Formalized Brouwerian Real Analysis using the Nuprl proof assistant

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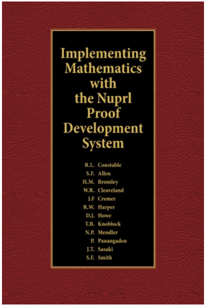
Introduction



Implementing Mathematics with The Nuprl Proof Development System
By the PRL Group:
R. L. Constable, S. F. Allen,
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- Inspired by the work of Errett Bishop and with NSF funding to implement *proofs-as-programs*, Bob Constable and his group at Cornell developed Nuprl, a proof assistant now based on extensional, constructive type theory.
- We have formalized all of Chapter 2 of Bishop and Bridges, *Constructive Analysis* in Nuprl.

Brouwerian Nuprl

- From studying Kleene we became convinced that both bar induction and the continuity principle should be true in Nuprl and that they would be very useful for formalizing constructive analysis.
- In the last few years we have formalized the semantics of Nuprl in Coq. This allowed us to extend Nuprl in new ways and formally confirm our conviction.
  - We added named exceptions and a "fresh" name binding operation, and proved rules for reasoning about them. These allow us to prove a *strong continuity principle* for Nuprl.
  - We added a version of *free choice sequences* to the Nuprl semantics. This allows us to justify a strong *bar induction* rule, and use that rule to prove a general form of *bar recursion*.
- Using these new features we prove Brouwer's theorem that all real functions on a proper, compact interval are continuous. This allows us to simplify several aspects of our formalization of *Constructive Analysis*.

Nuprl in a nutshell

Nuprl built its own programming logic and later evolved towards the work of Per Martin-Löf. The basic concept is $x = y \in T$ where x and y are *terms* and T is a *type*.

Computation comes first: the terms are definitional extensions of a primitive, untyped programming language that includes

- numbers and tokens: $\ldots, -1, 0, 1, 2 \ldots,$ 'abc', ...
- $\lambda x.t$, $\langle t_1, t_2 \rangle$, $\mathit{inl}\ t$, $\mathit{inr}\ t$, $+, *, -, \mathit{div}, \mathit{rem}$
- $t_1(t_2)$, $\mathit{spread}(t_1; a, b.t_2)$, $\mathit{decide}(t_1; a.t_2; b.t_3)$, $\mathbf{fix}\ F$
- exceptions, try?catch, $\nu x.t$, and several more

We then define $t_1 \mapsto t_2$ (*computes to*) and $t_1 \sim t_2$ (computational bi-simulation).

A *type* T "is" a partial equivalence relation R_T on terms that respects the bi-simulation relation. Then $x \in T$ if $(x\ R_T\ x)$ and $x = y \in T$ if $(x\ R_T\ y)$.

Every type is a member of a *universe* $\mathbb{U}_i$ where $i \in 0, 1, 2, \ldots$

Nuprl is an extensional type theory

- Nuprl is *extensional*: if $x = y \in A$ and $A = B \in \mathbb{U}_i$ then $x = y \in B$.
- Nuprl has many types: quotients $x, y.T//E(x, y)$, intersection $\bigcap_{x:A} B(x)$, "partial" types $\overline{T}$ where **fix** $(\lambda x.x) = \perp \in \overline{T}$ ...
- The same term can be proved to have many different types.
- Type membership is undecidable, so there is no type-checking algorithm.
- Function extensionality:
  $f = g \in x : A \to B(x) \Leftrightarrow \forall x : A.\ f(x) = g(x) \in B(x)$
- Nuprl is expressive enough to formalize category theory and (the semantics of) homotopy type theory – viz. Cubical type theory.

Squashing

- The "set" type $\{x\colon T \mid P(x)\}$ is the subtype of $x \in T$ for which $P(x)$ is true.
  - A member x of $\{x\colon T \mid P(x)\}$ does not "come with" a proof of $P(x)$.
  - This type is useful mainly when $P(x)$ is "squash stable" – we can construct a witness for $P(x)$ from x when we know only that $P(x)$ is true (i.e inhabited).
  - This allows us to omit unneeded evidence.
- $\downarrow T$, the "squash" of type T is $\{x : \mathit{Unit} \mid T\}$
  - $\downarrow T$ is inhabited if and only if T is inhabited, but it contains no other information (about what terms inhabit T).

Truncation

- $\downharpoonright T$, the “truncation” of T is the quotient type $T//\textbf{true}$
  - The inhabitants of $\downharpoonright T$ are the same as the inhabitants of T, but they are all equal.
  - $\downharpoonright T$ can express the existence of something of type T that is not extensional w.r.t. its parameters.
- The *choice principle* for type T is
  $\forall P : T \to \mathbb{P}.\ (\forall x : T.\ \downharpoonright P[x]) \Leftrightarrow \downharpoonright (\forall x : T.\ P[x])$
- In Nuprl we can prove
  - The choice principle is true for $\mathbb{N}$ and for $\mathbb{N} \to \mathbb{N}$
  - The choice principle is false for $(\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$

How does Nuprl prove Brouwer's theorem?

