A Family of Graphs on the Cantor set

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Graph and graph colorings

A *graph* $\mathcal{G} = (X, R)$, is a pair where X is a set and R is a binary irreflexive and symmetric binary relation on X.

A coloring of a graph $\mathcal{G} = (X, R)$ is a function $c : X \to K$ such that $c(x) \neq c(y)$ if $\{x, y\} \in R$;

(it is a k-coloring if the cardinality of K is k).

The chromatic number of $\mathcal{G}$, denoted by $\chi(\mathcal{G})$, is the least k such that there is a k-coloring of $\mathcal{G}$.

Let X be a standard Borel space, e.g. a complete metrizable separable space together with its Borel structure, and let $R \subseteq X^2$ be irreflexive and symmetric.

If R is a Borel (or analytic) subset of X^2, we say that the graph (X, R) is a Borel (or analytic) graph.

We work mostly with $X = 2^{\mathbb{N}}$ or $X \subseteq [\omega]^\omega$,

Definition

The Borel chromatic number of $\mathcal{G}$, $\chi_B(\mathcal{G})$, is the least $k \leq \aleph_0$ such that there is a Borel measurable coloring $c : X \to k$ of $\mathcal{G}$ (giving k the discrete topology).

If such a Borel coloring does not exist, then it is said that $\mathcal{G}$ has uncountable Borel chromatic number, expressed by $\chi_B(\mathcal{G}) > \aleph_0$.

We say that a graph $\mathcal{G}$ is infinitely chromatic if its Borel chromatic number is infinite.

Shifts and their graphs

The shift function $S : [\omega]^\omega \to [\omega]^\omega$ is defined by $S(X) = X \setminus \{min(X)\}$.

The shift graph on $[\omega]^\omega$ is the graph obtained by $\{X, Y\}$ is an edge if $Y = S(X)$ or $X = S(Y)$.

The graph $([\omega]^\omega, S)$ has chromatic number 2, while its Borel chromatic number is $\aleph_0$.

The corresponding graph $\mathcal{G}_S$ on $2^{\mathbb{N}}$, obtained identifying infinite subsets of $\mathbb{N}$ with their characteristic functions, is given by the shift function on $2^{\mathbb{N}}$, namely,

$S : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ defined as follows: for $x \in 2^{\mathbb{N}}$, $S(x)(i) = 0$ if i is the first natural number such that $x(i) = 1$, and $S(x)(i) = x(i)$ for every other i.

The graph $\mathcal{G}_S = (2^{\mathbb{N}}, S)$ is then defined by the edges $\{x, y\} \subseteq 2^{\mathbb{N}}$ such that $y = S(x)$ or $x = S(y)$.

The restriction of this graph to the elements of the Cantor space that are characteristic functions of infinite subsets of $\mathbb{N}$, is isomorphic to the shift graph on $[\omega]^{\omega}$.

The study of Borel chromatic numbers was initiated in Kechris, Solecki and Todorcevic, Borel chromatic numbers, Advances in Math. 1999;

Di Prisco, Todorcevic, Basis problems for Borel graphs, Zbornik Radova Serbian Academy of Sc., 2015, analyzes graphs defined on Borel subsets of $[\omega]^\omega$ and their Borel chromatic numbers.

Other aspects of chromatic numbers of graphs defined on Polish spaces have been studied, for example see Miller, B., Measurable chromatic numbers, JSL, 2008.

Definition

Given two graphs $G = (X, R)$ and $G' = (X', R')$, a graph-homomorphism from G into G', is a map $f : X \to X'$ such that $xRy \Rightarrow f(x)R'f(y)$.

A graph-embedding or simply an embedding is an injective homomorphism, in other words, an isomorphism of G with a subgraph of G'.

If $\mathcal{C}$ is a class of functions, $G \leq_{\mathcal{C}} G'$ expresses that there is a homomorphism of G into G' which belongs to $\mathcal{C}$; and $G \sqsubseteq_{\mathcal{C}} G'$ if there is an embedding of G into G' which is in $\mathcal{C}$.

