Algebra and Arithmetic of Plane Binary Trees:

Theory & Applications of Mapped Regular Pavings

Raazesh Sainudiin

Jennifer Harlow, Kenneth Kuhn, Dominic Lee, Carey Priebe, Gloria Teng and Warwick Tucker

School of Mathematics and Statistics, University of Canterbury,
Christchurch, New Zealand

April 2014,

UC Maths & Stats Primer

Fri Apr 4 11:18:48 NZDT 2014

Extending Arithmetic:

reals $\rightarrow$ intervals $\rightarrow$ mapped partitions of interval

1. arithmetic over reals

Extending Arithmetic:

reals $\rightarrow$ intervals $\rightarrow$ mapped partitions of interval

1. arithmetic over reals
2. naturally extends to
   arithmetic over intervals

Extending Arithmetic:

reals $\rightarrow$ intervals $\rightarrow$ mapped partitions of interval

1. arithmetic over reals
2. naturally extends to
   arithmetic over intervals
3. **Our Main Idea:**
   – **is** to further naturally extend to
   arithmetic over mapped partitions of an interval called
   Mapped Regular Pavings (MRPs)

Extending Arithmetic:

reals $\rightarrow$ intervals $\rightarrow$ mapped partitions of interval

1. arithmetic over reals
2. naturally extends to
   arithmetic over intervals
3. **Our Main Idea:**
   – **is** to further naturally extend to
   arithmetic over mapped partitions of an interval called
   Mapped Regular Pavings (MRPs)
4. – **by** exploiting the *algebraic structure of partitions formed by finite-rooted-binary (frb) trees*

Extending Arithmetic:

reals $\rightarrow$ intervals $\rightarrow$ mapped partitions of interval

1. arithmetic over reals
2. naturally extends to
   arithmetic over intervals
3. **Our Main Idea:**
   – **is** to further naturally extend to
   arithmetic over mapped partitions of an interval called
   Mapped Regular Pavings (MRPs)
4. – **by** exploiting the *algebraic structure of partitions formed by finite-rooted-binary (frb) trees*
5. – **thereby** provide algorithms for several algebras and their inclusions over frb tree partitions

arithmetic from intervals to their frb-tree partitions

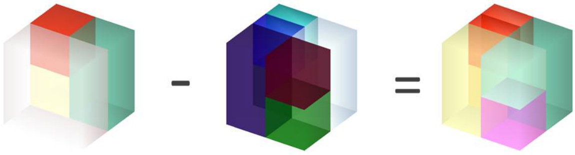


Figure : Arithmetic with coloured spaces.

arithmetic from intervals to their frb-tree partitions

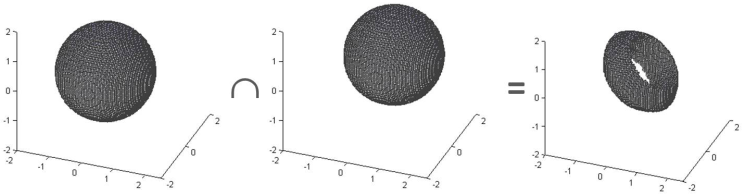


Figure : Intersection of enclosures of two hollow spheres.

arithmetic from intervals to their frb-tree partitions

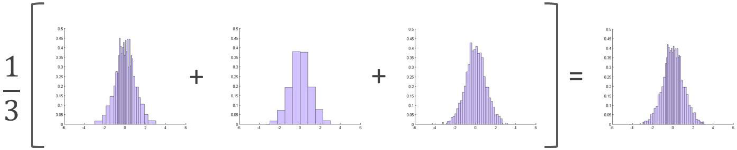


Figure : Histogram averaging.

Why Mapped Regular pavings (MRPs)?

MRPs allow any arithmetic defined over elements in $\mathbb{Y}$ to be extended point-wise to $\mathbb{Y}$-MRPs.

1. Arithmetic on piece-wise constant functions and interval-valued functions;

Why Mapped Regular pavings (MRPs)?

MRPs allow any arithmetic defined over elements in $\mathbb{Y}$ to be extended point-wise to $\mathbb{Y}$-MRPs.

1. Arithmetic on piece-wise constant functions and interval-valued functions;
2. Exploiting the tree-based structure to obtain interval enclosures of real-valued functions efficiently

Why Mapped Regular pavings (MRPs)?

MRPs allow any arithmetic defined over elements in $\mathbb{Y}$ to be extended point-wise to $\mathbb{Y}$-MRPs.

1. Arithmetic on piece-wise constant functions and interval-valued functions;
2. Exploiting the tree-based structure to obtain interval enclosures of real-valued functions efficiently
3. Statistical set-processing operations like marginal density, conditional density and highest coverage regions, visualization, etc

Why Mapped Regular pavings (MRPs)?

MRPs allow any arithmetic defined over elements in $\mathbb{Y}$ to be extended point-wise to $\mathbb{Y}$-MRPs.

1. Arithmetic on piece-wise constant functions and interval-valued functions;
2. Exploiting the tree-based structure to obtain interval enclosures of real-valued functions efficiently
3. Statistical set-processing operations like marginal density, conditional density and highest coverage regions, visualization, etc
4. Other Possibilities: “Tree’d” Contractor Programs and Constraint Propagators (Bounded-error Robotics)

An RP tree a root interval $\boldsymbol{x}_\rho \in \mathbb{IR}^d$

The regularly paved boxes of $\boldsymbol{x}_\rho$ can be represented by nodes of finite rooted binary (frb-trees) of geometric group theory

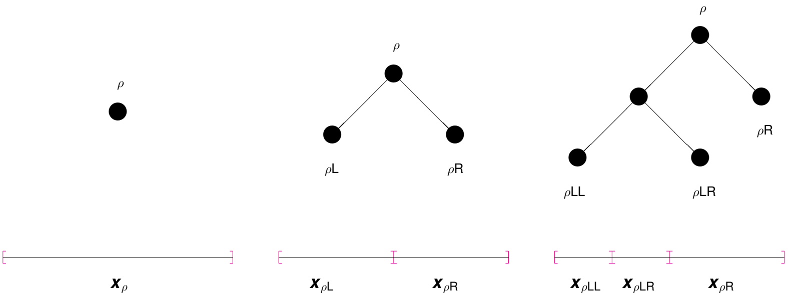
An operation of bisection on a box is equivalent to performing the operation on its corresponding node in the tree:

An RP tree a root interval $\boldsymbol{x}_\rho \in \mathbb{IR}^d$

The regularly paved boxes of $\boldsymbol{x}_\rho$ can be represented by nodes of finite rooted binary (frb-trees) of geometric group theory

An operation of bisection on a box is equivalent to performing the operation on its corresponding node in the tree:

Leaf boxes of RP tree partition the root interval $\boldsymbol{x}_\rho \in \mathbb{IR}^1$

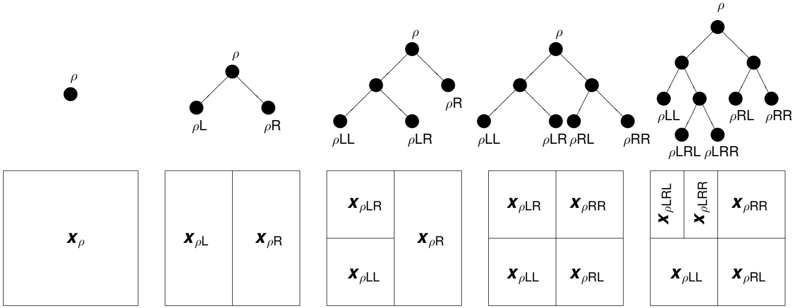


An RP tree a root interval $\boldsymbol{x}_\rho \in \mathbb{IR}^d$

The regularly paved boxes of $\boldsymbol{x}_\rho$ can be represented by nodes of finite rooted binary (frb-trees) of geometric group theory

An operation of bisection on a box is equivalent to performing the operation on its corresponding node in the tree:

Leaf boxes of RP tree partition the root interval $\boldsymbol{x}_\rho \in \mathbb{IR}^2$