- It satisfies both weak and strong forms of the *continuity principle* for numbers.
  - Let $\mathcal{B} = \mathbb{N} \to \mathbb{N}$ and $\mathbb{N}_n = \{0, 1, \ldots n-1\}$
  - If $F \in \mathcal{B} \to \mathbb{N}$ then

weak $\forall f : \mathcal{B}. \downarrow\exists n : \mathbb{N}.\ \forall g : \mathcal{B}.\ F(g) = F(f)$ if $(f = g \in \mathbb{N}_n \to \mathbb{N})$

strong $\downarrow\exists M : n{:}\mathbb{N} \to (\mathbb{N}_n \to \mathbb{N}) \to (\mathbb{N} \bigcup Unit).\ \forall f{:}\mathcal{B}.\ (\downarrow\exists n : \mathbb{N}.\ M(n, f) = F(f)) \wedge (\forall n{:}\mathbb{N}.\ M(n, f) = F(f)\ \textit{if}\ M(n, f) \in \mathbb{N})$

- Nuprl has two (and only two) induction principles
  - Induction on $\mathbb{N}$
  - Bar Induction
- Using Bar Induction we prove FAN, and from FAN and the strong continuity principle we prove Brouwer's theorem that all functions from $[0, 1] \to \mathbb{R}$ are uniformly continuous.

The constructive real numbers

$$
\begin{array}{rcl}
\mathbb{R} & = & \{r:\mathbb{N}^+ \to \mathbb{Z} \mid \\
 & & \quad \forall n, m:\mathbb{N}.\ |n * r(m) - m * r(n)| \leq 2(n+m)\ \} \\
\text{then} & & |\frac{r(n)}{2n} - \frac{r(m)}{2m}| \leq \frac{1}{n} + \frac{1}{m} \\
\text{so} & & \lambda n.\frac{r(n)}{2n} \text{ is Bishop's regular} \\
(r = s) & = & \forall n.\ |r(n) - s(n)| \leq 4 \\
(r < s) & = & \exists n.\ r(n) + 4 < s(n) \\
(r \neq s) & = & (r < s) \vee (s < r) \\
Fun(f, I) & = & \forall x, y:\{r:\mathbb{R} \mid r \in I\}.(x = y) \Rightarrow (f(x) = f(y)) \\
SFun(f, I) & = & \forall x, y:\{r:\mathbb{R} \mid r \in I\}.(f(x) \neq f(y)) \Rightarrow (x \neq y)
\end{array}
$$

Brouwer's theorem

$$\begin{aligned} Cont(f, I) \quad = \quad & \forall \epsilon > 0.\ \exists \delta > 0.\ \forall x, y{:}\{r{:}\mathbb{R} \mid r \in I\}. \\ & |x - y| \le \delta \Rightarrow |f(x) - f(y)| \le \epsilon \end{aligned}$$

Theorem 1: $\forall a, b{:}\mathbb{R}.\ (a < b) \Rightarrow (Cont(f, [a, b]) \Leftrightarrow Fun(f, [a, b]))$
Theorem 2: $\forall a, b{:}\mathbb{R}.\ (a \le b) \Rightarrow (Cont(f, [a, b]) \Leftrightarrow SFun(f, [a, b]))$

Bar Induction Rule

$$\begin{array}{rcll}
H & \vdash & T \in \mathit{Type} & (1) \\
H,\ n{:}\mathbb{N},\ s:(\mathbb{N}_n \to T) & \vdash & B(n,s) \vee \neg B(n,s) & (2) \\
H,\ a{:}(\mathbb{N} \to T) & \vdash & \downarrow\exists n{:}\mathbb{N}.\ B(n,a) & (3) \\
H,\ n{:}\mathbb{N}, s{:}(\mathbb{N}_n \to T), x{:}B(n,s) & \vdash & f(n,s) \in X(n,s) & (4) \\
H,\ n{:}\mathbb{N}, s{:}(\mathbb{N}_n \to T), x{:}P & \vdash & f(n,s) \in X(n,s) & (5) \\
P = \forall t{:}T.\ f(n+1, s.t) & \in & X(n+1, s.t) & \\
s.t = \lambda m.\ \mathit{if}\ \ m = n & & \mathit{then}\ t\ \mathit{else}\ s(m) & (6) \\
\hline
H & \vdash & f(0,c) \in X(0,c) & (7)
\end{array}$$

The continuity principle is constructive

- For $F \in \mathcal{B} \to \mathbb{N}$, $\bullet \in \mathit{Unit}$ the constructive content of strong continuity is an $M_F \in n : \mathbb{N} \to (\mathbb{N}_n \to \mathbb{N}) \to (\mathbb{N} \bigcup \mathit{Unit})$
- $M_F = \lambda n, f.\ \nu e.(F(\overline{f})?e : \bullet)$ where $\overline{f}(x) =$ if $x < 0$ then $\perp$ else if $x < n$ then $f(x)$ else $\mathit{exception}(e; \bullet)$
- Kreisel and others (e.g. Escardo and Xu) have shown that no such M can be extensional. This means that extensionally equal $F = G \in \mathcal{B} \to \mathbb{N}$ can have $M_F(n, f) \neq M_G(n, f)$.
- We can (and must) truncate the strong continuity proposition.
- The truncated version is still strong enough for many purposes, in particular, Brouwer's theorem.

How do we know that Nuprl is consistent?

- Doug Howe defined the $t_1 \sim t_2$ (computational bi-simulation) relation and proved its crucial properties.
- Stuart Allen gave an "inductive-recursive" definition of the partial equivalence relation semantics for the types in a Nuprl universe. He then converted this into a recursive definition. Using this semantics he defined the truth of a Nuprl sequent $H \models C \; ext \; t$.
- Vincent Rahli and Abhishek Anand have formalized all of this in Coq and proved (most of) the rules of Nuprl (work still ongoing).
- Bar Induction is powerful enough to prove that the predicative part of Coq is consistent. Therefore its soundness proof needs some more powerful principle. We used $P \vee \neg P$ in the impredicative *Prop* universe. *We believe bar induction for other reasons due to Brouwer and Kleene.*