Strong Failures of Higher Analogs of Hindman's Theorem

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$$\mathrm{FS}(X) = \{x_1 + \cdots + x_n \big| n \in \mathbb{N} \text{ and } x_1, \ldots, x_n \in X \text{ are distinct}\}.$$

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Theorem (Galvin/Glazer/Hindman)

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For every commutative cancellative semigroup G and every colouring $c : G \longrightarrow 2$ with two colours, there exists an infinite $X \subseteq G$ such that $\mathrm{FS}(X)$ is monochromatic.

In all known proofs of this result, the set X is constructed by means of a recursion with ω steps.

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Theorem

For every uncountable commutative cancellative semigroup G *there exists a colouring* $c : G \longrightarrow 2$ *such that whenever* $X \subseteq G$ *is uncountable,* $\mathrm{FS}(X)$ *is not monochromatic.*

Our key algebraic tool to treat these problems is the following result:

Theorem

Let G be any commutative cancellative semigroup of cardinality $\kappa > \omega$. Then there are countable abelian groups G_α, $\alpha < \kappa$, such that G embeds into

$$\bigoplus_{\alpha<\kappa} G_\alpha = \left\{ x \in \prod_{\alpha<\kappa} G_\alpha \middle| x(\alpha) = 0 \text{ for all but finitely many } \alpha < \kappa \right\}.$$

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Note that, if $c : \bigoplus_{\alpha<\kappa} G_\alpha \longrightarrow 2$ is a "bad" colouring, then so is $c \upharpoonright G$. Thus from now on we will assume without loss of generality that $G = \bigoplus_{\alpha<\kappa} G_\alpha$, where each G_α is countable.

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Given $x \in \bigoplus_{\alpha<\kappa} G_\alpha$, we will define the **support** of x to be

$$\mathrm{supp}(x) = \{\alpha < \kappa | x(\alpha) \neq 0\} \in [\kappa]^{<\omega}.$$

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We denote the statement above by $G \nrightarrow [\omega_1]^{\mathrm{FS}}_{\omega}$ (recall the square-bracket notation for higher analogs of Ramsey's theorem).

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Modulo large cardinals it is consistent (e.g. in a model of Martin's Maximum) that $\mathbb{R} \rightarrow [\omega_1]^{\mathrm{FS}}_{\omega_1}$.

Definition

Given cardinals $\kappa \geq \theta$, the symbol $S(\kappa, \theta)$ will denote the following statement: there exists a colouring $d : [\kappa]^{<\omega} \longrightarrow \theta$ such that, whenever $\mathcal{X} \subseteq [\kappa]^{<\omega}$ satisfies $|\mathcal{X}| = \kappa$, for every $\delta < \theta$ it is possible to find two distinct $x, y \in \mathcal{X}$ such that $d(z) = \delta$ whenever $x \triangle y \subseteq z \subseteq x \cup y$.

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Theorem

Let $\kappa = \mathrm{cf}(\kappa) \geq \theta \geq \omega_1$. If $S(\kappa, \theta)$ holds, then for every commutative cancellative G with $|G| = \kappa$, $G \nrightarrow [\kappa]_{\theta}^{\mathrm{FS}_2}$.

Here $\mathrm{FS}_2(X) = \{x + y \big| x, y \in X \text{ are distinct}\}$ for every $X \subseteq G$.

Recall that the combinatorial principle $\mathrm{Pr}_1(\kappa, \lambda, \theta, \chi)$ states the existence of a colouring $c : [\kappa]^2 \longrightarrow \theta$ such that, whenever $\mathcal{X} \subseteq [\kappa]^{<\chi}$ has size λ and is pairwise disjoint, for all $\delta < \theta$ we can find two distinct $x, y \in \mathcal{X}$ such that $c[x \times y] = \{\delta\}$.

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Fact

If $\mathrm{cf}(\kappa) = \kappa > \omega_1$ *admits a nonreflecting stationary set, then* $\mathrm{Pr}_1(\kappa, \kappa, \kappa, \omega)$ *holds (for example, if* $\kappa = \lambda^+$ *for* $\lambda = \mathrm{cf}(\lambda) \geq \omega_1$*).*

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If $\kappa = \mathrm{cf}(\kappa) \geq \omega_1$ and $\theta \leq \kappa$, then $\mathrm{Pr}_1(\kappa, \kappa, \theta, \omega)$ implies $S(\kappa, \theta)$. In particular, if $\mathrm{Pr}_1(\kappa, \kappa, \theta, \omega)$ holds then $G \not\rightarrow [\kappa]^{\mathrm{FS}_2}_{\theta}$ whenever $|G| = \kappa$.

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In fact, more is true.

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$S(\omega_1, \omega_1)$ holds. In particular, whenever $|G| = \omega_1$, it is the case that $G \nrightarrow [\omega_1]_{\omega_1}^{\mathrm{FS}_2}$.

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If $\kappa = \mathrm{cf}(\kappa) \geq \omega_1$ *and* $\theta \leq \kappa$*, then* $\mathrm{Pr}_1(\kappa, \kappa, \theta, \omega)$ *implies the existence of a* $d : [\kappa]^{<\omega} \longrightarrow \theta$ *such that, for all families* $\mathcal{X}, \mathcal{Y} \subseteq [\kappa]^{<\omega}$ *satisfying* $|\mathcal{X}| = |\mathcal{Y}| = \kappa$ *and every* $\delta < \theta$*, there are* $x \in \mathcal{X}$ *and* $y \in \mathcal{Y}$ *such that* $d(z) = \delta$ *whenever* $x \triangle y \subseteq z \subseteq x \cup y$*.*

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If $\kappa = \mathrm{cf}(\kappa) \geq \omega_1$ and $\theta \leq \kappa$, then $\mathrm{Pr}_1(\kappa, \kappa, \theta, \omega)$ implies the existence of a $d : [\kappa]^{<\omega} \longrightarrow \theta$ such that, for all families $\mathcal{X}, \mathcal{Y} \subseteq [\kappa]^{<\omega}$ satisfying $|\mathcal{X}| = |\mathcal{Y}| = \kappa$ and every $\delta < \theta$, there are $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ such that $d(z) = \delta$ whenever $x \triangle y \subseteq z \subseteq x \cup y$.

Theorem

If $\kappa = \mathrm{cf}(\kappa) \geq \omega_1$ and $\theta \leq \kappa$ satisfy the conclusion of the above theorem, then whenever $|G| = \kappa$ there is a colouring $c : G \longrightarrow \theta$ such that for every $n \in \mathbb{N}$ and every choice of $X_1, \ldots, X_n \subseteq G$ with $|X_1| = \cdots = |X_n| = \kappa$, the sumset

$$X_1 + \cdots + X_n = \{x_1 + \cdots + x_n \big| x_1 \in X_1, \ldots, x_n \in X_n\}$$

meets all colours.

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The conclusion of the theorem at the top also holds if $\kappa = \theta = \omega_1$. In particular, whenever $|G| = \omega_1$ there is a colouring $c : G \longrightarrow \omega_1$ such that every sumset $X_1 + \cdots + X_n$ in which $|X_1| = \cdots = |X_n| = \omega_1$ must meet all colours.