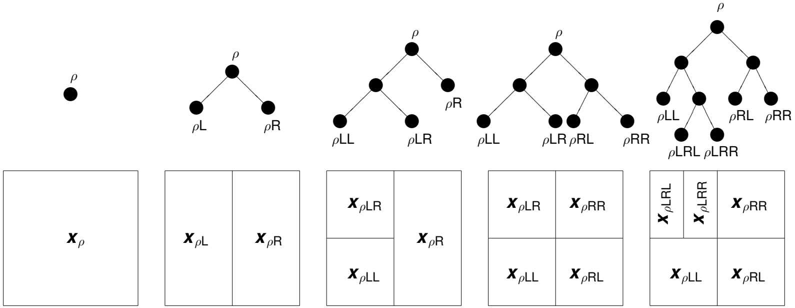


An RP tree a root interval $\boldsymbol{x}_\rho \in \mathbb{IR}^d$

The regularly paved boxes of $\boldsymbol{x}_\rho$ can be represented by nodes of finite rooted binary (frb-trees) of geometric group theory

An operation of bisection on a box is equivalent to performing the operation on its corresponding node in the tree:

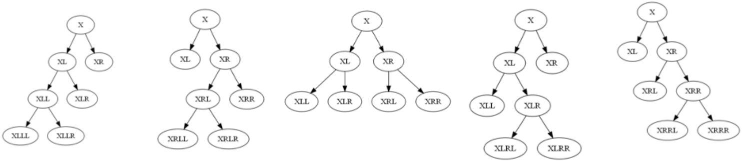
Leaf boxes of RP tree partition the root interval $\boldsymbol{x}_\rho \in \mathbb{IR}^2$



By this “RP Peano's curve” frb-trees encode paritions of $\boldsymbol{x}_\rho \in \mathbb{IR}^d$

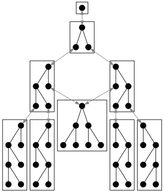
Algebraic Structure and Combinatorics of RPs

Leaf-depth encoded RPs

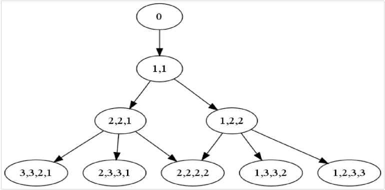


(3, 3, 2, 1) (1, 3, 3, 2) (2, 2, 2, 2) (2, 3, 3, 1) (1, 2, 3, 3)

There are C_k RPs with k splits

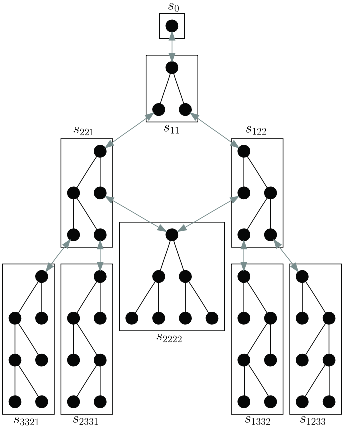


$$
\begin{aligned}
C_0 &= 1\\
C_1 &= 1\\
C_2 &= 2\\
C_3 &= 5\\
C_4 &= 14\\
C_5 &= 42\\
\dots &= \dots\\
C_k &= \frac{(2k)!}{(k+1)!k!}\\
\dots &= \dots\\
C_{15} &= 9694845\\
\dots &= \dots\\
C_{20} &= 6564120420\\
\dots &= \dots
\end{aligned}
$$



Hasse (transition) Diagram of Regular Pavings

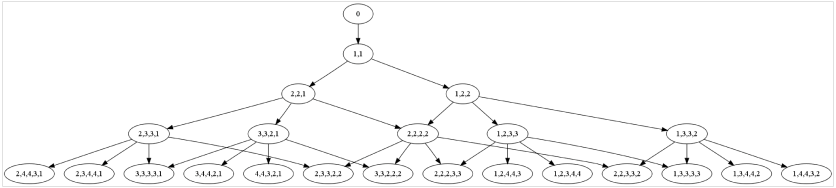
Transition diagram over $\mathbb{S}_{0:3}$ with split/reunion operations



RS, W.Taylor and G.Teng, Catalan Coefficients, Sequence A185155 in The On-Line Encyclopedia of Integer Sequences, 2012, `http://oeis.org`

Hasse (transition) Diagram of Regular Pavings

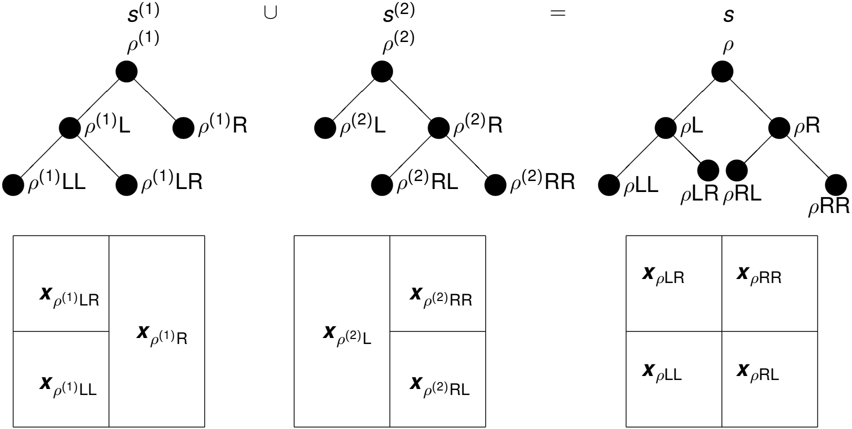
Transition diagram over $\mathbb{S}_{0:4}$ with split/reunion operations



1. The above state space is denoted by $\mathbb{S}_{0:4}$
2. Number of RPs with k splits is the Catalan number C_k
3. There is more than one way to reach a RP by k splits
4. Randomized enclosure algorithms are Markov chains on $\mathbb{S}_{0:\infty}$

RPs are closed under union operations

$s^{(1)} \cup s^{(2)} = s$ is union of two RPs $s^{(1)}$ and $s^{(2)}$ of $\boldsymbol{x}_\rho \in \mathbb{R}^2$.



RPs are closed under union operations

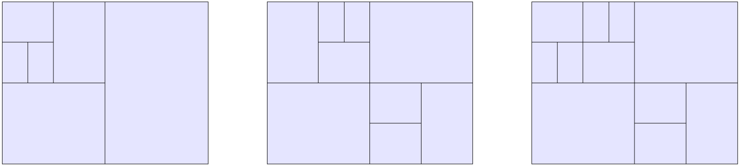
Lemma 1: The algebraic structure of frb-trees (underlying Thompson's group) is closed under union operations.

RPs are closed under union operations

Lemma 1: The algebraic structure of frb-trees (underlying Thompson's group) is closed under union operations.

Proof: by a "transparency overlay process" argument (cf. Meier 2008).

$s^{(1)} \cup s^{(2)} = s$ is union of two RPs $s^{(1)}$ and $s^{(2)}$ of $\boldsymbol{x}_\rho \in \mathbb{R}^2$.



