Bases of homogeneous families below the first Mahlo cardinal

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Set Theory and its Applications in Topology

Introduction

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- κ *is not* ω*-Erdös;*
- *there is a hereditary, compact and large family* $\mathcal{F}$ *on* κ*;*
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For a whole separable reflexive space with no subsymmetric basic sequences (Tsirelson space), finite powers of the Schreier family were used.

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A Schreier sequence is defined inductively for $\alpha < \omega_1$ by

(a) $\mathcal{S}_0 := [\omega]^{\leq 1}$,
(b) $\mathcal{S}_{\alpha+1} := \mathcal{S}_\alpha \otimes \mathcal{S}$,
(c) $\mathcal{S}_\alpha := \bigcup_{n<\omega} (\mathcal{S}_{\alpha_n} \upharpoonright \omega \setminus n)$ where $(\alpha_n)_n$ is such that $\sup_n \alpha_n = \alpha$, if α is limit;

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where, given $\mathcal{F}, \mathcal{G} \subseteq [\omega]^{<\omega}$,

$$\mathcal{F} \otimes \mathcal{G} = \{\bigcup_{i=1}^{n} s_i : s_1 < \cdots < s_n \text{ in } \mathcal{F} \text{ and } \{\min s_i : 1 \leq i \leq n\} \in \mathcal{G}\}.$$

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Corollary (B., Lopez-Abad, Todorcevic)

For every cardinal κ below the first Mahlo cardinal, there is a reflexive Banach space of density κ with no subsymmetric basic sequences.

Basic definitions

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Given $\alpha < \omega_1$, $\mathcal{F}$ is α-homogeneous if $\alpha = \mathrm{srk}(\mathcal{F}) \leq \mathrm{rk}(\mathcal{F}) < \iota(\alpha)$, where

$$\mathrm{srk}(\mathcal{F}) := \inf\{\mathrm{rk}(\mathcal{F} \upharpoonright C) : C \text{ is an infinite subset of } \kappa\},$$

$$\iota(\alpha) = \min\{\text{exponentially indecomposible ordinal larger than } \alpha\}.$$

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If $\mathcal{F}$ is homogeneous on κ and $\mathcal{H}$ is homogeneous on ω, a family $\mathcal{G}$ on κ is a multiplication of $\mathcal{F}$ by $\mathcal{H}$ when

- *$\mathcal{G}$ is homogeneous and $\iota(\mathrm{srk}(\mathcal{G})) = \iota(\mathrm{srk}(\mathcal{F}) \cdot \mathrm{srk}(\mathcal{H}))$.*
- *Every sequence $(s_n)_{n<\omega}$ in $\mathcal{F}$ has an infinite subsequence $(t_n)_n$ such that for every $x \in \mathcal{H}$ one has that $\bigcup_{n\in x} t_n \in \mathcal{G}$.*

Definition

A basis on κ is a pair $(\mathfrak{B}, \times)$ such that:

- $\mathfrak{B}$ is a collection of homogeneous families on κ containing all cubes and for all $\omega \leq \alpha < \omega_1$, there is a α-homogeneous family on κ in $\mathfrak{B}$.
- $\mathfrak{B}$ is closed under $\cup$ and $\sqcup$.
- $\times : \mathfrak{B} \times \mathfrak{S} \to \mathfrak{B}$ is such that for every $\mathcal{F} \in \mathfrak{B}$ and every $\mathcal{H} \in \mathfrak{S}$ one has that $\mathcal{F} \times \mathcal{H}$ is a multiplication of $\mathcal{F}$ by $\mathcal{H}$.

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Remark: Schreier families are spreading and uniform, so that, in particular, any element of the basis can be multiplied (within the basis) by a Schreier family.

Example Given $\mathcal{F}$ on ω, let $\langle \mathcal{F} \rangle_{\mathrm{spr}}$ be the set of all $\{n_1 < \cdots < n_k\}$ such that there is $\{m_1 < \cdots < m_k\} \in \mathcal{F}$ such that $m_i \le n_i$.

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together with $\times : \mathfrak{B} \times \mathfrak{S} \to \mathfrak{B}$ defined by

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$$\mathcal{F} \oplus \mathcal{G} = \{s \cup t : s < t,\ s \in \mathcal{G},\ t \in \mathcal{F}\}.$$

$$\mathcal{F} \otimes \mathcal{G} = \{\bigcup_{k<n} s_k : n \in \omega,\ s_k < s_{k+1},\ s_k \in \mathcal{F},\ \{\min s_k : k < n\} \in \mathcal{G}\}.$$

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A basis of families on $\mathcal{P}$ is defined analagously.

