zeros of X_μ determine ranges of x_μ:

- x_μ from spatial sector (angles) run between two roots of X_μ (two semi-axes)
- x_μ from Lorentzian sector (radius) is not restricted by roots of X_μ (horizons)

Kerr–NUT–(A)dS metric

— rewinding Killing angles

$$\boldsymbol{g} = \sum_\mu \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \left(\sum_k A_\mu^{(k)} \boldsymbol{d}\psi_k \right)^2 \right]$$

Any constant linear combination of Killing coordinates gives Killing coordinates:

$$\phi_\mu = \sum_k \alpha_\mu^k \, \psi_k$$

Which coordinates should be periodic?

Kerr–NUT–(A)dS metric

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Any constant linear combination of Killing coordinates gives Killing coordinates:

$$\phi_\mu = \sum_k \alpha_\mu^k \psi_k$$

rewinding Killing coordinates ⇒ measurable changes of geometry:

- conical singularities — superposing cosmic string
- mixing time and angles — superposing rotating strings
- mixing angles — twisting geometry

in general, regularity of all axes cannot be achieved!

Kerr–NUT–(A)dS metric

— rewinding Killing angles

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coordinates associated with Killing vectors vanishing at axes:

$$\phi_\mu = \sum_k \mathring{A}_\mu^{(k)} \psi_k \qquad \Rightarrow \qquad \boldsymbol{\partial}_{\phi_\mu}\Big|_{x_\mu = \mathring{x}_\mu} = 0$$

$$x_\mu = \mathring{x}_\mu \quad \text{axis} \qquad\qquad \big(\,\text{i.e., } X_\mu(\mathring{x}_\mu) = 0\,\big)$$

$$\mathring{A}_\mu^{(k)} = \sum_{\substack{\nu_1,\dots,\nu_k \\ \nu_1<\dots<\nu_k,\ \nu_i\neq\mu}} \mathring{x}_{\nu_1}^2 \dots \mathring{x}_{\nu_k}^2$$

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rewinding $\phi_\mu = \sum_k \mathring{A}_\mu^{(k)} \psi_k$ parametrized by $\mathring{x}_\mu$

$$\Downarrow$$

$$\boldsymbol{g} = \sum_\mu \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \left(\sum_\nu \frac{J_\mu(\mathring{x}_\nu^2)}{\mathring{U}_\nu} \boldsymbol{d}\phi_\nu \right)^2 \right]$$

where $J_\mu(x^2) = \prod_{\substack{\nu \\ \nu \neq \mu}} (x_\nu^2 - x^2)$

Kerr–NUT–(A)dS metric

— restriction to axes and horizons

$$\boldsymbol{g} = \sum_\mu \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_\nu \frac{J_\mu(\mathring{x}_\nu^2)}{\mathring{U}_\nu} \boldsymbol{d}\phi_\nu \Big)^2 \right]$$

$\tilde{\mathcal{A}}$ — **intersection of $\bar{N}$ axes or horizons**

- axes conditions: $x_{\bar{\mu}} = \mathring{x}_{\bar{\mu}}$ ($\mathring{x}_{\tilde{\mu}}$ is zero of $X_{\tilde{\mu}}$)
- corresponding coordinates $\phi_{\bar{\mu}}$ degenerate
- $x_{\tilde{\mu}}$ and $\phi_{\tilde{\mu}}$ – coordinates on $\tilde{\mathcal{A}}$

Kerr–NUT–(A)dS metric

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- corresponding coordinates $\phi_{\bar{\mu}}$ degenerate
- $x_{\tilde{\mu}}$ and $\phi_{\tilde{\mu}}$ – coordinates on $\tilde{\mathcal{A}}$

Metric restricted to $\tilde{\mathcal{A}}$ is Kerr-NUT-(A)dS with $\tilde{X}_{\tilde{\mu}} = \dfrac{X_{\tilde{\mu}}}{\mathring{\bar{J}}(x_{\tilde{\mu}}^2)}$

$$\boldsymbol{g}|_{\tilde{\mathcal{A}}} = \sum_{\tilde{\mu}} \Big[\frac{\tilde{U}_{\tilde{\mu}}}{\tilde{X}_{\tilde{\mu}}} \boldsymbol{d}x_{\tilde{\mu}}^2 + \frac{\tilde{X}_{\tilde{\mu}}}{\tilde{U}_{\tilde{\mu}}} \Big(\sum_{\tilde{\nu}} \frac{\tilde{J}_{\tilde{\mu}}(\mathring{x}_{\tilde{\nu}}^2)}{\mathring{\tilde{U}}_{\tilde{\nu}}} \boldsymbol{d}\phi_{\tilde{\nu}} \Big)^2 \Big]$$

Kerr–NUT–(A)dS metric

$(D = 2N)$

$$\boldsymbol{g} = \sum_{\mu} \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \left(\sum_k A_\mu^{(k)} \boldsymbol{d}\psi_k \right)^2 \right]$$

$$X_\mu = X_\mu(x_\mu)$$

off-shell metric

On-shell Kerr–NUT–(A)dS metric

Kerr–NUT–(A)dS metric

$(D = 2N)$

$$\boldsymbol{g} = \sum_{\mu} \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_k A_\mu^{(k)} \boldsymbol{d}\psi_k \Big)^2 \right]$$

Einstein equations

with $\Lambda = (2N-1)(N-1)\lambda$

$\Downarrow$

$$X_\mu = \lambda \, \mathcal{J}(x_\mu^2) - 2\, b_\mu \, x_\mu$$

$\mathcal{J}(x^2)$ polynomial of degree N in x^2

$$\mathcal{J}(x^2) = \prod_\mu (a_\mu^2 - x^2) = \sum_{k=0}^{N} \mathcal{A}^{(k)} (-x^2)^{N-k}$$

