Bases and selectors for cofinal families

Carlos Uzcátegui Aylwin
Escuela de Matemáticas
Universidad Industrial de Santander
Bucaramanga, Colombia

Oaxaca, september 2016

Bases for cofinal families

$\mathbb{N}^{[\infty]}$ the collection of infinite subsets of $\mathbb{N}$ as subspace of $\{0,1\}^{\mathbb{N}}$.

A collection $\mathcal{C} \subseteq \mathbb{N}^{[\infty]}$ is cofinal, if for all $A \in \mathbb{N}^{[\infty]}$ there is $B \subseteq A$ such that $B \in \mathcal{C}$.

A base for $\mathcal{C}$ is a family $\mathcal{B} \subseteq \mathcal{C}$ which is cofinal in $\mathcal{C}$, i.e., for all $A \in \mathcal{C}$, there is $B \in \mathcal{B}$ such that $B \subseteq A$.

Bases for cofinal families

$\mathbb{N}^{[\infty]}$ the collection of infinite subsets of $\mathbb{N}$ as subspace of $\{0,1\}^{\mathbb{N}}$.

A collection $\mathcal{C} \subseteq \mathbb{N}^{[\infty]}$ is cofinal, if for all $A \in \mathbb{N}^{[\infty]}$ there is $B \subseteq A$ such that $B \in \mathcal{C}$.

A base for $\mathcal{C}$ is a family $\mathcal{B} \subseteq \mathcal{C}$ which is cofinal in $\mathcal{C}$, i.e., for all $A \in \mathcal{C}$, there is $B \in \mathcal{B}$ such that $B \subseteq A$.

Questions:

(1) $\mathcal{C}$ analytic or co-analytic cofinal. Does $\mathcal{C}$ has a Borel base?

(2) Which cofinal families admit a (topologically) closed base?

(3) How "simple" can a base be? Simple = of a canonical form.

(4) Existence of Borel selectors for cofinal families.

An example

Let $\mathbb{N} = \bigcup_n K_n$ be a partition of $\mathbb{N}$ with each K_n finite.

$$\mathcal{C}(K_n)_n = \{A \in \mathbb{N}^{[\infty]} : \ |A \cap K_n| \leq 1 \ \text{ for all } n\}.$$

An example

Let $\mathbb{N} = \bigcup_n K_n$ be a partition of $\mathbb{N}$ with each K_n finite.

$$\mathcal{C}(K_n)_n = \{A \in \mathbb{N}^{[\infty]} : \ |A \cap K_n| \leq 1 \ \text{ for all } n\}.$$

$$\mathcal{I} = \{A \in \mathbb{N}^{[\infty]} : \sum_{n \in A} 1/n < \infty\}.$$

An example

Let $\mathbb{N} = \bigcup_n K_n$ be a partition of $\mathbb{N}$ with each K_n finite.

$$\mathcal{C}(K_n)_n = \{A \in \mathbb{N}^{[\infty]} : \ |A \cap K_n| \leq 1 \ \text{ for all } n\}.$$

$$\mathcal{I} = \{A \in \mathbb{N}^{[\infty]} : \sum_{n \in A} 1/n < \infty\}.$$

Let $K_n = \{m \in \mathbb{N} : \ n^2 \leq m < (n+1)^2\}$, then

$$\mathcal{C}(K_n)_n \subseteq \mathcal{I}.$$

Convergent sequences in sequentially compact spaces

X a Polish space. $\mathcal{B}_1(X)$= Real valued function on X of the first Baire class. $\mathcal{B}_1(X)$ as a subspace of $\mathbb{R}^X$.

K is a Rosenthal compact, if it is homeormorphic to a compact subset of $\mathcal{B}_1(X)$. Every Rosenthal compact is sequentially compact. Let $(f_n)_n$ be a sequence of $K \subseteq \mathcal{B}_1(X)$.

$$\mathcal{C}(f_n)_n = \{A \in \mathbb{N}^{[\infty]} : (f_n)_{n\in A} \text{ is pointwise convergent}\}.$$

As K is sequentially compact, then $\mathcal{C}(f_n)_n$ is cofinal.