Families and bases on trees

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Let T be a tree and given families $\mathcal{A}$ and $\mathcal{C}$ on T, let $\mathcal{A} \odot_T \mathcal{C}$ be the family on T of all $s \subseteq T$ such that:

- $\langle s \rangle \cap Ch_a \subseteq \mathcal{A}$, that is, for every $t \in T$, the set of immediate successors of t with respect to s belongs to $\mathcal{A}$;
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Proposition

If $\mathcal{A}$ and $\mathcal{C}$ are homogeneous families on $(T, <_a)$ and $(T, <_c)$, respectively, then $\mathcal{A} \odot_T \mathcal{C}$ is a homogeneous family on T.

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$$\mathrm{srk}(\mathcal{F}) = \inf\{\mathrm{rk}(\mathcal{F} \upharpoonright X) : X \text{ is an infinite chain, comb or fan}\}.$$

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Lemma

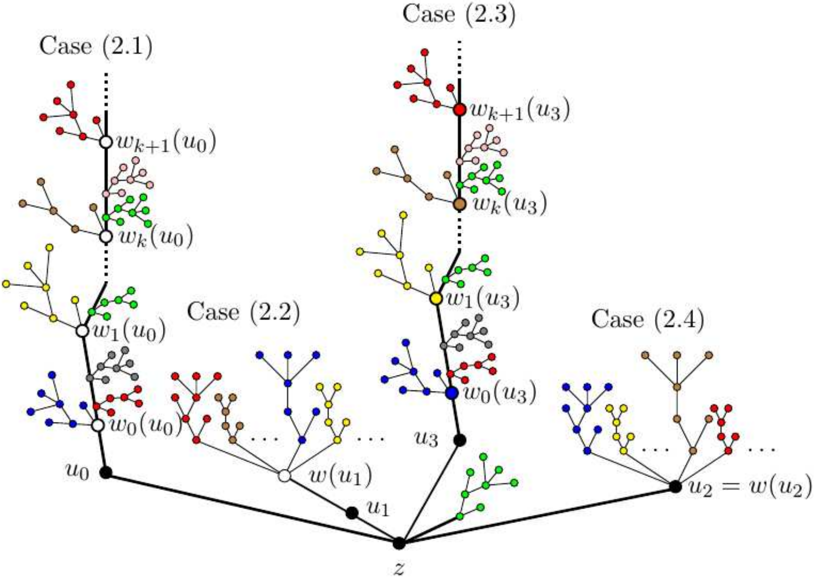
If $\mathcal{B}^a$ and $\mathcal{B}^c$ are bases on $(T, <_a)$ and $(T, <_c)$, respectively, let $\mathfrak{B}$ be the collection of all homogeneous families $\mathcal{F}$ on T such that

- *$\langle \mathcal{F} \rangle$ is homogeneous and $\mathrm{rk}(\langle \mathcal{F} \rangle) < \iota(\mathrm{rk}(\mathcal{F}))$;*
- *$\mathcal{A} := \langle \mathcal{F} \rangle \cap Ch_a \in \mathfrak{B}^a$ and $\mathcal{C} := \langle \mathcal{F} \rangle \cap Ch_c \in \mathfrak{B}^c$.*

Given $\mathcal{F} \in \mathfrak{B}$ and a hereditary, spreading, uniform family $\mathcal{H}$ on ω, then

$$\mathcal{F} \times \mathcal{H} = ((\mathcal{A} \times_a \mathcal{H}) \sqcup_a [T]^{\leq 1}) \odot_T ((\mathcal{C} \times_c \mathcal{H}) \boxtimes_c 5)$$

is a multiplication such that $\mathfrak{B}$ is a basis on T.



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Given a basis $\mathfrak{B}$ on κ, the collection of families of the form

$$\mathcal{G} = \{s \subset T : s \text{ is a chain and } ht''s \in \mathcal{F}\}$$

for some $\mathcal{F} \in \mathfrak{B}$ (with some suitable multiplication) is a basis on $(T, <_c)$.

Definition

$(C_\alpha)_{\alpha<\kappa}$ is a small C-sequence on κ if

- each C_α is a club in α with $otp(C_\alpha) = cof(\alpha)$;
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$$\rho_0(\alpha, \beta) := (C_\beta \cap \alpha)^\frown \rho_0(\alpha, \min(C_\beta \setminus \alpha))$$
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Corollary

Every cardinal below the first Mahlo cardinal has a basis.

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- Can we get better bounds?