Kerr–NUT–(A)dS metric

— explicit parameters of the solutions

$$\boldsymbol{g} = \sum_{\mu} \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \left(\sum_k A_\mu^{(k)} \boldsymbol{d}\psi_k \right)^2 \right]$$

$$X_\mu = \lambda \prod_\mu (a_\mu^2 - x_\mu^2) - 2\, b_\mu\, x_\mu$$

λ	$\propto$ cosmological constant	1
b_μ	mass and NUT charges	N
a_μ	rotational parameters	N

Kerr–NUT–(A)dS metric
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scaling freedom

$$x_\mu \to s x_\mu\,, \quad \psi_k \to s^{-(k+1)}\psi_k\,,$$
$$a_\mu \to s a_\mu\,, \quad b_\mu \to s^{2N-1} b_\mu\,, \quad \lambda \to \lambda$$

Kerr–NUT–(A)dS metric

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$$x_\mu \to s x_\mu\ , \quad \psi_k \to s^{-(k+1)} \psi_k\ ,$$
$$a_\mu \to s a_\mu\ , \quad b_\mu \to s^{2N-1} b_\mu\ , \quad \lambda \to \lambda$$

$\Rightarrow$ elimination of one parameter by a gauge condition $\quad - 1$

$$\lambda a_N^2 = -1$$

Kerr–NUT–(A)dS metric

— signature

$$\boldsymbol{g} = \sum_\mu \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_k A_\mu^{(k)} \boldsymbol{d}\psi_k \Big)^2 \right]$$

$$X_\mu = \lambda \prod_\mu (a_\mu^2 - x_\mu^2) - 2\, b_\mu\, x_\mu$$

Signature is given by:

- signs of X_μ and U_μ
- Wick rotations of coordinates and parameter

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Signature is given by:

- signs of X_μ and U_μ
- Wick rotations of coordinates and parameter

Wick rotation:

- for $x_\mu^2 < 0$ and $b_\mu\, x_\mu \in \mathbb{R}$
- for $a_\mu^2 < 0$

the metric remains real

Examples

Sphere

$$\boldsymbol{g} = \sum_\mu \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_k A_\mu^{(k)} \boldsymbol{d}\psi_k \Big)^2 \right]$$

$$b_\mu = 0 \qquad \lambda > 0 \qquad \Rightarrow \qquad X_\mu = \lambda\, \mathcal{J}(x_\mu^2) = \lambda \prod_\mu (a_\mu^2 - x_\mu^2)$$

$\pm a_\mu$ are roots of X_μ $\quad\Rightarrow\quad$ natural ranges of coordinates:

$$-a_1 < x_1 < a_1 < x_2 < a_2 < x_3 < \cdots < x_N < a_N$$

Sphere

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rewinding angles:

$$\phi_\mu = \lambda a_\mu^2 \sum_k \frac{(-a_\mu^2)^{N-1-k}}{\mathcal{U}_\mu} \psi_k$$

and

Jacobi transformation:

$$\rho_\mu^2 = \frac{J(a_\mu^2)}{-\lambda a_\mu^2 U_\mu} \qquad\qquad \rho_0^2 = \frac{A^{(N)}}{\mathcal{A}^{(N)}}$$

$$\Downarrow$$

Sphere

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$$\Downarrow$$

$$\boldsymbol{g} = \boldsymbol{d}\rho_0^2 + \sum_\mu \left[\boldsymbol{d}\rho_\mu^2 + \rho_\mu^2\, \boldsymbol{d}\phi_\mu^2 \right]$$

$$\sum_{\mu=0}^{N} \rho_\mu^2 = \frac{1}{\lambda}$$

$2N$ dimensional sphere in $2N+1$ dimensional Euclidian space

Sphere

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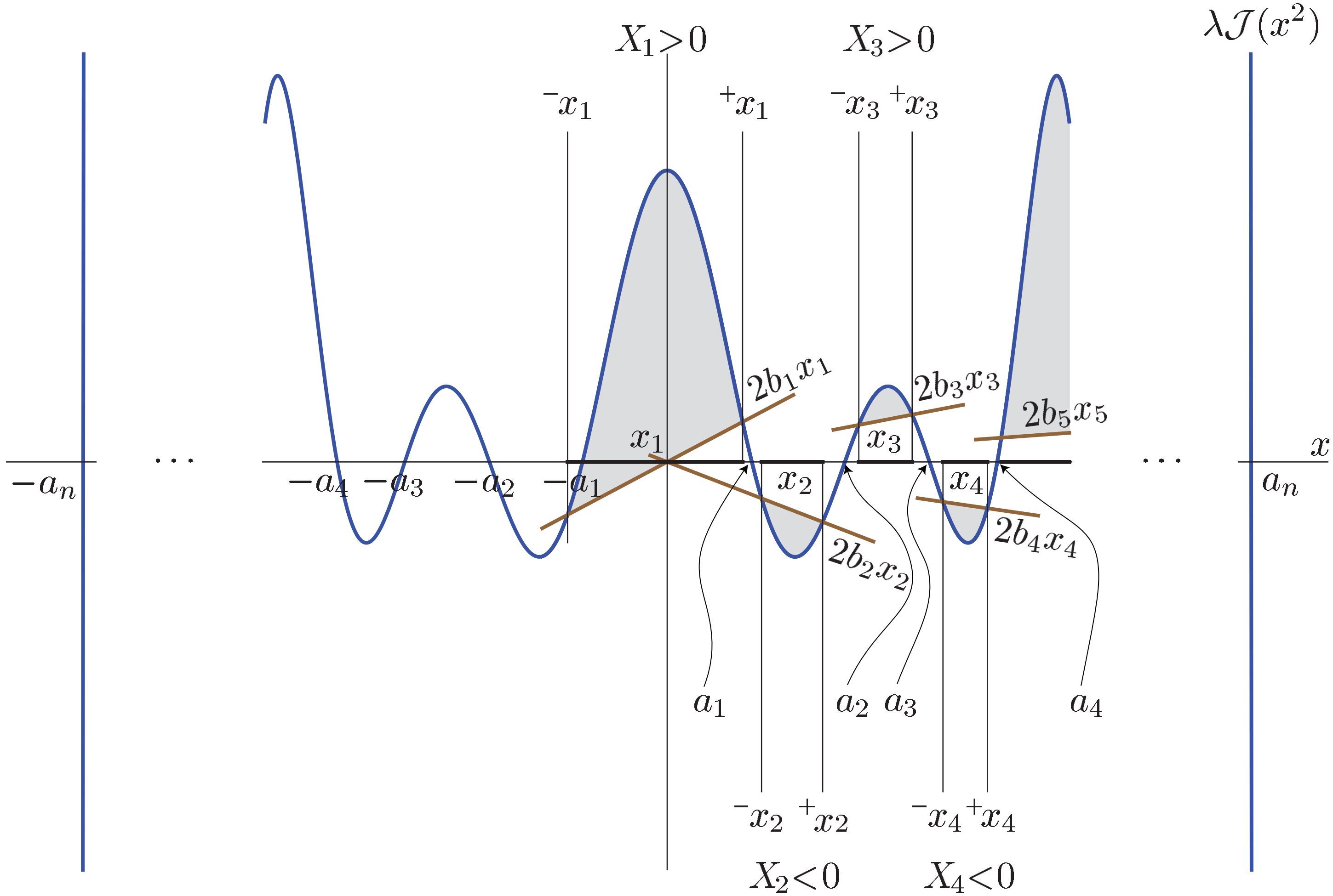
geometry does not depend on a_μ!

