Reentrant phase transitions and van der Waals behaviour for hairy black holes

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Background: Black Hole Chemistry

Black Hole Thermodynamics

Thermodynamics		**Gravity**
Energy E	$\longleftrightarrow$	M Mass
Temperature T	$\longleftrightarrow$	$\hbar\kappa/2\pi$ Surface Gravity
Entropy S	$\longleftrightarrow$	$A/4\hbar$ Horizon Area
$dE = TdS - PdV + \text{work terms}$	$\longleftrightarrow$	$dM = (\kappa/8\pi)dA + \text{work terms}$

Where is the PdV term in the first law of black hole mechanics?

Extended Phase Space Thermodynamics

- Geometric and scaling arguments suggest that the cosmological constant, Λ, should be considered as a thermodynamic parameter in the first law [0904.2765].
- Mathematically it is found that,

$$dM = TdS + VdP + \sum_i \Omega_i dJ_i + \Phi dQ$$

where

$$P = -\frac{\Lambda}{8\pi} = \frac{(d-1)(d-2)}{16\pi\ell^2}$$

$\Rightarrow$ mass is enthalpy.

- Why? The first law and Smarr formula are related via Eulerian scaling

$$(d-3)M = (d-2)TS - 2VP + (d-2)\sum_i \Omega_i J_i + (d-3)\Phi Q$$

Black Hole Chemistry: Results

- Charged AdS black holes $\Rightarrow$ van der Waals analogy [1205.0559]:
  - liquid/gas transition $\longleftrightarrow$ small/large BH transition,
  - a single critical point,
  - in both cases, $P_c v_c / T_c = 3/8$,
  - law of corresponding states,
  - mean field theory critical exponents, $\alpha = 0$, $\beta = \frac{1}{2}$, $\gamma = 1$, $\delta = 3$.
- Higher d rotating black holes [1401.2586] and black holes in higher curvature gravity theories [1406.7015, 1505.05517] provide examples of triple points and (multiple)-reentrant phase transitions.
- Thermodynamically inspired black holes, e.g. the van der Waals black hole [1408.1105].
- An entropy inequality for the thermodynamic volume [1012.2888, 1411.4309].

Black hole chemistry with scalar hair

Hairy Black Holes

- For Einstein gravity conformally coupled to a scalar field in asymptotically flat spacetime there exist well-known no-hair theorems. "Black holes have no hair."
- In the presence of a cosmological constant, the no hair theorems can be evaded. Hairy black holes with conformal scalar hair are known to exist in $d = 3$ and $d = 4$ [hep-th/0205319, 0907.0219].

$$\Box\phi - \frac{d-2}{4(d-1)}R\phi = 0$$

- In $d > 4$ a special approach is needed $\Rightarrow$ couple the scalar field to the higher dimensional Euler densities using a rank 4 tensor which transforms appropriately [Oliva, Ray; 1112.4112].

An Aside: Lovelock Gravity

- The focus here will be on a special class of hairy black holes where the scalar field is conformally coupled to higher curvature terms.
- Higher curvature gravity modifies the standard Einstein-Hilbert action through the addition of higher curvature terms,

$$\mathcal{I} = \int d^d x \sqrt{g} \left(c_0 + c_1 R + c_2 \mathcal{L}(\mathcal{R}^2) + c_3 \mathcal{L}(\mathcal{R}^3) + \cdots \right)$$

- Generically, the field equations will no longer be second order differential equations.
- Lovelock gravity is the most general higher curvature theory of gravity which maintains second order field equations,

$$\mathcal{L}^{(k)} = \frac{2k!}{2^k} \delta^{a_1}_{[c_1} \delta^{b_1}_{d_1} \cdots \delta^{a_k}_{c_k} \delta^{b_k}_{d_k]} R_{a_1 b_1}{}^{c_1 d_1} \cdots R_{a_k b_k}{}^{c_k d_k}$$

- Appear in perturbative approaches to quantizing gravity.

