# Moduli of regularity and rates of convergence for Fejér monotone sequences

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**Joint work with U. Kohlenbach and A. Nicolae**

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# Introduction

Many problems in applied mathematics can be brought into the following format:

Let $(X,d)$ be a metric space and $F : X \to \mathbb{R}$ be a function: find a zero $z \in X$ of $F$.

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# Rate of convergence for approximate sequence

1. Numerical methods, e.g. based on suitable iterative techniques, usually yield sequences $(x_n)$ in $X$ of approximate zeros, i.e. $|F(x_n)| < 1/n$.
2. Based on extra assumptions (e.g. the compactness of $X$, the Fejér monotonicity of $(x_n)$ and the continuity of $F$) one then shows that $(x_n)$ converges to an actual zero $z$ of $F$.

**An obvious question then concerns the speed of the convergence of $(x_n)$ towards $z$ and whether there is an effective rate of convergence.**

(i) if the zero for $F$ is unique it usually is possible to give an explicit effective rate of convergence,

(ii) if $F$ has many zeros, one usually can use the non-uniqueness to define a (computable) function $F$ for which $(x_n)$ does not have a computable rate of convergence.

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# Modulus of uniqueness

Even though sometimes left implicit, the effectivity of iterative procedures in the case of unique zeros rests on the existence of an effective so-called modulus of uniqueness:

*Definition 1 (Kohlenbach).*

Let $(X,d)$ be a metric space, $F : X \to \mathbb{R}$ with $\text{zer}\, F = \{z\}$ and $r > 0$. We say that $\phi : (0,\infty) \to (0,\infty)$ is a *modulus of uniqueness* for $F$ w.r.t. zer $F$ and $\overline{B}(z,r)$ if for all $\varepsilon > 0$ and $x \in \overline{B}(z,r)$ we have the following implication

$$|F(x)| < \phi(\varepsilon) \Rightarrow d(x,z) < \varepsilon.$$

Suppose now that $(x_n)$ is a sequence of $(1/n)$-approximate zeros contained in $\overline{B}(z,r)$ for some $r > 0$. If $\phi$ is a modulus of uniqueness for $F$ w.r.t. zer $F$ and $\overline{B}(z,r)$, then

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**Definition 2.**

Let $(X,d)$ be a metric space and $F: X \to \mathbb{R}$ with $\operatorname{zer} F \neq \emptyset$. Fixing $p \in \operatorname{zer} F$ and $r > 0$, we say that $\phi : (0,\infty) \to (0,\infty)$ is a *modulus of regularity* for $F$ w.r.t. $\operatorname{zer} F$ and $\overline{B}(p,r)$ if for all $\varepsilon > 0$ and $x \in \overline{B}(p,r)$ we have the following implication

$$|F(x)| < \phi(\varepsilon) \Rightarrow \operatorname{dist}(x, \operatorname{zer} F) < \varepsilon.$$

If there exists $p \in \operatorname{zer} F$ such that $\phi : (0,\infty) \to (0,\infty)$ is a modulus of regularity for $F$ w.r.t. $\operatorname{zer} F$ and $\overline{B}(p,r)$ for any $r > 0$, then $\phi$ is said to be a *modulus of regularity* for $F$ w.r.t. $\operatorname{zer} F$.

**Proposition 3.**

If $X$ is proper and $F$ is continuous, then for any $p \in \operatorname{zer} F$ and $r > 0$, $F$ has a modulus of regularity w.r.t. $\operatorname{zer} F$ and $\overline{B}(p,r)$.

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While the concept of a modulus of regularity and Proposition 3 had been used in various special situations in

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we develop it in this talk as a general tool towards a unified treatment of a number of concepts studied in convex optimization and fixed point theory such as metric subregularity, Hölder regularity, weak sharp minima etc. which can be seen as instances of the concept of regularity w.r.t. zer $F$ for suitable choices of $F$.

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# Fixed point

For nonempty, closed and convex subsets $C_1, C_2 \subseteq \mathbb{R}^n$ consider

$$T := R_{N_{C_2}} R_{N_{C_1}},$$

where $R_{N_C} = 2P_C - 1$ is the reflected resolvent.