Euclidian instanton

$$\boldsymbol{g} = \sum_{\mu} \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_k A_\mu^{(k)} \boldsymbol{d}\psi_k \Big)^2 \right]$$

$b_\mu \neq 0 \qquad \lambda > 0 \qquad \Rightarrow \qquad X_\mu = \lambda\, \mathcal{J}(x_\mu^2) - 2\, b_\mu\, x_\mu$



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let ${}^{(-)}x_\mu$, ${}^{(+)}x_\mu$ be roots of X_μ close to $a_{\mu-1}$ and a_μ

$\Rightarrow$ natural ranges of coordinates:

$${}^{(-)}x_\mu < x_\mu < {}^{(+)}x_\mu$$

for suitable signs of b_μ one get $x_1 < x_2 < \cdots < x_N$

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- Euclidian signature

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- Euclidian signature
- geometry depend both on b_μ and a_μ

Black hole

$$\boldsymbol{g} = \sum_{\mu} \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_k A_\mu^{(k)} \boldsymbol{d}\psi_k \Big)^2 \right]$$

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Wick rotation of radial coordinate:

$$x_N = ir \qquad b_N = im$$

Gauge condition:

$$a_N^2 = -\frac{1}{\lambda}$$

Black hole

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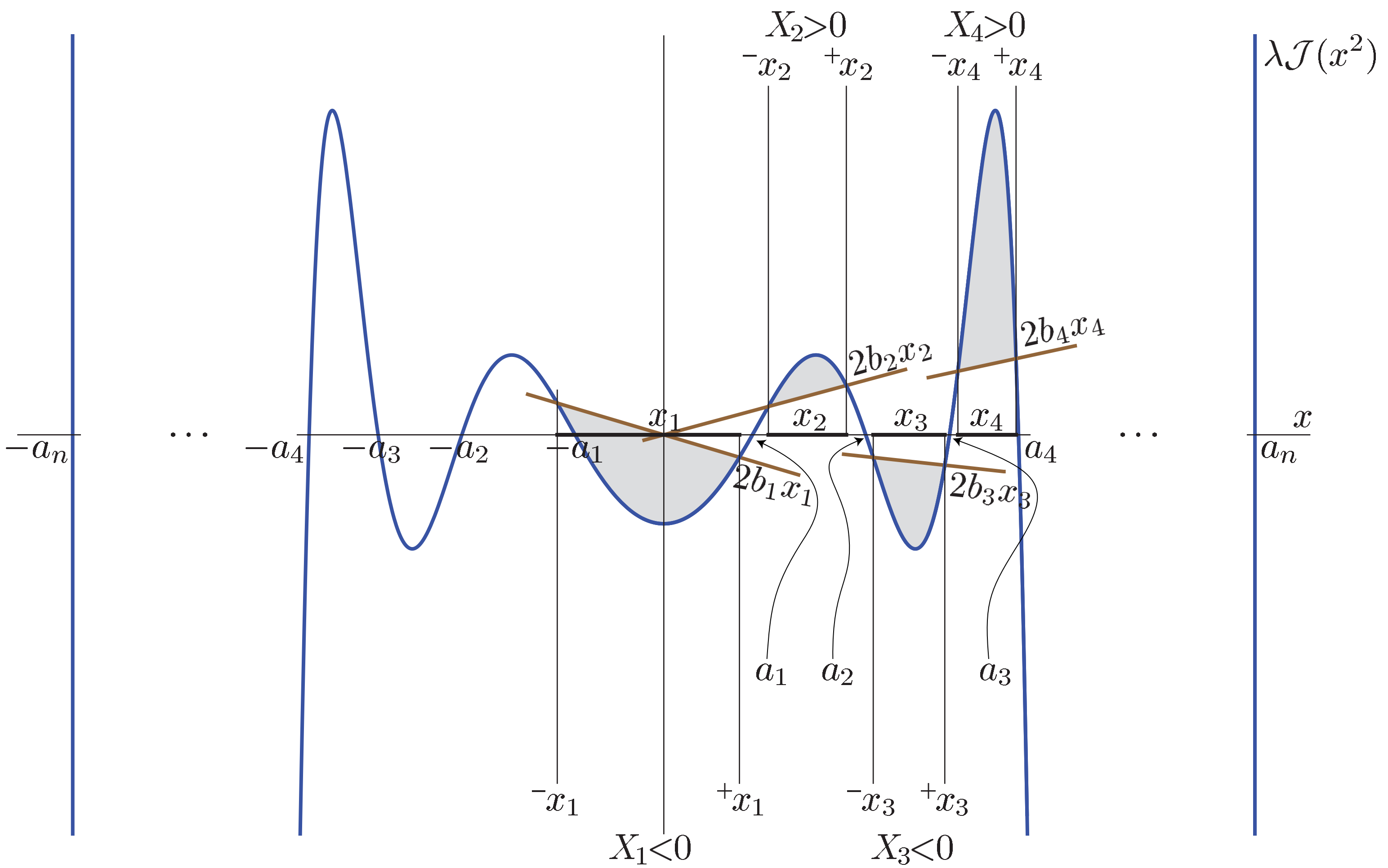