For two graphs $\mathcal{G} = (X, R)$ and $\mathcal{H} = (Y, S)$,
a **Borel homomorphism** from $\mathcal{G} = (X, R)$ into $\mathcal{H} = (Y, S)$ is a Borel mapping $f : X \to Y$ such that $(x, y) \in R$ implies $(f(x), f(y)) \in S$.

Kechris, Solecky and Todorcevic present an interesting dichotomy which explains when an analytic graph has uncountable Borel chromatic number.
They show the existence of a graph $\mathcal{G}_0$ such that $\chi_{\mathcal{B}}(\mathcal{G}_0) > \aleph_0$ and $\mathcal{G}_0$ is the minimal analytic graph with uncountable Borel chromatic number in the following sense.

Theorem

(KST; 6.3) Let X be a Polish space and $\mathcal{G} = (X, R)$ an analytic graph. Then exactly one of the following holds.

1. $\chi_{\mathcal{B}}(\mathcal{G}) \leq \aleph_0$,
2. *there is a continuous homomorphism from $\mathcal{G}_0$ into $\mathcal{G}$.*

$\mathcal{G}_0$ will be described below.

The original proof uses a Baire category argument and effective descriptive set theory. B. Miller has found a proof reminiscent of Cantor's proof of the perfect set theorem for closed sets using derivatives.

This dichotomy has been carefully studied in several subsequent papers (notably by B. Miller) and it has been used to explain other dichotomies in the context of descriptive set theory. For example, the $\mathcal{G}_0$-dichotomy gives a natural short proof of Silver's dichotomy theorem for co-analytic equivalence relations.

An open problem.

KST asked for the existence of a graph characterizing infinite Borel chromaticity, and in particular they asked if the following dichotomy is true: If $\mathcal{G} = (X, R)$ is an analytic graph on a Polish space X, then exactly one of the following holds:

1. $\chi_{\mathcal{B}}(\mathcal{G}) < \aleph_0$,
2. there is a continuous homomorphism of $\mathcal{G}_S$ into $\mathcal{G}$.

It is shown in KST that when the graph is of the form (X, F), defined by a $\leq \aleph_0$-to-1 Borel function $F : X \to X$, the problem can be reduced to understanding when $\chi_{\mathcal{B}}(\mathcal{A}, S)$ is infinite, for the graph defined by the shift S on a Borel $\mathcal{A} \subseteq [\omega]^\omega$.

Theorem

(KST , Theorem 5.1) Let $F : X \to X$ be a Borel mapping defined on a Borel space X. Then, the Borel chromatic number of the graph (X, F) belongs to the set $\{1, 2, 3, \aleph_0\}$.

In KST it is asked if any Borel set with infinite Borel chromatic number contains a set of the form $[x]^\infty$.

Proposition

There is a Borel subset $\mathcal{A}$ of $[\omega]^\omega$ such that $\chi_{\mathcal{B}}(\mathcal{A}, S)$ is infinite but $\mathcal{A}$ contains no set of the form $[x]^\infty$.

(For $x \subseteq \omega$ infinite, $[x]^\infty$ denotes the collection of all infinite subsets of x)

Conjecture:
If $\mathcal{A} \subseteq [\omega]^\omega$ is a Borel set, the graph $(\mathcal{A}, S)$ has infinite Borel chromatic number if and only if there is a continuous homomorphism

$$h : \mathcal{G}_S \to (\mathcal{A}, S).$$

For closed sets some interesting facts can be shown. For example,

Theorem

Let $C \subseteq [\omega]^\omega$ be closed and infinitely chromatic. Then there is a tree $T \subseteq [\mathbb{N}]^{<\infty}$ such that $[T] \subseteq C$, $[T]$ is infinitely chromatic, and every node of T has infinitely many immediate successors in T.

In other words, every closed infinitely chromatic set is strongly dominating.

Graphs on $2^{\mathbb{N}}$ defined by finite binary sequences.

Consider graphs on $2^{\mathbb{N}}$ determined by a family $\mathcal{F} = \{t_i\}_{i=0}^{\infty}$ of finite sequences of 0's and 1's with the property that if $i < j$, then $|t_i| < |t_j|$.