**Borwein-Li-Tam** have shown that if $C_1, C_2$ are convex semialgebraic sets with $O \in C_1 \cap C_2$ which can be described by polynomials on $\mathbb{R}^n$ of degree $d > 1$, then (in our terminology), given $r > 0$, $T$ admits the following modulus of regularity w.r.t. Fix $T$ and $\overline{B}(O,r)$

$$\phi(\varepsilon) := 2(\varepsilon/\mu)^\gamma,$$

for suitable $\mu > 0$ and $\gamma \geq 1$ depending on $r$ and $d$.

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# Zeros of set valued operators

Let $X$ be a Banach spaces, $X^*$ its dual and $J : X \to 2^{X^*}$ the normalized duality mapping.

An operator $A : D \subset X \to 2^X \setminus \emptyset$ is called *$\psi$-strongly accretive*, where $\psi : [0,\infty) \to [0,\infty)$ is an increasing function with $\psi(0) = 0$, if

$$\langle x^* - y^*, x - y\rangle_+ \geq \psi(\|x-y\|)\|x-y\|, \tag{1}$$

for all $x, y \in X$, $x^* \in A(x)$, $y^* \in A(y)$, where $\langle v, u\rangle_+ = \max\{j(v) : j \in J(u)\}$.

Assume that $\operatorname{zer} A \neq \emptyset$ (hence it is a singleton) and let $x \in X, x \notin \operatorname{zer} A$. Taking in (1) $y \in \operatorname{zer} A$, we obtain

$$\|x^*\| \geq \frac{\langle x^*, x-y\rangle_+}{\|x-y\|} \geq \psi(\|x-y\|),$$

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# Minimization problems

Let $(X,d)$ metric space and $f: X \to (-\infty,\infty]$. Suppose that set of solutions $S$ of the associated minimization problem is nonempty and denote $m = \min_{x\in X} f(x)$.

The set $S$ is called a set of *$\psi$-boundedly weak sharp minima* for $f$, that is, for any bounded set $C \subseteq X$ with $C \cap S \neq \emptyset$, there exists an increasing function $\psi = \psi_C : [0,\infty) \to [0,\infty)$ satisfying $\psi(0) = 0$ such that

$$f(x) \geq m + \psi_C(\mathrm{dist}(x,S)), \quad (2)$$

holds for all $x \in C$.
Fixing $p \in S$ and $r > 0$, a modulus of regularity for $f$ w.r.t. $S$ and $\overline{B}(p,r)$ can be defined by $\phi : (0,\infty) \to (0,\infty)$, $\phi(\varepsilon) := \psi_C(\varepsilon)$, where $C = \overline{B}(p,r)$.

The case $\psi_C(\varepsilon) = \alpha\varepsilon$ with $\alpha > 0$, was introduced by **Burke-Ferris**.

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# Quantitative version

Let $H$ be a Hilbert space and $f : H \to (-\infty, \infty]$ a proper, convex and lower semi-continuous function which attains its minimum. Take $p \in \operatorname{argmin} f$ and $r, r' > 0$. Consider the following statements:

1. The function $f$ admits a modulus of regularity w.r.t. $\operatorname{argmin} f$ and $\overline{B}(p,r)$.
2. For $\gamma > 0$, the resolvent of $f$, $J_{\gamma\partial f}$, admits a modulus of regularity w.r.t. $\operatorname{Fix} J_{\gamma\partial f}$ and $\overline{B}(p,r)$.
3. The subdifferential of $f$, $\partial f$, admits a modulus of regularity w.r.t. $\operatorname{zer} \partial f$ and $\overline{B}(p,r')$.

The following theorem can be considered a quantitative version of the following well known fact:

$$\operatorname{argmin} f = \operatorname{zer} \partial f = \operatorname{Fix} J_{\gamma\partial f}.$$

# Quantitative version

*Theorem 4.*

(i) If $f|_{\overline{B}(p,r+1)}$ is additionally uniformly continuous admitting a modulus of uniform continuity, then 1 implies 2 for all $\gamma > 0$. If $\phi$ is a modulus of regularity for $f$ w.r.t. $\operatorname{argmin} f$ and $\overline{B}(p,r)$, and $\rho$ is a modulus of uniform continuity for $f|_{\overline{B}(p,r+1)}$, then

$$\phi^*(\varepsilon) = \min\left\{\rho\left(\frac{\phi(\varepsilon)}{2}\right), \frac{\gamma\phi(\varepsilon)}{2(r+1)}, 1\right\}$$

is a modulus of regularity for $J_{\gamma\partial f}$ w.r.t. Fix $J_{\gamma\partial f}$ and $\overline{B}(p,r)$.