Black hole

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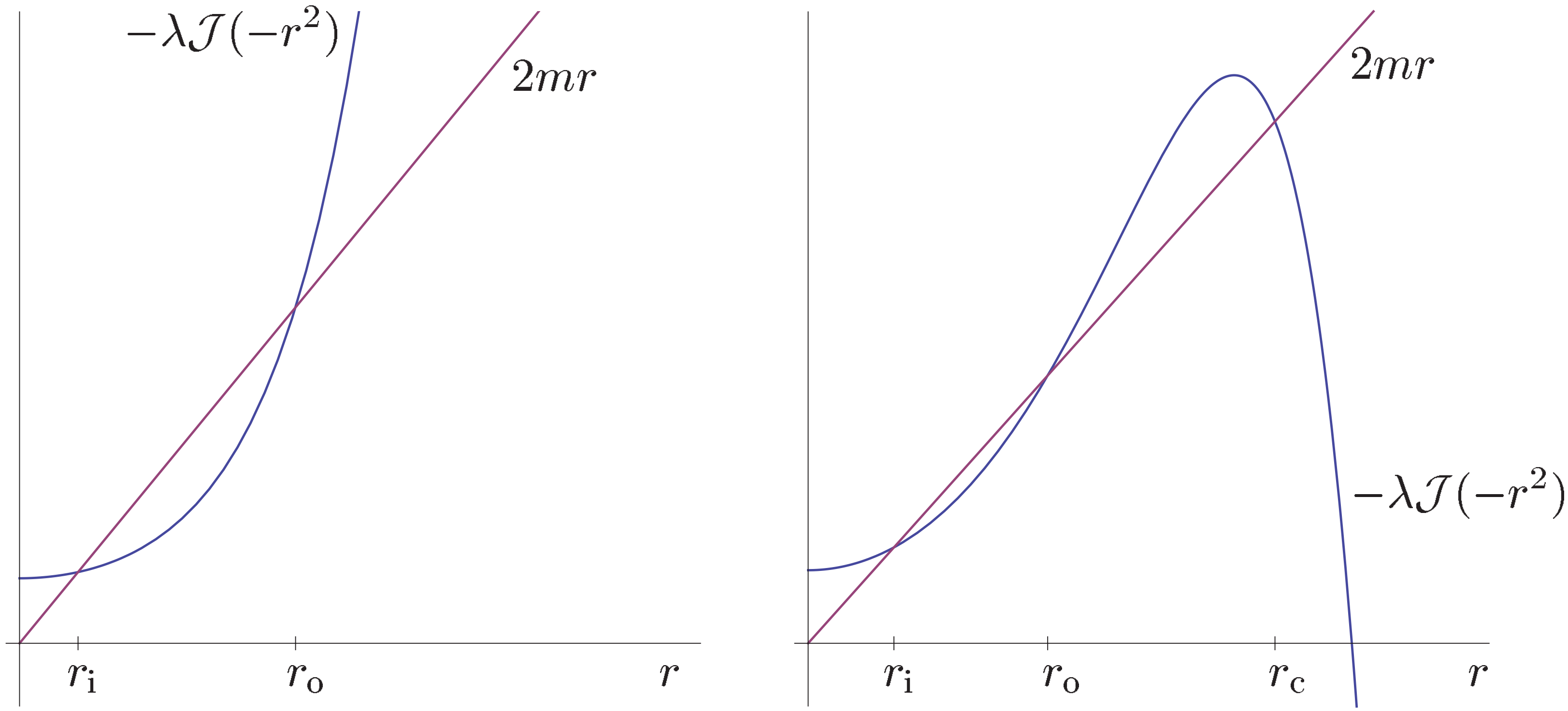
$$X_N = \lambda\, \mathcal{J}(-r^2) + 2\, m\, r \qquad x_N = ir \qquad b_N = im \qquad a_N^2 = -\lambda^{-1}$$



$\lambda \leq 0$ $\qquad$ $\lambda > 0$

Limit of vanishing rotations

- Krtouš P., Kubizňák D., Frolov V. P., Kolář I.: *Deformed and twisted black holes with NUTs*, Class. Quant. Grav. 33 (2016) 115016

Limit of vanishing rotations

— parametrization of rotations

- for $m \neq 0$, $b_\mu = 0$, $\lambda = 0$

  – rotations parametrized by a_μ

  (comparing generators of horizon and static observers at infinity)

- for nontrivial b_μ

  – unclear identification of rotations

  (how to distinguish local nutty rotations from global rotations?)

- qualitatively estimated by a 'length' of the x-coordinate range

  $^{(+)}x_\mu - {}^{(-)}x_\mu$

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vanishing rotations in $\bar{N}$ directions

$\Updownarrow$

$a_{\bar{\mu}} \to 0$ **for these directions**

Limit of vanishing rotations
— scaling of coordinates and parameters

$$\boldsymbol{g} = \sum_\mu \left[\frac{U_\mu}{X_\mu} \boldsymbol{d}x_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_\nu \frac{J_\mu(a_\nu^2)}{\mathcal{U}_\nu} \boldsymbol{d}\phi_\nu \Big)^2 \right]$$

$$X_\mu = \lambda\, \mathcal{J}(x_\mu^2) - 2\, b_\mu\, x_\mu$$

- setting $a_{\tilde{\mu}} = 0$ yields degenerate ranges of coordinates
- setting $a_{\tilde{\mu}} = 0$ yields degenerate metric
- a suitable scaling of coordinates necessary!