Algorithm 1: $\texttt{RPUnion}(\rho^{(1)}, \rho^{(2)})$

input : Root nodes $\rho^{(1)}$ and $\rho^{(2)}$ of RPs $s^{(1)}$ and $s^{(2)}$, respectively, with root box $\boldsymbol{x}_{\rho^{(1)}} = \boldsymbol{x}_{\rho^{(2)}}$

output : Root node ρ of RP $s = s^{(1)} \cup s^{(2)}$

if $\texttt{IsLeaf}(\rho^{(1)})$ & $\texttt{IsLeaf}(\rho^{(2)})$ **then**

  $\rho \leftarrow \texttt{Copy}(\rho^{(1)})$

  return ρ

end

else if $!\texttt{IsLeaf}(\rho^{(1)})$ & $\texttt{IsLeaf}(\rho^{(2)})$ **then**

  $\rho \leftarrow \texttt{Copy}(\rho^{(1)})$

  return ρ

end

else if $\texttt{IsLeaf}(\rho^{(1)})$ & $!\texttt{IsLeaf}(\rho^{(2)})$ **then**

  $\rho \leftarrow \texttt{Copy}(\rho^{(2)})$

  return ρ

end

else

  $!\texttt{IsLeaf}(\rho^{(1)})$ & $!\texttt{IsLeaf}(\rho^{(2)})$

end

Make ρ as a node with $\boldsymbol{x}_\rho \leftarrow \boldsymbol{x}_{\rho^{(1)}}$

Graft onto ρ as left child the node $\texttt{RPUnion}(\rho^{(1)}\mathrm{L}, \rho^{(2)}\mathrm{L})$

Graft onto ρ as right child the node $\texttt{RPUnion}(\rho^{(1)}\mathrm{R}, \rho^{(2)}\mathrm{R})$

return ρ

Note: this is not the minimal union of the (Boolean mapped) RPs of Jaulin et. al. 2001

Dfn: Mapped Regular Paving (MRP)

- Let $\boldsymbol{s} \in \mathbb{S}_{0:\infty}$ be an RP with root node ρ and root box $\boldsymbol{x}_\rho \in \mathbb{IR}^d$

Dfn: Mapped Regular Paving (MRP)

- Let $\boldsymbol{s} \in \mathbb{S}_{0:\infty}$ be an RP with root node ρ and root box $\boldsymbol{x}_\rho \in \mathbb{IR}^d$
- and let $\mathbb{Y}$ be a non-empty set.

Dfn: Mapped Regular Paving (MRP)

- Let $s \in \mathbb{S}_{0:\infty}$ be an RP with root node ρ and root box $\boldsymbol{x}_\rho \in \mathbb{IR}^d$
- and let $\mathbb{Y}$ be a non-empty set.
- Let $\mathbb{V}(s)$ and $\mathbb{L}(s)$ denote the sets all nodes and leaf nodes of s, respectively.

Dfn: Mapped Regular Paving (MRP)

- Let $s \in \mathbb{S}_{0:\infty}$ be an RP with root node ρ and root box $\boldsymbol{x}_\rho \in \mathbb{IR}^d$
- and let $\mathbb{Y}$ be a non-empty set.
- Let $\mathbb{V}(s)$ and $\mathbb{L}(s)$ denote the sets all nodes and leaf nodes of s, respectively.
- Let $f : \mathbb{V}(s) \to \mathbb{Y}$ map each node of s to an element in $\mathbb{Y}$ as follows:

$$\{\rho \mathsf{v} \mapsto f_{\rho \mathsf{v}} : \rho \mathsf{v} \in \mathbb{V}(s), f_{\rho \mathsf{v}} \in \mathbb{Y}\} \ .$$

Dfn: Mapped Regular Paving (MRP)

- Let $s \in \mathbb{S}_{0:\infty}$ be an RP with root node ρ and root box $\boldsymbol{x}_\rho \in \mathbb{IR}^d$
- and let $\mathbb{Y}$ be a non-empty set.
- Let $\mathbb{V}(s)$ and $\mathbb{L}(s)$ denote the sets all nodes and leaf nodes of s, respectively.
- Let $f : \mathbb{V}(s) \to \mathbb{Y}$ map each node of s to an element in $\mathbb{Y}$ as follows:
$$\{\rho\mathsf{v} \mapsto f_{\rho\mathsf{v}} : \rho\mathsf{v} \in \mathbb{V}(s), f_{\rho\mathsf{v}} \in \mathbb{Y}\} \ .$$
- Such a map f is called a $\mathbb{Y}$*-mapped regular paving (*$\mathbb{Y}$*-MRP)*.

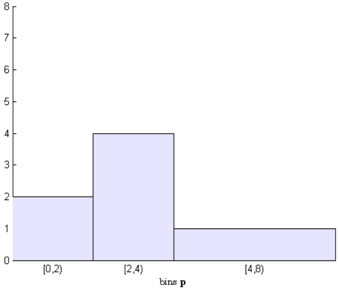
Dfn: Mapped Regular Paving (MRP)

- Let $s \in \mathbb{S}_{0:\infty}$ be an RP with root node ρ and root box $\boldsymbol{x}_\rho \in \mathbb{IR}^d$
- and let $\mathbb{Y}$ be a non-empty set.
- Let $\mathbb{V}(s)$ and $\mathbb{L}(s)$ denote the sets all nodes and leaf nodes of s, respectively.
- Let $f : \mathbb{V}(s) \to \mathbb{Y}$ map each node of s to an element in $\mathbb{Y}$ as follows:
$$\{\rho\mathsf{v} \mapsto f_{\rho\mathsf{v}} : \rho\mathsf{v} \in \mathbb{V}(s), f_{\rho\mathsf{v}} \in \mathbb{Y}\} \ .$$
- Such a map f is called a $\mathbb{Y}$*-mapped regular paving* *($\mathbb{Y}$-MRP)*.
- Thus, a $\mathbb{Y}$-MRP f is obtained by augmenting each node $\rho\mathsf{v}$ of the RP tree s with an additional data member $f_{\rho\mathsf{v}}$.

Examples of $\mathbb{Y}$-MRPs

If $\mathbb{Y} = \mathbb{R}$

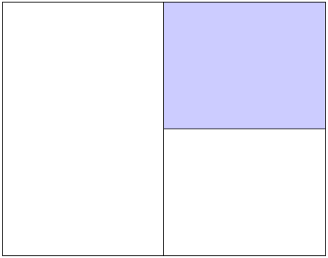
$\mathbb{R}$-MRP over s_{221} with $x_\rho = [0, 8]$



Examples of $\mathbb{Y}$-MRPs

If $\mathbb{Y} = \mathbb{B}$

$\mathbb{B}$-MRP over s_{122} with $x_\rho = [0, 1]^2$ (e.g. Jaulin et. al. 2001)

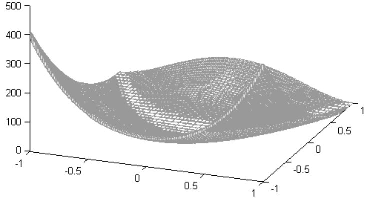


Examples of $\mathbb{Y}$-MRPs

If $\mathbb{Y} = \mathbb{IR}$

– frb tree representation for interval inclusion algebra

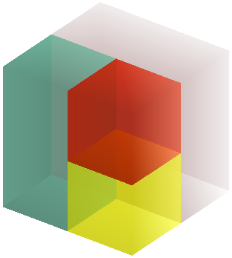
$\mathbb{IR}$-MRP enclosure of the Rosenbrock function with $x_\rho = [-1, 1]^2$



Examples of $\mathbb{Y}$-MRPs

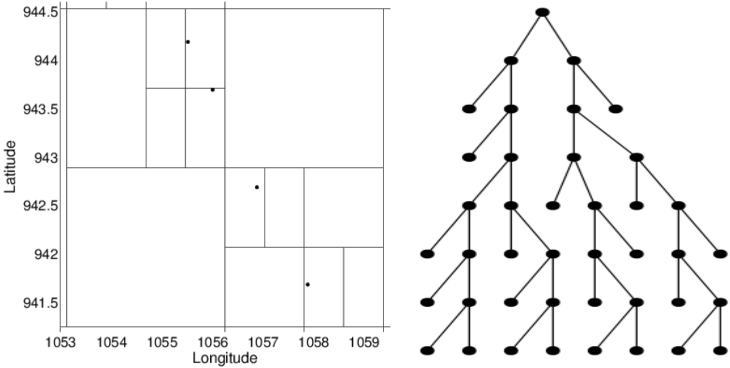
If $\mathbb{Y} = [0,1]^3$
– R G B colour maps

$[0,1]^3$-MRP over s_{3321} with $x_\rho = [0,1]^3$



If $\mathbb{Y} = \mathbb{Z}_+ := \{0, 1, 2, ...\}$
– radar-measured aircraft trajectory data