(ii) If there exists $\gamma > 0$ such that 2 holds, then 1 is satisfied. Moreover 3 holds too if $r' < r$.

(iii) If $\partial f$ is single-valued, $r' = r$ and $(\mathrm{Id} + \gamma\partial f)|_{\overline{B}(p,r+1)}$, $\gamma > 0$, is uniformly continuous admitting a modulus of uniform continuity, then 3 implies 2.

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$$\phi^*(\varepsilon) = \min\left\{\rho\left(\frac{\phi(\varepsilon)}{2}\right), \frac{\gamma\phi(\varepsilon)}{2(r+1)}, 1\right\}$$

is a modulus of regularity for $J_{\gamma\partial f}$ w.r.t. Fix $J_{\gamma\partial f}$ and $\overline{B}(p,r)$.

(ii) If there exists $\gamma > 0$ such that 2 holds, then 1 is satisfied. Moreover 3 holds too if $r' < r$.

(iii) If $\partial f$ is single-valued, $r' = r$ and $(\mathrm{Id} + \gamma\partial f)|_{\overline{B}(p,r+1)}$, $\gamma > 0$, is uniformly continuous admitting a modulus of uniform continuity, then 3 implies 2.

## Quantitative version

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# Rates of convergence

*Theorem 5.*

Let $(X,d)$ be a metric space and $F : X \to \mathbb{R}$ with $\operatorname{zer} F \neq \emptyset$. Suppose that $(x_n)$ is a sequence in $X$ which is Fejér monotone w.r.t. $\operatorname{zer} F$, $b > 0$ is an upper bound on $d(x_0, p)$ for some $p \in \operatorname{zer} F$ and there exists $\alpha : (0,\infty) \to \mathbb{N}$ such that

$$\forall \varepsilon > 0 \, \exists n \leq \alpha(\varepsilon) \, (|F(x_n)| < \varepsilon).$$

If $\phi$ is a modulus of regularity for $F$ w.r.t. $\operatorname{zer} F$ and $\overline{B}(p,b)$, then $(x_n)$ is a Cauchy sequence with Cauchy modulus

$$\forall \varepsilon > 0 \, \forall n, \tilde{n} \geq \alpha(\phi(\varepsilon/2)) \, (d(x_n, x_{\tilde{n}}) < \varepsilon) \tag{3}$$

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$$\forall \varepsilon > 0 \, \forall n \geq \alpha(\phi(\varepsilon)) \, (\operatorname{dist}(x_n, \operatorname{zer} F) < \varepsilon). \tag{4}$$

In particular, if $X$ is complete and $\operatorname{zer} F$ is closed, then $(x_n)$ converges to some $z \in \operatorname{zer} F$ with a rate of convergence $\alpha(\phi(\varepsilon/2))$.

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# Algorithms

1. **Picard:** Minimizing the distance between two nonintersecting sets in CAT(0) spaces
2. **Cyclic:** CFP.
3. **Mann:** Finding a zero of the sum of two monotone operators.
4. **Proximal point:** Finding a minimizer of a convex function.

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# Picard iteration

Let $X$ be a complete metric space and $T : X \to X$ a nonexpansive mapping with Fix $T \neq \emptyset$. The *Picard iteration* generates starting from $x_0 \in X$ the sequence given by

$$x_{n+1} = T^n x \quad \text{for any } n \geq 0. \tag{5}$$

It is well-known that $(x_n)$ is Fejér monotone w.r.t. Fix $T$. Moreover, Fix $T$ is closed .
Let $b > 0$ be an upper bound on $d(x_0, p)$ for some $p \in$ Fix $T$. By Fejér monotonicity, $(x_n) \subseteq \overline{B}(p, b)$. Considering $F : X \to \mathbb{R}, F(x) = d(x, Tx)$, if $\phi$ is a modulus of regularity for $F$ w.r.t. zer $F$ and $\overline{B}(p, b)$, and $\alpha$ is a rate of asymptotic regularity for $(x_n)$, then, applying Theorem 5, we can deduce that $(x_n)$ converges to a fixed point of $T$ with a rate of convergence $\alpha(\phi(\varepsilon/2))$.