Theorem (P. Dodos, 2006 based on a work of G. Debs)

(i) $\mathcal{C}(f_n)_n$ is co-analytic. If K is not first countable, then $\mathcal{C}(f_n)_n$ is not Borel.

(ii) $\mathcal{C}(f_n)_n$ has a Borel base.

Homogeneous sets for colorings

Ramsey's Theorem: Let $\varphi : \mathbb{N}^{[2]} \to 2$.

$$hom(\varphi) = \{H \in \mathbb{N}^{[\infty]} : \ H \text{ is } \varphi\text{-homogeneous}\}$$

is a closed cofinal family.

Homogeneous sets for colorings

Ramsey's Theorem: Let $\varphi : \mathbb{N}^{[2]} \to 2$.

$$hom(\varphi) = \{H \in \mathbb{N}^{[\infty]} : \ H \text{ is } \varphi\text{-homogeneous}\}$$

is a closed cofinal family.

Theorem: The following families admit a base of the form $hom(\varphi)$ for a coloring $\varphi : \mathbb{N}^{[2]} \to 2$.

(1) Tall p-ideals.

(2) $\{A \in \mathbb{N}^{[\infty]} : \ (x_n)_{n \in A} \text{ is convergent}\}$
where $(x_n)_n$ is a sequence in a compact metric space.

(3) $nwd(X, \tau)$ where (X, τ) is regular without isolated points.

Connection with Ramsey type properties of ideals

$\mathcal{I}^+$ all subsets of $\mathbb{N}$ not belonging to $\mathcal{I}$.

Kaketov preorder:

$\mathcal{I} \leq_K \mathcal{J}$, if there is $f : \omega \to \omega$ such that $f^{-1}(E) \in \mathcal{J}$ for all $E \in \mathcal{I}$

$\mathcal{R}$ ideal generated by the homogeneous sets of the random graph

Theorem (Hrusak-Meza) (i) $\omega \to (\mathcal{I}^+)^2_2$ iff $\mathcal{R} \not\leq_K \mathcal{I}$.

(ii) $\mathcal{I}^+ \to (\mathcal{I}^+)^2_2$ iff $\mathcal{R} \not\leq_K \mathcal{I}\lceil A$ for all $A \in \mathcal{I}^+$.

There is φ such that $hom(\varphi) \subseteq \mathcal{I}$ iff $\mathcal{R} \leq_K \mathcal{I}$.

Coloring of pairs does not suffice

Let $e = (r_n)_n$ be an enumeration of $\mathbb{Q}$.

Let $\psi : \mathbb{N}^{[3]} \to 2$ be given by

$$\psi\{k, l, m\} = 1 \quad \Leftrightarrow \quad |r_l - r_k| > |r_m - r_l|, \quad k < l < m.$$

There is No $\varphi : \mathbb{N}^{[2]} \to 2$ such that $hom(\varphi) \subseteq hom(\psi)$.

Coloring of pairs does not suffice

Let $e = (r_n)_n$ be an enumeration of $\mathbb{Q}$.

Let $\psi : \mathbb{N}^{[3]} \to 2$ be given by

$$\psi\{k,l,m\} = 1 \quad \Leftrightarrow \quad |r_l - r_k| > |r_m - r_l|, \quad k < l < m.$$

There is No $\varphi : \mathbb{N}^{[2]} \to 2$ such that $hom(\varphi) \subseteq hom(\psi)$.

Sierpinski's coloring of $\mathbb{N}^{[2]}$:

$$\varphi_e\{i,j\} = 1 \quad \Leftrightarrow \quad r_i < r_j, \quad i < j.$$

Theorem: Let $\varphi : \mathbb{N}^{[2]} \to 2$. There is $A \subseteq \mathbb{N}$ such that $(r_n)_{n \in A}$ is order isomorphic to $\mathbb{Q}$ and

$$hom(\varphi_e \lceil A) \subseteq hom(\varphi).$$

Local version

Theorem: For any F_σ tall (i.e cofinal) ideal $\mathcal{I}$ and any $A \in \mathcal{I}^+$ there is $B \in \mathcal{I}^+$ with $B \subseteq A$ and a coloring $\varphi : \mathbb{N}^{[2]} \to 2$ such that

$$hom(\varphi \lceil B) \subseteq \mathcal{I}.$$

It is an open question whether the previous fact holds for every tall Borel ideal (Hrusak-Meza).