Given such a family $\mathcal{F} = \{t_i\}$, define the graph $G_{\mathcal{F}} = (2^{\mathbb{N}}, R_{\{t_i\}})$, or simply $(2^{\mathbb{N}}, \{t_i\})$, as follows:

Given $x, y \in 2^{\mathbb{N}}$, put $\{x, y\} \in R_{\{t_i\}}$ if and only if there is i such that

(a) $x \restriction i = y \restriction i \in \mathcal{F}$,

(b) $x(i) \neq y(i)$,

(c) $x(j) = y(j)$ for every $j > i$.

The condition $|t_i| \neq |t_j|$ for $i \neq j$ on the family $\{t_i\}$ gives that all these graphs are acyclic so, in particular, they are 2-chromatic.

We are interested in how the Borel chromatic number of these graphs depends on properties of the family $\mathcal{F}$.

Definition

For a family $\mathcal{F} = \{t_i\}$ as above, let

$$\mathcal{A}_{\mathcal{F}} = \{x \in 2^{\mathbb{N}} : \exists^{\infty} i \, (t_i \sqsubset x)\}.$$

We will see, in particular, how some properties of the graph depend on the set $\mathcal{A}_{\mathcal{F}}$.

The graph $\mathcal{G}_0$ of KST is defined using a family $\mathcal{F} = \{t_i\}_i$ which satisfies:

(a) for every i, t_i is of length i

(b) $\{t_i\}$ is dense, that is, for every $s \in 2^{<\omega}$ there is some i such that t_i extends s.

This graph characterizes uncountable Borel chromaticity for analytic graphs in the sense that

- it has uncountable Borel chromatic number (i.e. there is no coloring $c : 2^{\mathbb{N}} \to \aleph_0$ of this graph), and
- for any analytic graph $\mathcal{G} = (X, R)$ defined on a Polish space X (with $R \subseteq X^2$ analytic), the Borel chromatic number of $\mathcal{G}$ is $\leq \aleph_0$, or there is a continuous homomorphism of $\mathcal{G}_0$ into $\mathcal{G}$.

Notice that for such a family, $\mathcal{A}_{\{t_i\}}$ is a dense G_δ subset of $2^{\mathbb{N}}$. In particular, it is uncountable.

The shift graph $\mathcal{G}_S$ defined on $2^{\mathbb{N}}$ can be obtained from a family of finite sequences:
Let $t_i = 0^i$ for all i, then $G_{\{0^i\}} = (2^{\mathbb{N}}, \{0^i\})$ restricted to the collection of characteristic functions of elements of $[\omega]^\omega$ is the shift graph $\mathcal{G}_S$.

Theorem

Let $\mathcal{F} = \{t_i\}_{i=0}^{\infty}$ be such that $\mathcal{A}_{\mathcal{F}} = \emptyset$, then $G_{\mathcal{F}}$ is Borel 2-chromatic.

In this case, there is a natural (trivial) Borel homomorphism of $G_{\mathcal{F}}$ into the shift graph $G_{\{0^i\}}$.

Moreover, there is a continuous 2-coloring of the graph $G_{\mathcal{F}}$, and a continuous homomorphism from the graph $G_{\mathcal{F}}$ into the shift graph $G_{\{0^i\}}$.

The proof goes by induction on the height of the tree formed by the family $\mathcal{F}$.

Non well founded trees

Consider now families $\{t_i\}$ with infinite $\sqsubseteq$-chains, i.e. families for which the set $\mathcal{A}_{\{t_i\}}$ is non-empty.

Theorem

If there is $a \in 2^{\mathbb{N}}$ such that $t_i = a \restriction i$ for every i, then the graph $G_{\{t_i\}}$ is isomorphic to the shift graph $G_{\{0^i\}}$.

The proof actually gives that if $a \in \mathcal{A}_\mathcal{F}$, is such that

- For every i, $a \restriction i \in \mathcal{F}$, then $G_{\{0^i\}} \sqsubset_c G_{\{t_i\}}$.
- For every i, $t_i \sqsubset a$, then $G_{\{t_i\}} \sqsubset_c G_{\{0^i\}}$.

Proposition

If for the family $\mathcal{F} = \{t_i\}_{i=0}^{\infty}$ the set $\mathcal{A}_{\mathcal{F}}$ is countable, then the corresponding graph $G_{\mathcal{F}}$ has countable Borel chromatic number.