Hairy Black Holes

- The tensor,

$$S_{\mu\nu}{}^{\gamma\delta} = \phi^2 R_{\mu\nu}{}^{\gamma\delta} - 12\delta^{[\gamma}_{[\mu}\delta^{\delta]}_{\nu]}\nabla_\rho\phi\nabla^\rho\phi - 48\phi\delta^{[\gamma}_{[\mu}\nabla_{\nu]}\nabla^{\delta]}\phi + 18\delta^{[\gamma}_{[\mu}\nabla_{\nu]}\phi\nabla^{\delta]}\phi$$

transforms as $S_{\mu\nu}{}^{\gamma\delta} \to \Omega^{-8/3} S_{\mu\nu}{}^{\gamma\delta}$ when $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$ and $\phi \to \Omega^{-1/3}\phi$.

- Can build terms where the scalar field is conformally coupled to the Euler densities [1112.4112, 1508.03780],

$$\mathcal{L}^{(k)}(\phi, \nabla\phi) = b_k \frac{2k!}{2^k} \phi^{3d-8k} \delta^{a_1}_{[c_1}\delta^{b_1}_{d_1} \cdots \delta^{a_k}_{c_k}\delta^{b_k}_{d_k]} S_{a_1 b_1}{}^{c_1 d_1} \cdots S_{a_k b_k}{}^{c_k d_k}$$

- Here [Hennigar, Mann; 1509.06798] we consider solutions of the theory,

$$\begin{aligned} \mathcal{I} &= \frac{1}{\kappa}\int d^5x\sqrt{-g}\Big[R - 2\Lambda - \frac{1}{4}F^2 + \kappa\Big(b_0\phi^{15} + b_1\phi^7 S_{\mu\nu}{}^{\mu\nu} \\ &\quad + b_2\phi^{-1}(S_{\mu\gamma}{}^{\mu\gamma}S_{\nu\delta}{}^{\nu\delta} - 4S_{\mu\gamma}{}^{\nu\gamma}S_{\nu\delta}{}^{\mu\delta} + S_{\mu\nu}{}^{\gamma\delta}S^{\nu\mu}{}_{\gamma\delta})\Big)\Big] \end{aligned}$$

Metric & Properties

- The metric is given by

$$ds^2 = -fdt^2 + \frac{dr^2}{f} + r^2 d\Sigma^2_{(k)3}\,, \quad f = k - \frac{m}{r^2} - \frac{q}{r^3} + \frac{e^2}{r^4} + \frac{r^2}{\ell^2}$$

with $k = -1, 0, 1$ while

$$q = \frac{64\pi}{5} k b_1 n^9\,, \quad n = \epsilon \left(-\frac{18 k b_1}{5 b_0}\right)^{1/6}\,, \quad \phi = \frac{n}{r^{1/3}}\,, \quad A = \frac{\sqrt{3} e}{r^2} dt\,, \quad F = dA\,,$$

here $\epsilon = -1, 0, 1$. The couplings have to obey the constraint $10 b_0 b_2 = 9 b_1^2$.

- Thermodynamic quantities which satisfy the first law & Smarr formula,

$$M = \frac{3\omega_{3(k)}}{16\pi} m\,, \quad Q = -\frac{\sqrt{3}\omega_{3(k)}}{16\pi} e\,, \quad S = \omega_{3(k)} \left(\frac{r_+^3}{4} - \frac{5}{8} q\right)\,, \quad V = \frac{\omega_{3(k)}}{4} r_+^4$$

$$T = \frac{1}{\pi \ell^2 r_+^4} \left[r_+^5 + \frac{k \ell^2 r_+^3}{2} + \frac{q \ell^2}{4} - \frac{e^2 \ell^2}{2 r_+}\right]\,, \quad \Phi = -\frac{2\sqrt{3}}{r_+^2} e$$

Equation of State and Gibbs Energy

- The equation of state is,

$$P = \frac{T}{v} - \frac{2k}{3\pi v^2} + \frac{512}{243}\frac{e^2}{\pi v^6} - \frac{64}{81}\frac{q}{\pi v^5}$$

  where the specific volume is $v = 4r_+/3$.