Galvin's Lemma and closed basis

Theorem: (Galvin, 1968) Let $\mathcal{O} \subseteq \mathbb{N}^{[\infty]}$ be an open set and $A \subseteq \mathbb{N}$ infinite. There is $B \subseteq A$ infinite such that

$$B^{[\infty]} \cap \mathcal{O} = \emptyset \quad \text{or} \quad B^{[\infty]} \subseteq \mathcal{O}.$$

Galvin's Lemma and closed basis

Theorem: (Galvin, 1968) Let $\mathcal{O} \subseteq \mathbb{N}^{[\infty]}$ be an open set and $A \subseteq \mathbb{N}$ infinite. There is $B \subseteq A$ infinite such that

$$B^{[\infty]} \cap \mathcal{O} = \emptyset \quad \text{or} \quad B^{[\infty]} \subseteq \mathcal{O}.$$

For each $\mathcal{F} \subseteq \mathrm{FIN}$, let $\mathcal{O}_{\mathcal{F}} = \bigcup_{s \in \mathcal{F}} \{A \in \mathbb{N}^{[\infty]} : \ s \sqsubset A\}$
Every open set $\mathcal{O} \subseteq \mathbb{N}^{[\infty]}$ is of the form $\mathcal{O}_{\mathcal{F}}$ for some $\mathcal{F} \subseteq \mathrm{FIN}$.

$hom(\mathcal{F}) =$ the homogeneous sets for the partition given by $\mathcal{O}_{\mathcal{F}}$.

Galvin's Lemma and closed basis

Theorem: (Galvin, 1968) Let $\mathcal{O} \subseteq \mathbb{N}^{[\infty]}$ be an open set and $A \subseteq \mathbb{N}$ infinite. There is $B \subseteq A$ infinite such that

$$B^{[\infty]} \cap \mathcal{O} = \emptyset \quad \text{or} \quad B^{[\infty]} \subseteq \mathcal{O}.$$

For each $\mathcal{F} \subseteq \mathsf{FIN}$, let $\mathcal{O}_{\mathcal{F}} = \bigcup_{s \in \mathcal{F}} \{A \in \mathbb{N}^{[\infty]} : \ s \sqsubset A\}$
Every open set $\mathcal{O} \subseteq \mathbb{N}^{[\infty]}$ is of the form $\mathcal{O}_{\mathcal{F}}$ for some $\mathcal{F} \subseteq \mathsf{FIN}$.

$hom(\mathcal{F}) =$ the homogeneous sets for the partition given by $\mathcal{O}_{\mathcal{F}}$.

Theorem: Let $\mathcal{C}$ be a cofinal family. If $\mathcal{C}$ has a closed base, then there is $\mathcal{F} \subseteq \mathsf{FIN}$ such that $hom(\mathcal{F}) \subseteq \mathcal{C}$.

Fact: There is a cofinal ideal $\mathcal{I}$ such that $hom(\mathcal{F}) \not\subseteq \mathcal{I}$ for all $\mathcal{F} \subseteq \mathsf{FIN}$.

Galvin's Lemma and closed basis

Theorem: (Galvin, 1968) Let $\mathcal{O} \subseteq \mathbb{N}^{[\infty]}$ be an open set and $A \subseteq \mathbb{N}$ infinite. There is $B \subseteq A$ infinite such that

$$B^{[\infty]} \cap \mathcal{O} = \emptyset \quad \text{or} \quad B^{[\infty]} \subseteq \mathcal{O}.$$

For each $\mathcal{F} \subseteq \mathsf{FIN}$, let $\mathcal{O}_{\mathcal{F}} = \bigcup_{s\in\mathcal{F}}\{A \in \mathbb{N}^{[\infty]} : \ s \sqsubset A\}$
Every open set $\mathcal{O} \subseteq \mathbb{N}^{[\infty]}$ is of the form $\mathcal{O}_{\mathcal{F}}$ for some $\mathcal{F} \subseteq \mathsf{FIN}$.