Limit of vanishing rotations
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regular sector

$$a_{\bar{N}+\tilde{\mu}} = \tilde{a}_{\tilde{\mu}}$$
$$x_{\bar{N}+\tilde{\mu}} = \tilde{x}_{\tilde{\mu}}$$
$$\phi_{\bar{N}+\tilde{\mu}} = \tilde{\phi}_{\tilde{\mu}}$$
$$\tilde{\mu} = 1, \dots, \tilde{N}$$

unspined sector

$$a_{\bar{\mu}} = \varepsilon\, \bar{a}_{\bar{\mu}}$$
$$x_{\bar{\mu}} = \varepsilon\, \bar{x}_{\bar{\mu}}$$
$$\phi_{\bar{\mu}} = \bar{\phi}_{\bar{\mu}}$$
$$\bar{\mu} = 1, \dots, \bar{N}$$

$$\varepsilon \longrightarrow 0$$

Limit of vanishing rotations

— limit of the metric

regular sector

$$a_{\bar{N}+\tilde{\mu}} = \tilde{a}_{\tilde{\mu}}$$
$$x_{\bar{N}+\tilde{\mu}} = \tilde{x}_{\tilde{\mu}}$$
$$\phi_{\bar{N}+\tilde{\mu}} = \tilde{\phi}_{\tilde{\mu}}$$
$$\tilde{\mu} = 1,\dots,\tilde{N}$$

unspined sector

$$a_{\bar{\mu}} = \varepsilon\,\bar{a}_{\bar{\mu}}$$
$$x_{\bar{\mu}} = \varepsilon\,\bar{x}_{\bar{\mu}}$$
$$\phi_{\bar{\mu}} = \bar{\phi}_{\bar{\mu}}$$
$$\bar{\mu} = 1,\dots,\bar{N}$$

$$\boldsymbol{g} = \tilde{\boldsymbol{g}} + \tilde{w}^2\bar{\boldsymbol{g}} \qquad \tilde{w}^2 = \frac{\tilde{A}^{(\tilde{N})}}{\tilde{\mathcal{A}}^{(\tilde{N})}}$$

$$\tilde{\boldsymbol{g}} = \sum_{\tilde{\mu}}\left[\frac{\tilde{U}_{\tilde{\mu}}}{\tilde{X}_{\tilde{\mu}}}\boldsymbol{d}\tilde{x}_{\tilde{\mu}}^2 + \frac{\tilde{X}_{\tilde{\mu}}}{\tilde{U}_{\tilde{\mu}}}\Big(\sum_{\tilde{\nu}}\frac{\tilde{J}_{\tilde{\mu}}(\tilde{a}_{\tilde{\nu}}^2)}{\tilde{\mathcal{U}}_{\tilde{\nu}}}\boldsymbol{d}\phi_{\tilde{\nu}}\Big)^2\right]$$

$$\tilde{X}_{\tilde{\mu}} = \lambda\,\tilde{\mathcal{J}}(\tilde{x}_{\tilde{\mu}}^2) - 2\,\tilde{b}_{\tilde{\mu}}\,\tilde{x}_{\tilde{\mu}}^{1-2\tilde{N}}$$

off-shell Kerr–NUT–(A)dS
Lorentzian part

$$\bar{\boldsymbol{g}} = \sum_{\bar{\mu}}\left[\frac{\bar{U}_{\bar{\mu}}}{\bar{X}_{\bar{\mu}}}\boldsymbol{d}\bar{x}_{\bar{\mu}}^2 + \frac{\bar{X}_{\bar{\mu}}}{\bar{U}_{\bar{\mu}}}\Big(\sum_{\bar{\nu}}\frac{\bar{J}_{\bar{\mu}}(\bar{a}_{\bar{\nu}}^2)}{\bar{\mathcal{U}}_{\bar{\nu}}}\boldsymbol{d}\phi_{\bar{\nu}}\Big)^2\right]$$

$$\bar{X}_{\bar{\mu}} = \lambda\,\bar{\mathcal{J}}(\bar{x}_{\bar{\mu}}^2) - 2\,\bar{b}_{\bar{\mu}}\,\bar{x}_{\bar{\mu}}$$

on-shell Kerr–NUT–(A)dS
Euclidian instanton

Limit of vanishing rotations

— counting parameters

$$\boldsymbol{g} = \tilde{\boldsymbol{g}} + \tilde{w}^2 \bar{\boldsymbol{g}}$$

parameters $\bar{a}_{\bar{\mu}}$ survive the limit $a_{\bar{\mu}} \to 0$

$$\frac{1}{\varepsilon} a_{\tilde{\mu}} \to \bar{a}_{\bar{\mu}}$$

Limit of vanishing rotations

— counting parameters

original metric

$$2N-1$$

$$N = \tilde{N} + \bar{N}$$

regular sector

$\tilde{N}$ parameters $\tilde{a}_{\tilde{\mu}}$

$\tilde{N}$ parameters $\tilde{b}_{\tilde{\mu}}$

unspined sector

$\bar{N}$ parameters $\bar{a}_{\bar{\mu}}$

$\bar{N}$ parameters $\bar{b}_{\bar{\mu}}$

Limit of vanishing rotations

— counting parameters

original metric

$$2N-1$$

$$N=\tilde{N}+\bar{N}$$

regular sector

$\tilde{N}$ parameters $\tilde{a}_{\tilde{\mu}}$

$\tilde{N}$ parameters $\tilde{b}_{\tilde{\mu}}$

-1 scaling freedom

$$2\bar{N}-1$$

unspined sector

$\bar{N}$ parameters $\bar{a}_{\bar{\mu}}$

$\bar{N}$ parameters $\bar{b}_{\bar{\mu}}$

-1 scaling freedom

$$2\tilde{N}-1$$

$$2N-2$$

1 parameter less after the limit

Limit of vanishing rotations

— interpretation

$$\boldsymbol{g} = \tilde{\boldsymbol{g}} + \tilde{w}^2 \bar{\boldsymbol{g}} \qquad \tilde{w}^2 = \frac{\tilde{A}^{(\tilde{N})}}{\tilde{\mathcal{A}}^{(\tilde{N})}}$$