- The Gibbs free energy is,

$$\begin{aligned} G = M - TS &= \omega_{3(k)}\left[\frac{9kv^2}{256\pi} - \frac{27v^4P}{1024} + \frac{40q^2}{81\pi v^4} + \left(\frac{5Pv}{8} + \frac{5k-4}{12\pi v}\right)q \right. \\ &\left. + \left(\frac{5}{9\pi v^2} - \frac{320q}{243\pi v^5}\right)e^2\right] . \end{aligned}$$

Results: Overview

- There are three topologies, $k = -1, 0, 1$: we found only $k = 1$ to yield interesting results.
- For $k = 1$ there are two cases: $e = 0$ and $e \neq 0$.
- Uncharged case: van der Waals behaviour only for $q < 0$ so entropy is always positive.
- Charged case: van der Waals behaviour for $q > 0$ and reentrant phase transitions for $q < 0$. In the $q > 0$ case, the entropy expression can be negative—we regard these cases as unphysical.

$$S = \omega_{3(k)} \left(\frac{r_+^3}{4} - \frac{5}{8} q \right)$$

Results: Uncharged Case

- For $q < 0$ and $k = 1$ there is a single critical point,

$$T_c = -\frac{3}{20\pi}\frac{(-5q)^{2/3}}{q}, \quad v_c = \frac{4}{3}(-5q)^{1/3}, \quad P_c = \frac{9}{200\pi}\left(-\frac{\sqrt{5}}{q}\right)^{2/3}$$

with mean field theory critical exponents, $\alpha = 0$, $\beta = \frac{1}{2}$, $\gamma = 1$, $\delta = 3$, $P_c v_c / T_c = 2/5$.

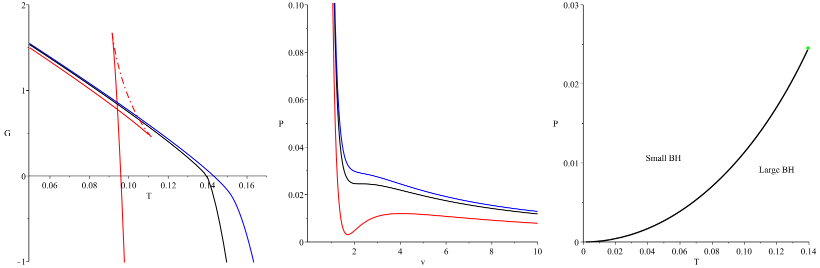

Figure: $q = -1$: red $\Rightarrow$ less than critical value, black $\Rightarrow$ critical value, blue $\Rightarrow$ greater than critical value.

Results: Charged Case I

- Recall $S = \omega_{3(k)} \left(\frac{r_+^3}{4} - \frac{5}{8} q \right) \Rightarrow$ possible for $S < 0$ when $q > 0$. We require positivity of entropy.

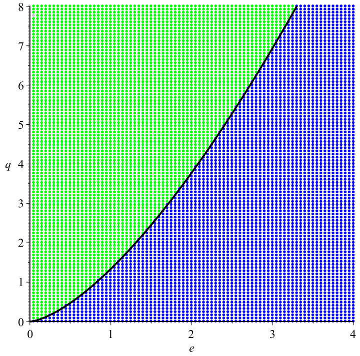

Figure: $k = 1$: q-e parameter space: blue dots indicate physical critical points, green dots indicate unphysical critical points (negative entropy). Black line: $q \approx 1.3375 e^{3/2}$.