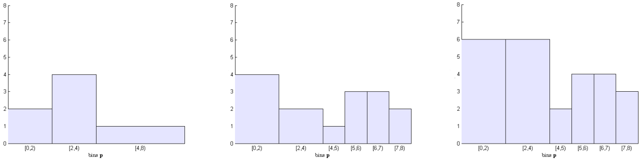
$\mathbb{Z}_+$-MRP trajectory of an aircraft and its tree



$\mathbb{Y}$-MRP Arithmetic

If $\star : \mathbb{Y} \times \mathbb{Y} \to \mathbb{Y}$ then we can extend $\star$ point-wise to two $\mathbb{Y}$-MRPs *f* and *g* with root nodes $\rho^{(1)}$ and $\rho^{(2)}$ via `MRPOperate`$(\rho^{(1)}, \rho^{(2)}, \star)$.
This is done using `MRPOperate`$(\rho^{(1)}, \rho^{(2)}, +)$

$f \qquad g \qquad f+g$



$\mathbb{R}$-MRP Addition by $\texttt{MRPOperate}(\rho^{(1)}, \rho^{(2)}, +)$

adding two piece-wise constant functions or $\mathbb{R}$-MRPs

Algorithm 2: `MRPOperate`$(\rho^{(1)}, \rho^{(2)}, \star)$

input : two root nodes $\rho^{(1)}$ and $\rho^{(2)}$ with same root box $\boldsymbol{x}_{\rho^{(1)}} = \boldsymbol{x}_{\rho^{(2)}}$ and binary operation $\star$.

output : the root node ρ of $\mathbb{Y}$-MRP $h = f \star g$.

Make a new node ρ with box and image
$\boldsymbol{x}_\rho \leftarrow \boldsymbol{x}_{\rho^{(1)}}; h_\rho \leftarrow f_{\rho^{(1)}} \star g_{\rho^{(2)}}$

if `IsLeaf`$(\rho^{(1)})$ & !`IsLeaf`$(\rho^{(2)})$ **then**
  Make temporary nodes L′, R′
  $\boldsymbol{x}_{\mathsf{L}'} \leftarrow \boldsymbol{x}_{\rho^{(1)}\mathsf{L}}; \boldsymbol{x}_{\mathsf{R}'} \leftarrow \boldsymbol{x}_{\rho^{(1)}\mathsf{R}}$
  $f_{\mathsf{L}'} \leftarrow f_{\rho^{(1)}}, f_{\mathsf{R}'} \leftarrow f_{\rho^{(1)}}$
  Graft onto ρ as left child the node `MRPOperate`$(\mathsf{L}', \rho^{(2)}\mathsf{L}, \star)$
  Graft onto ρ as right child the node `MRPOperate`$(\mathsf{R}', \rho^{(2)}\mathsf{R}, \star)$
end

else if !`IsLeaf`$(\rho^{(1)})$ & `IsLeaf`$(\rho^{(2)})$ **then**
  Make temporary nodes L′, R′
  $\boldsymbol{x}_{\mathsf{L}'} \leftarrow \boldsymbol{x}_{\rho^{(2)}\mathsf{L}}; \boldsymbol{x}_{\mathsf{R}'} \leftarrow \boldsymbol{x}_{\rho^{(2)}\mathsf{R}}$
  $g_{\mathsf{L}'} \leftarrow g_{\rho^{(2)}}, g_{\mathsf{R}'} \leftarrow g_{\rho^{(2)}}$
  Graft onto ρ as left child the node `MRPOperate`$(\rho^{(1)}\mathsf{L}, \mathsf{L}', \star)$
  Graft onto ρ as right child the node `MRPOperate`$(\rho^{(1)}\mathsf{R}, \mathsf{R}', \star)$
end

else if !`IsLeaf`$(\rho^{(1)})$ & !`IsLeaf`$(\rho^{(2)})$ **then**
  Graft onto ρ as left child the node `MRPOperate`$(\rho^{(1)}\mathsf{L}, \rho^{(2)}\mathsf{L}, \star)$
  Graft onto ρ as right child the node `MRPOperate`$(\rho^{(1)}\mathsf{R}, \rho^{(2)}\mathsf{R}, \star)$
end
return ρ

Unary transformations are easy too

Let $\texttt{MRPTransform}(\rho, \tau)$ apply the unary transformation $\tau : \mathbb{R} \to \mathbb{R}$ to a given $\mathbb{R}$-MRP f with root node ρ as follows:

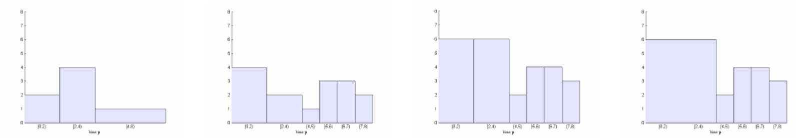
- copy f to g
- recursively set $f_{\rho v} = \tau(f_{\rho v})$ for each node ρv in g
- return g as $\tau(f)$

Minimal Representation of $\mathbb{R}$-MRP

Algorithm 3: `MinimiseLeaves`(ρ)

input : ρ, the root node of $\mathbb{R}$-MRP f.
output : Modify f into $\curlywedge(f)$, the unique $\mathbb{R}$-MRP with fewest leaves.

if !`IsLeaf`(ρ) **then**
    `MinimiseLeaves`$(\rho\mathsf{L})$
    `MinimiseLeaves`$(\rho\mathsf{R})$
    if `IsCherry`(ρ) & $(f_{\rho\mathsf{L}} = f_{\rho\mathsf{R}})$ **then**
        $f_\rho \leftarrow f_{\rho\mathsf{L}}$
        `Prune`$(\rho\mathsf{L})$
        `Prune`$(\rho\mathsf{R})$
    end
end



(a) f (b) g (c) $f+g$ (d) $\curlywedge(f+g)$

Arithmetic and Algebra of $\mathbb{R}$-MRPs

Thus, we can obtain arithmetical expressions specified by finitely many sub-expressions in a **directed acyclic graph** whose:

Arithmetic and Algebra of $\mathbb{R}$-MRPs

Thus, we can obtain arithmetical expressions specified by finitely many sub-expressions in a **directed acyclic graph** whose:

- inputs and output **nodes** are themselves $\mathbb{R}$-MRPs

Arithmetic and Algebra of $\mathbb{R}$-MRPs

Thus, we can obtain arithmetical expressions specified by finitely many sub-expressions in a **directed acyclic graph** whose:

- inputs and output **nodes** are themselves $\mathbb{R}$-MRPs
- and whose **edges** involve:

Arithmetic and Algebra of $\mathbb{R}$-MRPs

Thus, we can obtain arithmetical expressions specified by finitely many sub-expressions in a **directed acyclic graph** whose:

- inputs and output **nodes** are themselves $\mathbb{R}$-MRPs
- and whose **edges** involve:
  1. a binary arithmetic operation $\star \in \{+, -, \cdot, /\}$ over two $\mathbb{R}$-MRPs,
  2. a standard transformation of $\mathbb{R}$-MRP by elements of $\mathfrak{S} := \{\exp, \sin, \cos, \tan, \ldots\}$ and
  3. their compositions.

Stone-Wierstrass Theorem: $\mathbb{R}$-MRPs Dense in $C(\boldsymbol{x}_\rho, \mathbb{R})$

Theorem

Let $\mathcal{F}$ be the class of $\mathbb{R}$-MRPs with the same root box $\boldsymbol{x}_\rho$. Then $\mathcal{F}$ is dense in $C(\boldsymbol{x}_\rho, \mathbb{R})$, the algebra of real-valued continuous functions on $\boldsymbol{x}_\rho$.

Stone-Wierstrass Theorem: $\mathbb{R}$-MRPs Dense in $C(\boldsymbol{x}_\rho, \mathbb{R})$

Theorem

Let $\mathcal{F}$ be the class of $\mathbb{R}$-MRPs with the same root box $\boldsymbol{x}_\rho$. Then $\mathcal{F}$ is dense in $C(\boldsymbol{x}_\rho, \mathbb{R})$, the algebra of real-valued continuous functions on $\boldsymbol{x}_\rho$.