Lorentzian part

$$\tilde{\boldsymbol{g}} = \sum_{\tilde{\mu}} \left[\frac{\tilde{U}_{\tilde{\mu}}}{\tilde{X}_{\tilde{\mu}}} \boldsymbol{d}\tilde{x}_{\tilde{\mu}}^2 + \frac{\tilde{X}_{\tilde{\mu}}}{\tilde{U}_{\tilde{\mu}}} \Big(\sum_{\tilde{\nu}} \frac{\tilde{J}_{\tilde{\mu}}(\tilde{a}_{\tilde{\nu}}^2)}{\tilde{\mathcal{U}}_{\tilde{\nu}}} \boldsymbol{d}\phi_{\tilde{\nu}} \Big)^2 \right]$$

$$\tilde{X}_{\tilde{\mu}} = \lambda\, \tilde{\mathcal{J}}(\tilde{x}_{\tilde{\mu}}^2) - 2\, \tilde{b}_{\tilde{\mu}}\, \tilde{x}_{\tilde{\mu}}^{1-2\tilde{N}}$$

- $\tilde{b}_{\tilde{\mu}}$ mass and NUT charges
- $\tilde{a}_{\tilde{\mu}}$ rotations

Euclidian instanton

$$\bar{\boldsymbol{g}} = \sum_{\bar{\mu}} \left[\frac{\bar{U}_{\bar{\mu}}}{\bar{X}_{\bar{\mu}}} \boldsymbol{d}\bar{x}_{\bar{\mu}}^2 + \frac{\bar{X}_{\bar{\mu}}}{\bar{U}_{\bar{\mu}}} \Big(\sum_{\bar{\nu}} \frac{\bar{J}_{\bar{\mu}}(\bar{a}_{\bar{\nu}}^2)}{\bar{\mathcal{U}}_{\bar{\nu}}} \boldsymbol{d}\phi_{\bar{\nu}} \Big)^2 \right]$$

$$\bar{X}_{\bar{\mu}} = \lambda\, \bar{\mathcal{J}}(\bar{x}_{\bar{\mu}}^2) - 2\, \bar{b}_{\bar{\mu}}\, \bar{x}_{\bar{\mu}}$$

- $\bar{b}_{\bar{\mu}}$ deformation of the instanton
- $\bar{a}_{\bar{\mu}}$ twisting/intertwining of angles

Partially rotating deformed twisted black hole

Switching-off all rotations

regular sector	**unspined sector**
Lorentzian part	Euclidian instanton
$\tilde{N} = 1$	$\bar{N} = N - 1$

$$\boldsymbol{g} = -\big(1-\lambda r^2-2mr^{3-2N}\big)\boldsymbol{d}t^2 + \frac{1}{1-\lambda r^2-2mr^{3-2N}}\boldsymbol{d}r^2 + r^2\bar{\boldsymbol{g}}$$

$\bar{b}_{\bar{\mu}} = 0$ **Schwarzschild–Tangherlini–(A)dS black hole**

- $\bar{\boldsymbol{g}}$ sphere
- $\bar{a}_{\bar{\mu}}$ just a choice of coordinates

$\bar{b}_{\bar{\mu}} \neq 0$ **Non-rotating deformed twisted black hole**

- $\bar{\boldsymbol{g}}$ Euclidian instanton
- $\bar{b}_{\bar{\mu}}$ deformation parameters
- $\bar{a}_{\bar{\mu}}$ intertwining parameters

Deformed black hole

switching-off all rotations and twists by repeating the limiting procedure

$$\boldsymbol{g} = -f\,\boldsymbol{d}t^2 + \frac{1}{f}\boldsymbol{d}r^2 + r^2\Big[\boldsymbol{q}_{N-1} + \xi_{N-1}^2\Big(\boldsymbol{q}_{N-2} + \cdots + \xi_3^2\big(\boldsymbol{q}_2 + \xi_2^2\boldsymbol{q}_1\big)\Big)\Big]$$

$$f = 1 - \lambda r^2 - \frac{2m}{r^{2N-3}}$$

$$\boldsymbol{q}_{\bar\mu} = \frac{1}{\Delta_{\bar\mu}}\boldsymbol{d}\xi_{\bar\mu}^2 + \Delta_{\bar\mu}\,\boldsymbol{d}\phi_{\bar\mu}^2 \qquad\qquad \Delta_{\bar\mu} = 1 - \xi_{\bar\mu}^2 - 2\,\beta_{\bar\mu}\,\xi_{\bar\mu}^{-2\bar\mu+3}$$

- $\boldsymbol{q}_1$ is just a spherical geometry with a conical singularity
- $\boldsymbol{q}_{\bar\mu}$ with $\bar\mu > 1$ are nontrivial 2-dimensional geometries

Static deformed non-twisted black hole

Nutty spacetimes

- Kolář I., Krtouš P.: *Limits of Kerr-NUT-(A)dS spacetimes*, in preparation

Nutty spacetimes

— generalization of Taub-NUT-(A)dS

Limit near double roots of X_μ

$$X_\mu = \lambda\, \mathcal{J}(x_\mu^2) - 2\, b_\mu\, x_\mu$$

Polynomial $\mathcal{J}(x^2)$ can be parametrized by:

— by roots a_μ $\qquad \mathcal{J}(x^2) = \prod_\mu (a_\mu^2 - x^2)$

— by 'tangent points' $\hat{x}_\mu$ $\qquad \mathcal{J}(x^2) = \sum_{k=0}^{N} \frac{2N-1}{2N-2k-1} \hat{A}^{(k)} (-x^2)^{N-1-k}$

where $\hat{A}^{(k)} = \sum_{\substack{\nu_1,\dots,\nu_k \\ \nu_1<\dots<\nu_k}} \hat{x}_{\nu_1}^2 \dots \hat{x}_{\nu_k}^2$

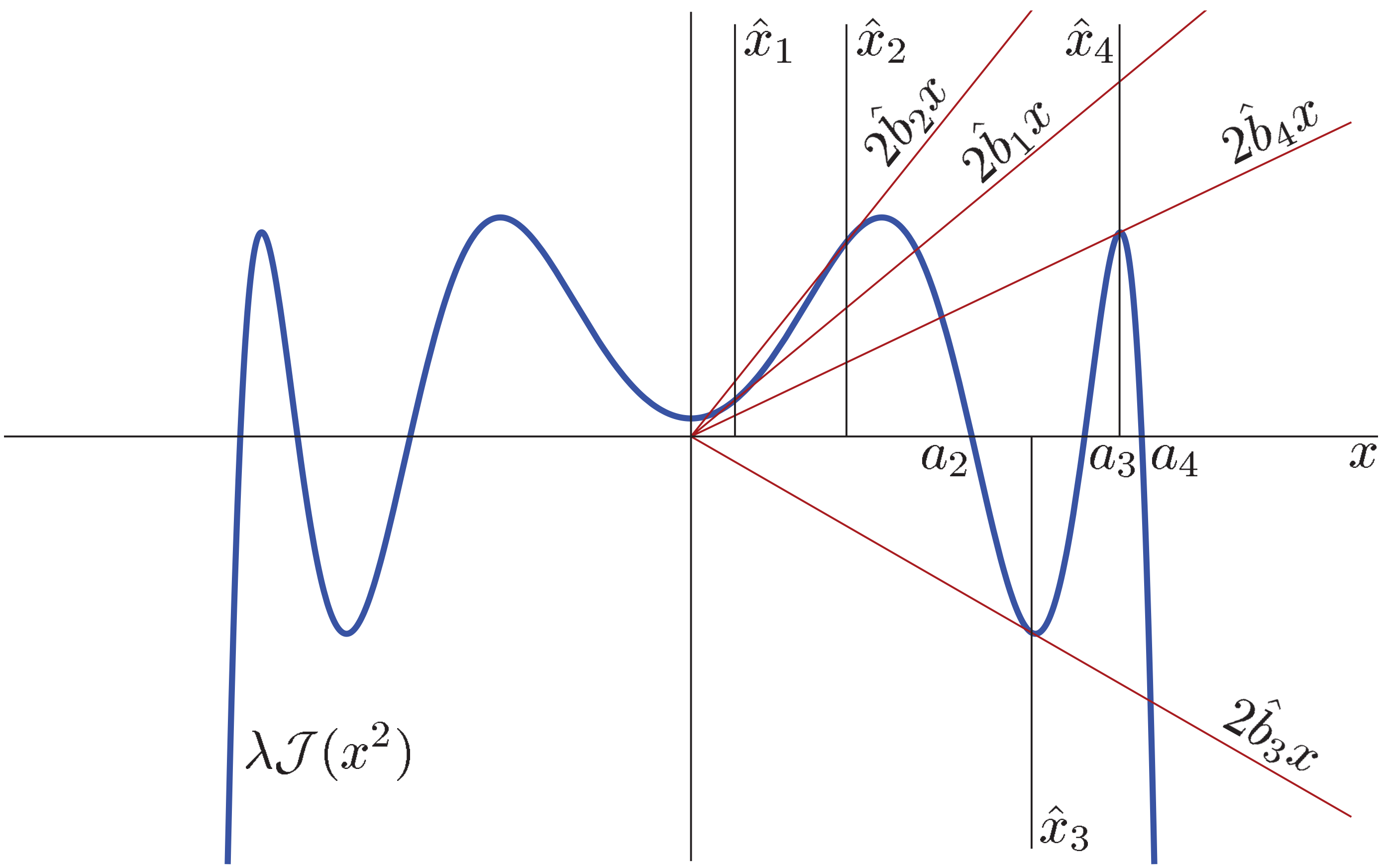
tangent points $\hat{x}_\mu$ are double roots of X_μ for suitable values $b_\mu = \hat{b}_\mu$

Nutty spacetimes

— generalization of Taub-NUT-(A)dS

tangent points $\hat{x}_\mu$ are double roots of X_μ for suitable values $b_\mu = \hat{b}_\mu$

$$X_\mu = \lambda \mathcal{J}(x_{\bar{\mu}}^2) - 2\hat{b}_{\bar{\mu}} x_{\bar{\mu}}$$



Nutty spacetimes

— generalization of Taub-NUT-(A)dS

N **roots** a_μ

gauge condition:

$$a_N = -\frac{1}{\lambda}$$

N **tangent points** $\hat{x}_\mu$

tangent point in Lorentzian sector:

$$\hat{r}^2 = \frac{1}{\lambda}\frac{\sum_k \frac{1}{2N-2k-1}\lambda^k \hat{\bar{A}}^{(k)}}{\sum_k \frac{1}{2N-2k-3}\lambda^k \hat{\bar{A}}^{(k)}}$$

$N-1$ **'nutty' parameters** $\hat{x}_{\bar{\mu}}$

(encode critical combination of rotations and NUT parameters)