Proposition

Let $\mathcal{F} = \{t_i\}$ be such that $\mathcal{A}_{\mathcal{F}} \neq \emptyset$. Then, the shift graph $G_{\{0^i\}}$ can be embedded into $G_{\mathcal{F}}$.

Proof.

Let $a \in 2^{\mathbb{N}}$ be such that the set $\{i \in \omega : t_i \sqsubset a\}$ is infinite. Let $n_0 < n_1 < \cdots < n_j < \ldots$ be such that for each j, $a \upharpoonright n_j$ is an element in the family $\{t_i\}$. Define $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ by

$$f(x)(n) = \begin{cases} a(n) & \text{if } n \neq n_i \text{ for every } i \\ a(n) + x(i) & \text{if } n = n_i. \end{cases}$$

In other words, we "copy" x in the positions of a given by the sequence $\{n_i\}_i$. □

Proof.

Cont.

Clearly, the function f is 1-1. To see that it is a homomorphism, given $x \in 2^{\mathbb{N}}$, and let j be the first i such that $x(i) = 1$, then

$$f(S(x)) \restriction n_j = f(x) \restriction n_j,$$

$$f(S(x))(n_j) \neq f(x)(n_j),$$

and for every $l > n_j$,

$$f(S(x))(l) = f(x)(l);$$

thus, $f(S(x))$ and $f(x)$ form an edge of $G_{\{t_i\}}$. □

Corollary

Let $\mathcal{G}$ be a graph on $2^{\mathbb{N}}$ determined by a family $\mathcal{F} = \{t_i\}_{i=0}^{\infty}$ of finite sequences of 0's and 1's with the property that if $i < j$, then $|t_i| < |t_j|$. Then The Borel chromatic number of $\mathcal{G}$ is either 2 or infinite.

Question

Is it true that for every $\mathcal{F} = \{t_i\}$ such that $\mathcal{A}_{\mathcal{F}} = \{a\}$ the graph $G_{\mathcal{F}}$ can be embedded in the shift graph $G_{\{0^i\}}$?

In some particular cases this is so.

We examine now continuous homomorphisms.

Proposition

Let $\mathcal{F} = \{t_i\} \subseteq 2^{<\omega}$ be such that $i < j$ implies that $|t_i| < |t_j|$. Suppose $h : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be a continuous homomorphic embedding of $G_{\mathcal{F}}$ into $G_{\{0^i\}}$. Then, for every $x \in \mathcal{A}_{\mathcal{F}}$, $h(x)$ must be constantly equal to 0.

Corollary

If the family $\mathcal{F} = \{t_i\}$ is such that there are $x, y \in \mathcal{A}_{\mathcal{F}}$ which form an edge, then there is no continuous homomorphism of $G_{\mathcal{F}}$ into $G_{\{0^i\}}$.

Corollary

Let $\mathcal{F} = \{t_i\} \subseteq 2^{<\omega}$ be an **antichain**, *satisfying that $i < j$ implies $|t_i| < |t_j|$.*

(a) *If $h : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a continuous homomorphism of $G_{\mathcal{F}}$ into $G_{\{0^i\}}$, then for every $x \in \overline{\cup_i[t_i]} \setminus \cup_i[t_i]$, $h(x)$ is the constant 0 sequence.*

(b) *There is a continuous homomorphism of $G_{\mathcal{F}}$ into $G_{\{0^i\}}$.*

Question

More generally,

Can every countably chromatic graph of the form $G_{\mathcal{F}}$ be homomorphically embedded in $G_{\{0^i\}}$?

In some cases there are homomorphic embeddings into $G_{\{0^i\}}$ but no continuous homomorphic embeddings.

Proposition

Suppose $\mathcal{A}_{\mathcal{F}} = \{a, b\}$, a and b form an edge, and for every i, $t_i \sqsubset a$ or $t_i \sqsubset b$. Then there is a Borel homomorphic embedding (necessarily not continuous) of $G_{\mathcal{F}}$ into the shift graph $G_{\{0^i\}}$.

Thank you