$hom(\mathcal{F}) =$ the homogeneous sets for the partition given by $\mathcal{O}_{\mathcal{F}}$.

Theorem: Let $\mathcal{C}$ be a cofinal family. If $\mathcal{C}$ has a closed base, then there is $\mathcal{F} \subseteq \mathsf{FIN}$ such that $hom(\mathcal{F}) \subseteq \mathcal{C}$.

Fact: There is a cofinal ideal $\mathcal{I}$ such that $hom(\mathcal{F}) \not\subseteq \mathcal{I}$ for all $\mathcal{F} \subseteq \mathsf{FIN}$.

¿Which cofinal families admit a base of the form $hom(\mathcal{F})$?

$F_{\sigma\delta}$ families

Theorem: Let $\mathcal{C}$ be a cofinal family such that

$$\mathcal{C} = \bigcap_n F_n$$

each F_n is F_σ hereditary and closed under finite changes. Then there is a $\mathcal{F} \subseteq \mathrm{FIN}$ such that $hom(\mathcal{F}) \subseteq \mathcal{C}$.

$F_{\sigma\delta}$ families

Theorem: Let $\mathcal{C}$ be a cofinal family such that

$$\mathcal{C} = \bigcap_n F_n$$

each F_n is F_σ hereditary and closed under finite changes. Then there is a $\mathcal{F} \subseteq \mathsf{FIN}$ such that $hom(\mathcal{F}) \subseteq \mathcal{C}$.

Questions:

(Farah) Is every $F_{\sigma\delta}$ ideal of the previous form?

(Hrusak) Does any Borel tall ideal contains a F_σ tall ideal?

Let $\mathcal{I}$ be a Borel ideal over $\mathbb{N}$. Is there $\mathcal{F} \subseteq \mathsf{FIN}$ such that

$$hom(\mathcal{F}) \subseteq \mathcal{I} \cup \mathcal{I}^\perp$$

$$\mathcal{I}^\perp = \{A \subseteq \mathbb{N} : \ A \cap B \text{ is finite for all } B \in \mathcal{I}\}$$

Borel selectors for cofinal families

A selector for a cofinal family $\mathcal{C}$ is $\Phi : \mathbb{N}^{[\infty]} \to \mathbb{N}^{[\infty]}$ such that

$$\Phi(A) \subseteq A \ \ \& \ \ \Phi(A) \in \mathcal{C}.$$

¿Which cofinal families admit Borel selectors?

Borel selectors for cofinal families

A selector for a cofinal family $\mathcal{C}$ is $\Phi : \mathbb{N}^{[\infty]} \to \mathbb{N}^{[\infty]}$ such that

$$\Phi(A) \subseteq A \;\;\&\;\; \Phi(A) \in \mathcal{C}.$$

¿Which cofinal families admit Borel selectors?

Theorem: *hom*(φ) admits a Borel selector for each $\varphi : \mathbb{N}^{[2]} \to 2$.

More generally

A selector for a cofinal family $\mathcal{C}$ is $\Phi : \mathbb{N}^{[\infty]} \to \mathbb{N}^{[\infty]}$ such that

$$\Phi(A) \subseteq A \ \ \& \ \ \Phi(A) \in \mathcal{C}.$$

¿Which cofinal families admit Borel selectors?

Theorem: *hom*(φ) admits a Borel selector for each $\varphi : \mathbb{N}^{[2]} \to 2$.

More generally

Theorem: If $\mathcal{O} \subseteq \mathbb{N}^{[\infty]}$ is clopen, then *hom*$(\mathcal{O})$ admits a Borel selector.

Borel selectors for cofinal families II

A $\sqsubseteq$-antichain $\mathcal{B} \subseteq \mathsf{FIN}$ is a *front*, if every infinite set has an initial segment in $\mathcal{B}$.

Borel selectors for cofinal families II

A $\sqsubseteq$-antichain $\mathcal{B} \subseteq$ FIN is a *front*, if every infinite set has an initial segment in $\mathcal{B}$.