Proof:

Since $\boldsymbol{x}_\rho \in \mathbb{IR}^d$ is a compact Hausdorff space, by the Stone-Weierstrass theorem we can establish that $\mathcal{F}$ is dense in $C(\boldsymbol{x}_\rho, \mathbb{R})$ with the topology of uniform convergence, provided that $\mathcal{F}$ is a sub-algebra of $C(\boldsymbol{x}_\rho, \mathbb{R})$ that separates points in $\boldsymbol{x}_\rho$ and which contains a non-zero constant function.

Stone-Wierstrass Theorem: $\mathbb{R}$-MRPs Dense in $C(\boldsymbol{x}_\rho, \mathbb{R})$

Theorem

Let $\mathcal{F}$ be the class of $\mathbb{R}$-MRPs with the same root box $\boldsymbol{x}_\rho$. Then $\mathcal{F}$ is dense in $C(\boldsymbol{x}_\rho, \mathbb{R})$, the algebra of real-valued continuous functions on $\boldsymbol{x}_\rho$.

Proof:

Since $\boldsymbol{x}_\rho \in \mathbb{IR}^d$ is a compact Hausdorff space, by the Stone-Weierstrass theorem we can establish that $\mathcal{F}$ is dense in $C(\boldsymbol{x}_\rho, \mathbb{R})$ with the topology of uniform convergence, provided that $\mathcal{F}$ is a sub-algebra of $C(\boldsymbol{x}_\rho, \mathbb{R})$ that separates points in $\boldsymbol{x}_\rho$ and which contains a non-zero constant function.

We will show all these conditions are satisfied by $\mathcal{F}$

Stone-Wierstrass Theorem Contd.: $\mathbb{R}$-MRPs Dense in $C(\boldsymbol{x}_\rho, \mathbb{R})$

- $\mathcal{F}$ is a sub-algebra of $C(\boldsymbol{x}_\rho, \mathbb{R})$ since it is closed under addition and scalar multiplication.

Stone-Wierstrass Theorem Contd.: $\mathbb{R}$-MRPs Dense in $C(\boldsymbol{x}_\rho, \mathbb{R})$

- $\mathcal{F}$ is a sub-algebra of $C(\boldsymbol{x}_\rho, \mathbb{R})$ since it is closed under addition and scalar multiplication.
- $\mathcal{F}$ contains non-zero constant functions

Stone-Wierstrass Theorem Contd.: $\mathbb{R}$-MRPs Dense in $C(\boldsymbol{x}_\rho, \mathbb{R})$

- $\mathcal{F}$ is a sub-algebra of $C(\boldsymbol{x}_\rho, \mathbb{R})$ since it is closed under addition and scalar multiplication.
- $\mathcal{F}$ contains non-zero constant functions
- Finally, RPs can clearly separate distinct points $x, x' \in \boldsymbol{x}_\rho$ into distinct leaf boxes by splitting deeply enough.

Stone-Wierstrass Theorem Contd.: $\mathbb{R}$-MRPs Dense in $C(\boldsymbol{x}_\rho, \mathbb{R})$

- $\mathcal{F}$ is a sub-algebra of $C(\boldsymbol{x}_\rho, \mathbb{R})$ since it is closed under addition and scalar multiplication.
- $\mathcal{F}$ contains non-zero constant functions
- Finally, RPs can clearly separate distinct points $x, x' \in \boldsymbol{x}_\rho$ into distinct leaf boxes by splitting deeply enough.
- Thus, $\mathcal{F}$, the class of $\mathbb{R}$-MRPs with the same root box $\boldsymbol{x}_\rho$, is dense in $C(\boldsymbol{x}_\rho, \mathbb{R})$, the algebra of real-valued continuous functions on $\boldsymbol{x}_\rho$.

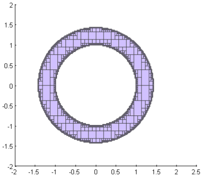
Stone-Wierstrass Theorem Contd.: $\mathbb{R}$-MRPs Dense in $C(\boldsymbol{x}_\rho, \mathbb{R})$

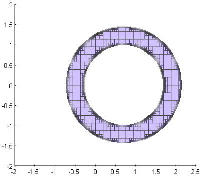
- $\mathcal{F}$ is a sub-algebra of $C(\boldsymbol{x}_\rho, \mathbb{R})$ since it is closed under addition and scalar multiplication.
- $\mathcal{F}$ contains non-zero constant functions
- Finally, RPs can clearly separate distinct points $x, x' \in \boldsymbol{x}_\rho$ into distinct leaf boxes by splitting deeply enough.
- Thus, $\mathcal{F}$, the class of $\mathbb{R}$-MRPs with the same root box $\boldsymbol{x}_\rho$, is dense in $C(\boldsymbol{x}_\rho, \mathbb{R})$, the algebra of real-valued continuous functions on $\boldsymbol{x}_\rho$.
- Q.E.D.

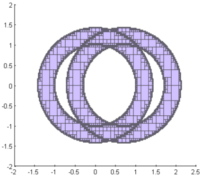
$\mathbb{B}$-MRP arithmetic – contractors, propagators & collaborators (bounded-error robotics)

Two Boolean-mapped regular pavings A_1 and A_2 and Boolean arithmetic operations with $+$ for set union, $-$ for symmetric set difference, $\times$ for set intersection, and $\div$ for set difference.

A_1 $\quad$ A_2 $\quad$ $A1 + A2$



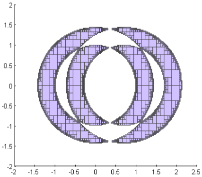




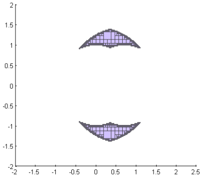
$\mathbb{B}$-MRP arithmetic – contractors, propagators & collaborators (bounded-error robotics)

Two Boolean-mapped regular pavings A_1 and A_2 and Boolean arithmetic operations with $+$ for set union, $-$ for symmetric set difference, $\times$ for set intersection, and $\div$ for set difference.

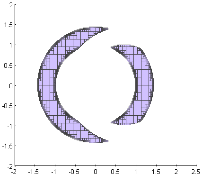
$A_1 - A2$



$A_1 \times A_2$

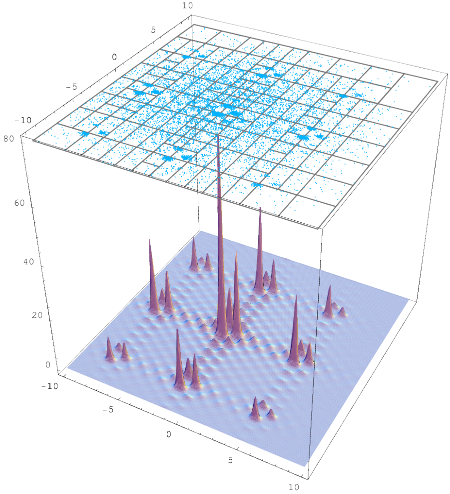


$A_1 \div A_2$



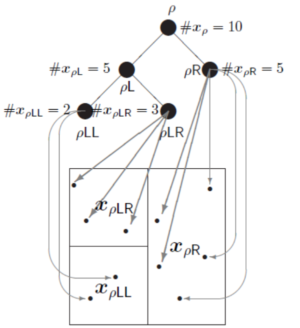
Nonparametric Density Estimation

Problem: Take samples from an unknown density f and consistently reconstruct f

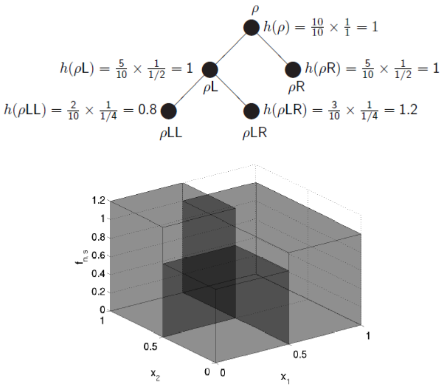


Nonparametric Density Estimation

Approach: Use statistical regular paving to get $\mathbb{R}$-MRP data-adaptive histogram



(a) An SRP tree and its constituents.