Results: Charged Case II

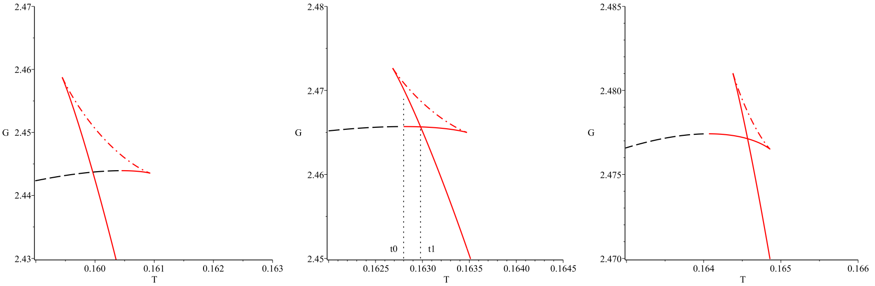

Figure: **Gibbs free energy**: $k = 1$, $e = q = 1$. Pressures $P = 0.031, 0.0313$, and 0.032 (left to right). The dashed black line corresponds to parameters yielding negative entropy. We see a large/small/large BH reentrant phase transition.

- Reentrant phase transition: A monotonic variation of a thermodynamic parameter yields two (or more) phase transitions, with the final state macroscopically similar to the initial state.

Results: Charged Case III

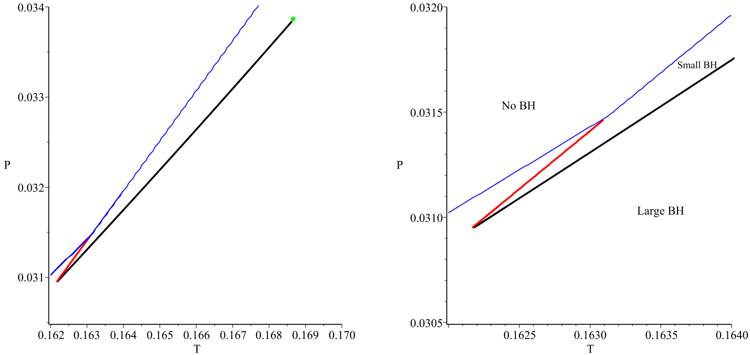

Figure: **Coexistence plots**: $k = 1$, $e = q = 1$. *Left*: $P - T$ coexistence plot showing zeroth and first order phase transitions (red and black curves, respectively). To the left of the blue line the black holes have negative entropy.

Results: Zero Entropy Case I

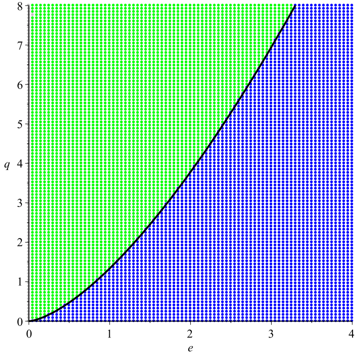

Figure: $k = 1$: q-e parameter space: blue dots indicate physical critical points, green dots indicate unphysical critical points (negative entropy). Black line: $q \approx 1.3375 e^{3/2}$.

Results: Zero Entropy Case II

- Phase transition occurs when the minimal branch of the Gibbs free energy changes.
- Expanding the equation of state near the critical point in terms of $\omega = v/v_c - 1$, $\rho = P/P_c - 1$ and $\tau = T/T_c - 1$ gives

$$\rho \approx A\tau - B\omega\tau - C\omega^3$$

with $A, B, C > 0$. Solving for ω there are three real solutions:

$$\omega_1 = \frac{2}{3}\sqrt{\frac{-3B\tau}{C}} \quad \omega_2 = \omega_3 = -\frac{1}{3}\sqrt{-\frac{3B\tau}{C}}$$

- Then the entropy is,

$$q = \frac{27}{160} v_c^3 \Rightarrow S_i = \frac{27\pi^2 v_c^3}{128}\left((\omega_i + 1)^3 - 1\right)$$

- Only one branch has positive entropy, so there are no phase transitions.

Conclusions

- We considered, for the first time, the extended phase space thermodynamics of black holes with conformal scalar hair.
- In the hyperbolic and flat cases, there were no interesting results.
- In the spherical case, without charge the solution exhibits van der Waals behaviour
- In the charged case, positivity of entropy results in reentrant phase transitions.
- In the case of zero entropy, the criticality ceases.