Theorem: Let $\mathcal{B}$ be a front. There is a Borel map $\Phi : 2^{\mathcal{B}} \times \mathbb{N}^{[\infty]} \to \mathbb{N}^{[\infty]}$ s.t. $\Phi(\mathcal{F}, A) \subseteq A$ and $\Phi(\mathcal{F}, A) \in hom(\mathcal{F})$.

Borel selectors for cofinal families II

A $\sqsubseteq$-antichain $\mathcal{B} \subseteq \mathrm{FIN}$ is a *front*, if every infinite set has an initial segment in $\mathcal{B}$.

Theorem: Let $\mathcal{B}$ be a front. There is a Borel map $\Phi : 2^{\mathcal{B}} \times \mathbb{N}^{[\infty]} \to \mathbb{N}^{[\infty]}$ s.t. $\Phi(\mathcal{F}, A) \subseteq A$ and $\Phi(\mathcal{F}, A) \in hom(\mathcal{F})$.

Theorem: For each front $\mathcal{B}$, there is a coanalytic cofinal family $\mathcal{I}$ such that $hom(\mathcal{F}) \not\subseteq \mathcal{I}$ for all $\mathcal{F} \subseteq \mathcal{B}$.
There is a co-analytic Ramsey tall ideal $\mathcal{I}$, i.e.

$$\mathcal{I}^+ \to (\mathcal{I}^+)^2_2.$$

Borel selectors for cofinal families II

A $\sqsubseteq$-antichain $\mathcal{B} \subseteq \mathrm{FIN}$ is a *front*, if every infinite set has an initial segment in $\mathcal{B}$.

Theorem: Let $\mathcal{B}$ be a front. There is a Borel map $\Phi : 2^{\mathcal{B}} \times \mathbb{N}^{[\infty]} \to \mathbb{N}^{[\infty]}$ s.t. $\Phi(\mathcal{F}, A) \subseteq A$ and $\Phi(\mathcal{F}, A) \in hom(\mathcal{F})$.

Theorem: For each front $\mathcal{B}$, there is a coanalytic cofinal family $\mathcal{I}$ such that $hom(\mathcal{F}) \not\subseteq \mathcal{I}$ for all $\mathcal{F} \subseteq \mathcal{B}$.
There is a co-analytic Ramsey tall ideal $\mathcal{I}$, i.e.

$$\mathcal{I}^+ \to (\mathcal{I}^+)^2_2.$$

Questions:
(1) Is there a Borel map $\Phi : 2^{\mathrm{FIN}} \times \mathbb{N}^{[\infty]} \to \mathbb{N}^{[\infty]}$ such that $\Phi(\mathcal{F}, A) \subseteq A$ and $\Phi(\mathcal{F}, A) \in hom(\mathcal{F})$?

(2) Does any closed cofinal family admit a Borel selector?

Uniform selectivity

An ideal $\mathcal{I}$ is uniformly selective if there is a Borel function Φ such that whenever $(D_n)_n$ is a decreasing sequence of sets not in $\mathcal{I}$, then $\Phi((D_n)_n) = D$ is an infinite set not in $\mathcal{I}$ such that $D/n \subseteq D_n$ for all $n \in D$.

Let $\mathcal{A}$ be an almost disjoint family of infinite subsets of $\mathbb{N}$. Then $\mathcal{I}(\mathcal{A})$, the ideal generated by $\mathcal{A}$, is selective (Mathias).

Uniform selectivity

An ideal $\mathcal{I}$ is uniformly selective if there is a Borel function Φ such that whenever $(D_n)_n$ is a decreasing sequence of sets not in $\mathcal{I}$, then $\Phi((D_n)_n) = D$ is an infinite set not in $\mathcal{I}$ such that $D/n \subseteq D_n$ for all $n \in D$.

Let $\mathcal{A}$ be an almost disjoint family of infinite subsets of $\mathbb{N}$. Then $\mathcal{I}(\mathcal{A})$, the ideal generated by $\mathcal{A}$, is selective (Mathias).

Theorem: Any F_σ selective ideal is uniformly selective.
This holds for $\mathcal{I}(\mathcal{A})$ when $\mathcal{A}$ is a closed almost disjoint family.

Question: Does the previous result hold for any analytic selective ideal?