Nutty spacetimes
— generalization of Taub-NUT-(A)dS

spacetime with near-critical NUTs
x-coordinate ranges near double roots $\hat{x}_{\bar{\mu}}$

$\mu = N$ – Lorentzian sector

$\bar{\mu} = 1, \ldots, N-1$ – spatial directions

non-critical value of mass:	$m \neq \hat{m}$
NUT charges near the critical values:	$b_{\bar{\mu}} = \hat{b}_{\bar{\mu}} + \mathcal{O}(\varepsilon^2)$
scaling coordinates near double roots:	$x_{\bar{\mu}} = \hat{x}_{\bar{\mu}} + \varepsilon\, \delta x_{\bar{\mu}}\, \xi_{\bar{\mu}}$
scaling angles:	$\psi_{\bar{k}+1} = \frac{1}{\varepsilon}\, \Psi_{\bar{k}}$
shifting time:	$t = \psi_0 + \frac{1}{\varepsilon} \sum_{\bar{k}} \mathring{\bar{A}}^{(\bar{k}+1)} \Psi_{\bar{k}}$
rewinding angles:	$\varphi_{\bar{\mu}} = \sum_{\bar{k}} \mathring{\bar{A}}^{(\bar{k})}_{\bar{\mu}}\, \Psi_{\bar{k}}$

Nutty spacetimes

— generalization of Taub-NUT-(A)dS

spacetime with near-critical NUTs

x-coordinate ranges near double roots $\hat{x}_{\bar\mu}$

non-critical value of mass:	$m \neq \hat{m}$
NUT charges near the critical values:	$b_{\bar\mu} = \hat{b}_{\bar\mu} + \mathcal{O}(\varepsilon^2)$
scaling coordinates near double roots:	$x_{\bar\mu} = \hat{x}_{\bar\mu} + \dfrac{\varepsilon}{\delta_{\bar\mu}}\,\xi_{\bar\mu}$
scaling angles:	$\psi_{\bar k+1} = \frac{1}{\varepsilon}\,\Psi_{\bar k}$
shifting time:	$t = \psi_0 + \frac{1}{\varepsilon}\,\sum_{\bar k} \mathring{\hat{A}}^{(\bar k+1)}\Psi_{\bar k}$
rewinding angles:	$\varphi_{\bar\mu} = \sum_{\bar k} \mathring{\hat{A}}^{(\bar k)}_{\bar\mu}\,\Psi_{\bar k}$

$$\boldsymbol{g} = -\frac{\Delta}{\Sigma}\Big(\boldsymbol{d}t + \sum_{\bar\mu} \frac{2\hat{x}_{\bar\mu}}{\delta_{\bar\mu}}\,\big(\xi_{\bar\mu} - \mathring{\xi}_{\bar\mu}\big)\boldsymbol{d}\varphi_{\bar\mu}\Big)^2 + \frac{\Sigma}{\Delta}\boldsymbol{d}r^2 + \sum_{\bar\mu}\frac{r^2 + \hat{x}_{\bar\mu}^2}{\delta_{\bar\mu}}\left(\frac{1}{1-\xi_{\bar\mu}^2}\,\boldsymbol{d}\xi_{\bar\mu}^2 + \big(1-\xi_{\bar\mu}^2\big)\boldsymbol{d}\varphi_{\bar\mu}^2\right)$$

$$\Delta = -\lambda\mathcal{J}(-r^2) - 2mr \qquad \Sigma = \prod_{\bar\nu}(r^2 + \hat{x}_{\bar\nu}^2) \qquad \delta_\mu = \lambda(2N-1)(\hat{r}^2 + \hat{x}_{\bar\mu}^2)$$

- Mann R. B., Stelea C.: *New multiply nutty spacetimes*, Phys.Lett. B634 (2006) 448

Nutty spacetimes

— generalization of Taub-NUT-(A)dS

spacetime with near-critical NUTs
x-coordinate ranges near double roots $\hat{x}_{\bar{\mu}}$

$$\boldsymbol{g} = -\frac{\Delta}{\Sigma}\Big(\boldsymbol{d}t + \sum_{\bar{\mu}} \frac{2\hat{x}_{\bar{\mu}}}{\delta_{\bar{\mu}}}\big(\xi_{\bar{\mu}} - \mathring{\xi}_{\bar{\mu}}\big)\boldsymbol{d}\varphi_{\bar{\mu}}\Big)^2 + \frac{\Sigma}{\Delta}\boldsymbol{d}r^2 + \sum_{\bar{\mu}} \frac{r^2 + \hat{x}_{\bar{\mu}}^2}{\delta_{\bar{\mu}}}\Big(\frac{1}{1-\xi_{\bar{\mu}}^2}\boldsymbol{d}\xi_{\bar{\mu}}^2 + \big(1-\xi_{\bar{\mu}}^2\big)\boldsymbol{d}\varphi_{\bar{\mu}}^2\Big)$$

$$\Delta = -\lambda\mathcal{J}(-r^2) - 2mr \qquad \Sigma = \prod_{\bar{\nu}}(r^2 + \hat{x}_{\bar{\nu}}^2) \qquad \delta_{\mu} = \lambda(2N-1)(\hat{r}^2 + \hat{x}_{\bar{\mu}}^2)$$

Parametrized by:

- mass m
- $N-1$ of new nutty parameters $\hat{x}_{\bar{\mu}}$
- $N-1$ of rewinding parameters $\mathring{\xi}_{\bar{\nu}}$
  — controlling regularity of semiaxes?

Summary

- Kerr–NUT–(A)dS describes wide family of geometries
- a full interpretation still open
- non-rotating limits leads to deformed and twisted black holes
- interpretation of spatial NUT parameters — deformation (?)
- interpretation of rewinding Killing angles — twisting (?)
- nutty spacetimes as double-root limits of Kerr–NUT–(A)dS
- analogous limit in Lorentzian sector

  — near horizon limits of Kerr–NUT–(A)dS