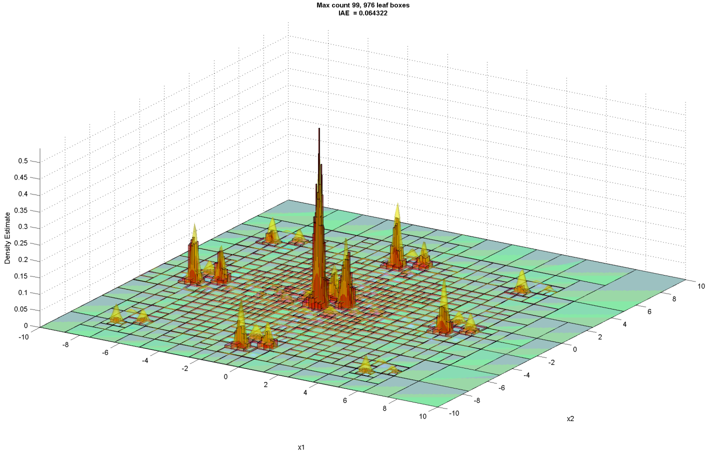


(b) An SRP histogram and its tree.

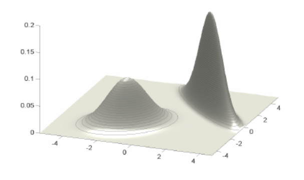
Nonparametric Density Estimation

Solution: $\mathbb{R}$-MRP histogram averaging allows us to produce a consistent Bayesian estimate of the density (up to 10 dimensions)

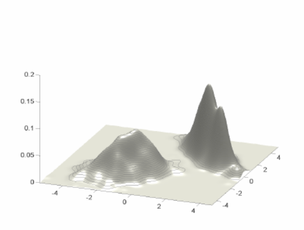
(Teng, Harlow, Lee and S., *ACM Trans. Mod. & Comp. Sim.*, [r. 2] 2012)



Kernel Density Estimate (visualization of a procedure)

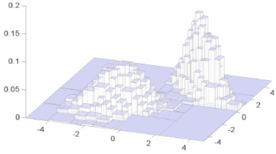


(a) True density.

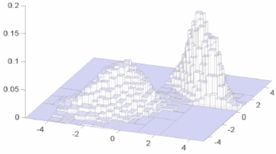


(c) MCMC bandwidth KDE.

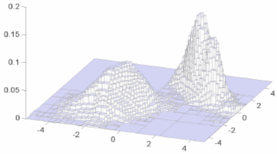
Approximating Kernel Density Estimates by $\mathbb{R}$-MRPs



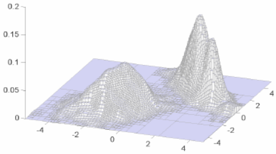
(a) $\overline{\psi} = 0.001$ (187 leaves).



(b) $\overline{\psi} = 0.005$ (316 leaves).



(c) $\overline{\psi} = 0.0001$ (919 leaves).



(d) $\overline{\psi} = 0.00001$ (4420 leaves).

Approximating Kernel Density Estimates by $\mathbb{R}$-MRPs

Table J.4: 5-d case: estimated errors for KDE and RMRP-KDE approximations.

	$\hat{d}_{KL}$	$\hat{L}_1$ error	Time (s)	Leaves
KDE ($n_K = 2,000$)	0.41	0.66	7,350–8,880	n/a
RMRP-KDE approximations				
$\overline{\psi} = 0.0001$	5.06	0.96	1.0	$2,363$
$\overline{\psi} = 0.00005$	4.85	0.91	2.3	$4,639$
$\overline{\psi} = 0.00001$	4.51	0.85	8.7	$17,759$
$\overline{\psi} = 0.000005$	4.49	0.84	17.2	$31,335$
$\overline{\psi} = 0.000001$	3.33	0.76	66.1	$133,493$
$\overline{\psi} = 0.0000005$	3.31	0.75	131.0	$237,561$
$\overline{\psi} = 0.0000001$	3.54	0.74	470.0	$895,012$

Finding image of $\mathbb{R}$-MRP is by fast look-ups

Algorithm 4: `PointWiseImage`(ρ, x)

input : ρ with box $\boldsymbol{x}_\rho$, the root node of $\mathbb{R}$-MRP f with RP s, and a point $x \in \boldsymbol{x}_\rho$.
output : Return $f_{\eta(x)}$ at the leaf node $\eta(x)$ that is associated with the box $\boldsymbol{x}_{\eta(x)}$ containing x.

```
if IsLeaf(ρ) then
    return f_ρ
end
else
    if x ∈ x_ρR then
        PointWiseImage(ρR, x)
    end
    else
        PointWiseImage(ρL, x)
    end
end
```

Finding image of $\mathbb{R}$-MRP is by fast look-ups

Algorithm 5: `PointWiseImage`(ρ, x)

input : ρ with box $\boldsymbol{x}_\rho$, the root node of $\mathbb{R}$-MRP f with RP s, and a point $x \in \boldsymbol{x}_\rho$.
output : Return $f_{\eta(x)}$ at the leaf node $\eta(x)$ that is associated with the box $\boldsymbol{x}_{\eta(x)}$ containing x.

if `IsLeaf`(ρ) **then**
    return f_ρ
end
else
    if $x \in \boldsymbol{x}_{\rho R}$ **then**
        `PointWiseImage`$(\rho R, x)$
    end
    else
        `PointWiseImage`$(\rho L, x)$
    end
end

- Cost of KDE image $\sim O(n)$ **KFLOPs** (FLOPs for kernel evaluation procedure)

Finding image of $\mathbb{R}$-MRP is by fast look-ups

Algorithm 6: `PointWiseImage`(ρ, x)

input : ρ with box $\boldsymbol{x}_\rho$, the root node of $\mathbb{R}$-MRP f with RP s, and a point $x \in \boldsymbol{x}_\rho$.
output : Return $f_{\eta(x)}$ at the leaf node $\eta(x)$ that is associated with the box $\boldsymbol{x}_{\eta(x)}$ containing x.

if `IsLeaf`(ρ) **then**
  return f_ρ
end
else
  if $x \in \boldsymbol{x}_{\rho \mathrm{R}}$ **then**
    `PointWiseImage`$(\rho\mathrm{R}, x)$
  end
  else
    `PointWiseImage`$(\rho\mathrm{L}, x)$
  end
end

- Cost of KDE image $\sim O(n)$ **KFLOPs** (FLOPs for kernel evaluation procedure)
- 10-fold CV cost $\sim 10 \times O\left(\frac{1}{10}n\frac{9}{10}n\right) = O(n^2)$ **KFLOPs**

Finding image of $\mathbb{R}$-MRP is by fast look-ups

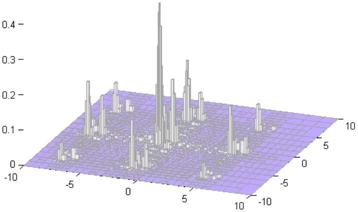
Algorithm 7: `PointWiseImage`(ρ, x)

input : ρ with box $\boldsymbol{x}_\rho$, the root node of $\mathbb{R}$-MRP f with RP s, and a point $x \in \boldsymbol{x}_\rho$.
output : Return $f_{\eta(x)}$ at the leaf node $\eta(x)$ that is associated with the box $\boldsymbol{x}_{\eta(x)}$ containing x.

if `IsLeaf`(ρ) **then**
  return f_ρ
end
else
  if $x \in \boldsymbol{x}_{\rho\mathrm{R}}$ **then**
    `PointWiseImage`$(\rho\mathrm{R}, x)$
  end
  else
    `PointWiseImage`$(\rho\mathrm{L}, x)$
  end
end