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# Distance between two sets

Let $X$ is a complete CAT$(0)$ space and $U, V \subseteq X$ are nonempty, closed and convex with $U \cap V = \emptyset$, then one aims to find best approximation pairs $(u, v) \in U \times V$ such that $d(u, v) = \text{dist}(U, V)$. This problem was studied in Hilbert spaces by **Bauschke and Borwein.**

Denote $\rho = \text{dist}(U, V)$ and suppose that $S = \{(u, v) \in U \times V : d(u, v) = \rho\} \neq \emptyset$. Given $x_0 \in X$, we consider the sequence $(x_n)$ given by (5), where $T : H \to H$, $T = P_U \circ P_V$.

- $T$ is nonexpansive.
- If $(u, v) \in S$, then $u \in \text{Fix } T$, so $\text{Fix } T \neq \emptyset$. At the same time, if $u \in \text{Fix } T$, then $(u, P_V u) \in S$.
- $(x_n)$ is asymptotically regular with a rate of asymptotic regularity

$$\alpha(\varepsilon) = \left[\frac{\rho^2 + b^2}{\varepsilon^2}\right] + 1,$$

were $b$ an upper bound on $d(x_0, p)$ for some fixed $p \in \text{Fix } T$.

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# Distance between two sets

Assume that the sets $U$ and $V$ are additionally *boundedly regular* which means that for any bounded set $K \subseteq X$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in K$ we have the following implication

$$\mathrm{dist}(x,U) < \delta \wedge \mathrm{dist}(x,V) < \rho + \delta \Rightarrow \mathrm{dist}(x, \mathrm{Fix}\, T) < \varepsilon. \tag{6}$$

Let $\varepsilon > 0$ and consider $K = \overline{B}(p,b)$. Since $U,V$ are boundedly regular, there exists $\delta = \delta(\varepsilon) > 0$ such that (6) holds for $x \in K$. Then

$$\phi(\varepsilon) = \frac{\rho\delta}{b+\rho},$$

is a modulus of regularity for $T$. w.r.t. Fix $T$ and $K$.

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# Proximal point algorithm

Let $H$ be a Hilbert space and $A : D \subset H \to 2^H$ a maximal monotone operator with $\mathrm{zer}\,A \neq \emptyset$. Note that $\mathrm{zer}\,A$ is closed. Given $x_0 \in H$ and a sequence of positive numbers $(\gamma_n)$, the *proximal point algorithm* (PPA) generates the sequence defined by

$$x_{n+1} = J_{\gamma_n A} x_n. \tag{7}$$

It is well-known that $(x_n)$ is Fejer monotone w.r.t. $\mathrm{zer}\,A$.
Denoting $F : D \to \mathbb{R}$, $F(x) = \mathrm{dist}(O, A(x))$ and $u_n = \frac{x_n - x_{n+1}}{\gamma_n}$, we have $F(x_{n+1}) \leq \|u_n\|$ for all $n \in \mathbb{N}$.

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Take $b > 0$ an upper bound of $\|x_0 - p\|$ for some $p \in \text{zer}\, A$. If $\sum_{n=0}^{\infty} \gamma_n^2 = \infty$ with a rate of divergence $\theta$, then,

$$\theta\left(\left\lceil \frac{b^2}{\varepsilon^2} \right\rceil\right)$$

is a rate of convergence of $(\|u_n\|)$ towards 0. Therefore,

$$\forall \varepsilon > 0 \forall n \geq \theta\left(\left\lceil \frac{b^2}{\varepsilon^2} \right\rceil\right) + 1 \; (F(x_n) \leq \varepsilon)$$

and so $\forall \varepsilon > 0 \; \left(F(x_{\alpha(\varepsilon)}) < \varepsilon\right)$, where $\alpha(\varepsilon) = \theta\left(\left\lceil \frac{2b^2}{\varepsilon^2} \right\rceil\right) + 1$.

Thus, if $\phi$ is a modulus of regularity for $A$ w.r.t. zer $A$ and $\overline{B}(p,b)$, then $(x_n)$ converges to some $z \in \text{zer}\, A$ with a rate of convergence $\alpha(\phi(\varepsilon/2, b))$.

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Let $f : H \to (-\infty, \infty]$ be a proper, convex and uniformly continuous function with a modulus of uniform continuity $\rho$ and $S = \operatorname{argmin} f \neq \emptyset$.

If $S$ is a set of $\psi$-boundedly global weak sharp minima for $f$, then $\phi^* : (0,\infty) \to (0,\infty)$, $\phi^*(\varepsilon) = \psi_C(\varepsilon)$, where $C = \overline{B}(p, b+1)$, is a modulus of regularity for $f$ w.r.t. $S$ and $\overline{B}(p, b+1)$ .

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