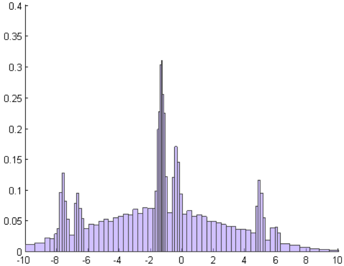
- Cost of KDE image $\sim O(n)$ **KFLOPs** (FLOPs for kernel evaluation procedure)
- 10-fold CV cost $\sim 10 \times O\left(\frac{1}{10}n\frac{9}{10}n\right) = O(n^2)$ **KFLOPs**
- But using $\mathbb{R}$-MRP approximation to KDE requires $10 \times O\left(\frac{1}{10}n\lg\left(\frac{9}{10}n\right)\right) = O(n\lg(n))$ **tree-look-ups**

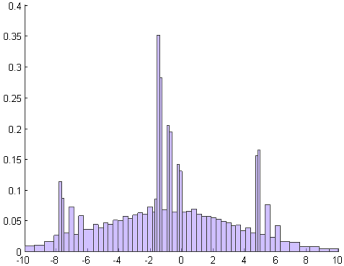
Coverage, Marginal & Slice Operators of $\mathbb{R}$-MRP



$\mathbb{R}$-MRP approximation to Levy density and its coverage regions with $\alpha = 0.9$ (light gray), $\alpha = 0.5$ (dark gray) and $\alpha = 0.1$ (black)

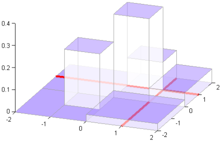
Coverage, Marginal & Slice Operators of $\mathbb{R}$-MRP

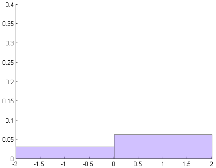


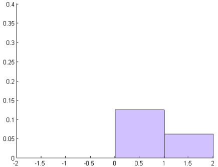


Marginal densities $f^{\{1\}}(x_1)$ and $f^{\{2\}}(x_2)$ along each coordinate of $\mathbb{R}$-MRP approximation

Coverage, Marginal & Slice Operators of $\mathbb{R}$-MRP







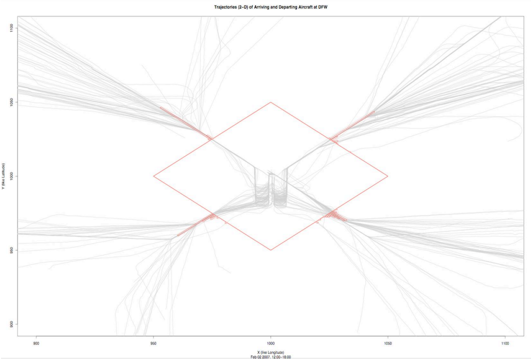
The slices of a simple $\mathbb{R}$-MRP in 2D

— "non-parametric regression arithmetic"

Air Traffic “Arithmetic” → dynamic air-space configuration

(G. Teng, K. Kuhn and RS, *J. Aerospace Comput., Inf. & Com.*, 9:1, 14–25, 2012.)

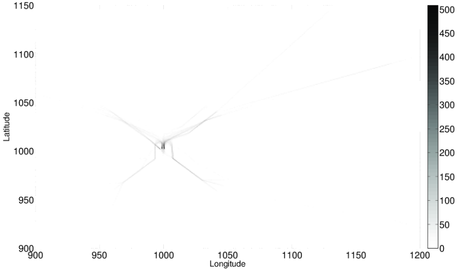
On a Good Day



Air Traffic “Arithmetic” $\rightarrow$ dynamic air-space configuration

(G. Teng, K. Kuhn and RS, *J. Aerospace Comput., Inf. & Com.*, 9:1, 14–25, 2012.)

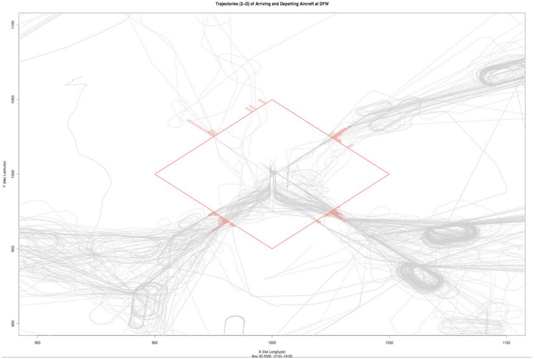
$\mathbb{Z}_+$-MRP On a Good Day



Air Traffic “Arithmetic” → dynamic air-space configuration

(G. Teng, K. Kuhn and RS, *J. Aerospace Comput., Inf. & Com.*, 9:1, 14–25, 2012.)

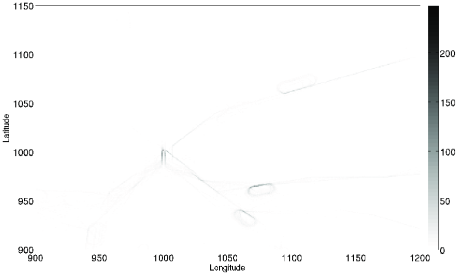
On a Bad Day



Air Traffic "Arithmetic" $\rightarrow$ dynamic air-space configuration

(G. Teng, K. Kuhn and RS, *J. Aerospace Comput., Inf. & Com.*, 9:1, 14–25, 2012.)

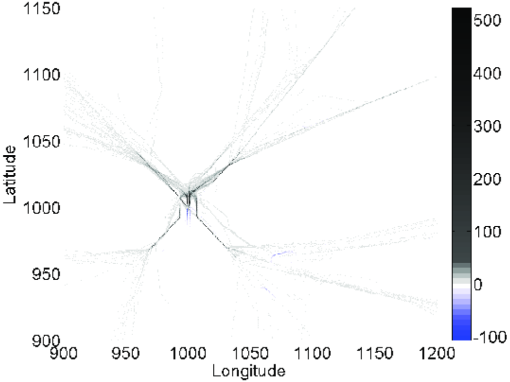
$\mathbb{Z}_+$-MRP On a Bad Day



Air Traffic "Arithmetic" $\rightarrow$ dynamic air-space configuration

(G. Teng, K. Kuhn and RS, *J. Aerospace Comput., Inf. & Com.*, 9:1, 14–25, 2012.)

$\mathbb{Z}_+$-MRP pattern for Good Day $-$ Bad Day



Example – Prioritised Splitting

inclusion function: $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{x}^2 + (\boldsymbol{x}+1)\sin(10\pi\boldsymbol{x})^2\cos(3\pi\boldsymbol{x})^2$
priority function: $\psi(\rho \mathsf{v}) = \mathrm{vol}\,(\rho \mathsf{v})\mathrm{wid}\,(\boldsymbol{g}(\boldsymbol{x}_{\rho \mathsf{v}}))$

To 50 leaves by	To 100 leaves by
$\texttt{RPQEnclose}^{\triangledown}(\rho, \boldsymbol{g}, \psi, \bar{\ell} = 50)$	$\texttt{RPQEnclose}^{\triangledown}(\rho, \boldsymbol{g}, \psi, \bar{\ell} = 100)$

Algorithm 8: `RPQEnclose`$^{\nabla}(\rho, \boldsymbol{g}, \psi, \bar{\ell})$

input : ρ, the root node of $\mathbb{IR}$-MRP $\boldsymbol{f}$ with RP s, root box $\boldsymbol{x}_{\rho}$ and $\boldsymbol{f}_{\rho} = \boldsymbol{g}(\boldsymbol{x}_{\rho})$,
$\psi : \mathbb{L}(s) \to \mathbb{R}$ such that
$\psi(\rho\mathsf{v}) = \mathrm{vol}\,(\boldsymbol{x}_{\rho\mathsf{v}})\,(\boldsymbol{g}(\boldsymbol{x}_{\rho\mathsf{v}}) - 0.5\,(\boldsymbol{g}(\boldsymbol{x}_{\rho\mathsf{vL}}) + \boldsymbol{g}(\boldsymbol{x}_{\rho\mathsf{vR}}))),$
$\bar{\ell}$ the maximum number of leaves.

output : $\boldsymbol{f}$ with modified RP s such that $|\mathbb{L}(s)| = \bar{\ell}$

if $|\mathbb{L}(s)| < \bar{\ell}$ **then**

$$\rho\mathsf{v} \leftarrow \texttt{random\_sample}\left(\operatorname*{argmax}_{\rho\mathsf{v}\in\mathbb{L}(s)} \psi(\rho\mathsf{v})\right)$$

Split $\rho\mathsf{v}$: $\triangledown(\rho\mathsf{v}) = \{\rho\mathsf{vL}, \rho\mathsf{vR}\}$ `// split the sampled node`

$\boldsymbol{f}_{\rho\mathsf{vL}} \leftarrow \boldsymbol{g}(\square(\boldsymbol{x}_{\rho\mathsf{vL}}))$

$\boldsymbol{f}_{\rho\mathsf{vR}} \leftarrow \boldsymbol{g}(\square(\boldsymbol{x}_{\rho\mathsf{vL}}))$

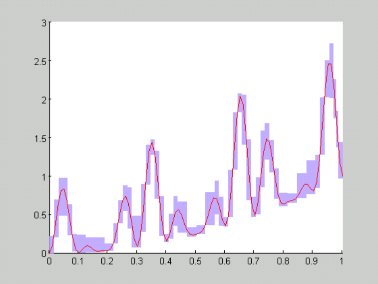
`RPQEnclose`$^{\nabla}(\rho, \psi, \bar{\ell})$

end

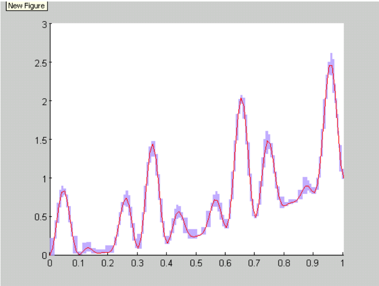
Example - Prioritised Splitting Continued

inclusion function: $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{x}^2 + (\boldsymbol{x}+1)\sin(10\pi\boldsymbol{x})^2\cos(3\pi\boldsymbol{x})^2$
priority function: $\psi(\rho\mathrm{v}) = \mathrm{vol}\,(\rho\mathrm{v})\mathrm{wid}\,(\boldsymbol{g}(\boldsymbol{x}_{\rho\mathrm{v}}))$

To 50 leaves by
$\texttt{RPQEnclose}^{\triangledown}(\rho, \boldsymbol{g}, \psi, \bar{\ell} = 50)$



To 100 leaves by
$\texttt{RPQEnclose}^{\triangledown}(\rho, \boldsymbol{g}, \psi, \bar{\ell} = 100)$



Can we get tighter enclosures using only 50 leaves by propagating the interval hull of 100-leaved $\mathbb{IR}$-MRP up the tree and then doing a prioritised merging of the cherries?

Hull Propagate up the tree via `HullPropagate`(ρ)

Algorithm 9: `HullPropagate`(ρ)

input : ρ, the root node of $\mathbb{IR}$-MRP $\boldsymbol{f}$ with RP $\boldsymbol{s}$.
output : Modify input MRP $\boldsymbol{f}$.

if !`IsLeaf`(ρ) **then**
    `HullPropagate`(ρL)
    `HullPropagate`(ρR)
    $\boldsymbol{f}_{\rho} \leftarrow \boldsymbol{f}_{\rho \mathsf{L}} \sqcup \boldsymbol{f}_{\rho \mathsf{R}}$
end

By calling `HullPropagate`(ρ) on our $\mathbb{IR}$-MRP of Example constructed by `RPQEnclose`$^{\bigtriangledown}(\rho, \boldsymbol{g}, \psi, \bar{\ell} = 100)$ we would have tightened the range enclosures of $\boldsymbol{g}$ in the internal nodes.

Prioritised Merging via $\texttt{RPQEnclose}^{\triangle}(\rho, \psi, \bar{\ell}')$

Algorithm 10: $\texttt{RPQEnclose}^{\triangle}(\rho, \psi, \bar{\ell}')$

input : ρ, the root node of $\mathbb{IR}$-MRP $\boldsymbol{f}$ with RP s, box $\boldsymbol{x}_\rho$,
$\psi : \mathbb{C}(s) \to \mathbb{R}$ as $\psi(\rho\mathsf{v}) = \mathrm{vol}(\boldsymbol{x}_{\rho\mathsf{v}})\,(\boldsymbol{f}_{\rho\mathsf{v}} - 0.5\,(\boldsymbol{f}_{\rho\mathsf{vL}} + \boldsymbol{f}_{\rho\mathsf{vR}}))$,
$\bar{\ell}'$ the maximum number of leaves.

output : modified $\boldsymbol{f}$ with RP s such that $|\mathbb{L}(s)| = \bar{\ell}'$ or $\mathbb{C}(s) = \emptyset$.

if $|\mathbb{L}(s)| \geq \bar{\ell}'$ & $\mathbb{C}(s) \neq \emptyset$ **then**

  $\rho\mathsf{v} \leftarrow \texttt{random\_sample}\left(\mathrm{argmin}_{\rho\mathsf{v} \in \mathbb{C}(s)} \psi(\rho\mathsf{v})\right)$ `// choose a random node with smallest` ψ

  $\texttt{Prune}(\rho\mathsf{L})$

  $\texttt{Prune}(\rho\mathsf{R})$

  $\texttt{RPQEnclose}^{\triangle}(\rho, \psi, \bar{\ell}')$

end

Example – Split, Propogating & Prune

Yes we can!

$\texttt{RPQEnclose}^{\bigtriangledown}(\rho, \boldsymbol{g}, \psi, \bar{\ell} = 100); \texttt{HullPropagate}(\rho); \texttt{RPQEnclose}^{\bigtriangleup}(\rho, \psi, \bar{\ell}' = 50)$

Conclusions

- $\mathbb{Y}$-MRPs provide frb-tree partition arithmetic
- $\mathbb{IY}$-MRPs allow efficient arithmetic for Neumaier's inclusion algebras
- $\mathbb{IY}$ can be $\mathbb{IR}$ for $\boldsymbol{f} : \mathbb{IR}^d \rightarrow \mathbb{IR}$
- $\mathbb{IY}$ can be $\mathbb{IR}^m$ for $\boldsymbol{f} : \mathbb{IR}^d \rightarrow \mathbb{IR}^m$
- $\mathbb{IY}$ can be $(\mathbb{IR}, \mathbb{IR}^m, \mathbb{IR}^{m^2})$ for range, gradient & Hessian of $\boldsymbol{f} : \mathbb{IR}^d \rightarrow \mathbb{IR}$
- Other obvious extensions include arithmetic over Taylor polynomial inclusion algebras
- In general the domain and range of $\boldsymbol{f}$ can be complete lattices with intervals and bisection operations
- We have seen several statistical applications of $\mathbb{Y}$-MRPs
- CODE: *mrs: a C++ class library for statistical set processing* by Bycroft, Harlow, Sainudiin, Teng and York.

References

Jaulin, L., Kieffer, M., Didrit, O. & Walter, E. (2001). *Applied interval analysis*. London: Springer-Verlag.

Meier, J., *Groups, graphs and trees: an introduction to the geometry of infinite groups*, CUP, Cambridge, 2008.

Neumaier, A., *Interval methods for systems of equations*, CUP, Cambridge, 1990.

Lugosi, G. and Nobel, A. (1996). Consistency of data-driven histogram methods for density estimation and classification. *The Annals of Statistics* **24** 687–706.

Sainudiin, R. and York, T. L. (2005). *An Auto-validating Rejection Sampler*. BSCB Dept. Technical Report BU-1661-M, Cornell University, Ithaca, New York.

Acknowledgements

- RS's external consulting revenues from the New Zealand Ministry of Tourism
- WT's Swedish Research Council Grant 2008-7510 that enabled RS's visits to Uppsala in 2006 and 2009
- Erskine grant from University of Canterbury that enabled WT's visit to Christchurch in 2011 & 2014
- EU Marie Curie International Research Staff Exchange Grants (CORCON 2014-2017 & CONSTRUMATH 2009-2012)
- University of Canterbury MSc Scholarship to JH.

Thank you!