Minimization of Quadratic Functions on Convex Sets without Asymptotes

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INTRODUCTION

$q : \mathbb{R}^n \to \mathbb{R}$
$q(x) := \frac{1}{2}x^\top Ax + b^\top x + c$
$A = A^\top \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$

F convex set in $\mathbb{R}^n$

F is an *FW*-set if every quadratic function q which is bounded below on F attains its infimum on F.

Every compact convex set is an *FW*-set.

Every convex polyhedron P is an *FW*-set.
(M. Frank, P. Wolfe, 1956).

F is a *qFW*-set if the property holds for every quadratic function q which is in addition quasiconvex on F.

F is a *cFW*-set if the property holds for every quadratic function q which is in addition convex on F.

PROPOSITION.
Affine images of cFW-sets are cFW-sets.
Affine images of qFW-sets are qFW-sets.
Affine images of FW-sets are FW-sets.

PROPOSITION.
If the union of two FW-sets is convex, then it is FW, too.
The analogous statement holds for qFW-sets.

f-ASYMPTOTES

M affine manifold in $\mathbb{R}^n$
F closed convex set in $\mathbb{R}^n$

M is called an *f-asymptote* (Klee, 1960) of F if $F \cap M = \emptyset$ and dist$(F, M) = 0$.

THEOREM.
Let F be a convex set in $\mathbb{R}^n$.
Then the following statements are equivalent:
(i) F is qFW.
(ii) F is cFW.
(iii) F has no f-asymptotes.
(iv) $P(F)$ is closed for every orthogonal projection P.

THEOREM. (Z.-Q. Luo, S. Zhang, 1999).
Under any linear (or affine) map, the image of a convex region in $\mathbb{R}^n$ defined by convex quadratic constraints is always a closed set.

COROLLARY (Z.-Q. Luo, S. Zhang, 1999).
A convex region in $\mathbb{R}^n$ defined by convex quadratic constraints is always qFW.

COROLLARY.
FW sets have no f-asymptotes.

COROLLARY.
Any finite intersection of qFW-sets is again qFW.

COROLLARY.
If $F_1, ..., F_m$ are qFW-sets,
then the Cartesian product $F := F_1 \times ... \times F_m$ is qFW.

PROOF.
Suppose $F_i \subset \mathbb{R}^{d_i}$.
Then

$$\begin{aligned} F &= \left(F_1 \times \mathbb{R}^{d_2} \times \cdots \times \mathbb{R}^{d_m}\right) \\ &\quad \cap \left(\mathbb{R}^{d_1} \times F_2 \times \mathbb{R}^{d_3} \times \cdots \times \mathbb{R}^{d_m}\right) \\ &\quad \cap \cdots \\ &\quad \cap \left(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_{m-1}} \times F_m\right). \end{aligned}$$

EXAMPLE (Z.-Q. Luo, S. Zhang, 1999).

$$\begin{array}{ll} \text{minimize} & q(x) := x_1^2 - 2x_1x_2 + x_3x_4 \\ \text{subject to} & c_1(x) := x_1^2 - x_3 \le 0 \\ & c_2(x) := x_2^2 - x_4 \le 0 \\ & x := (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \end{array}$$

$F := \{x \in \mathbb{R}^4 : c_1(x) \le 0, c_2(x) \le 0\}$ is a qFW-set.

$$\begin{aligned} \inf_{x \in F} q(x) &= \inf \left\{ x_1^2 - 2x_1x_2 + x_1^2 x_2^2 \right\} \\ &= \inf \left\{ x_1^2 + (1 - x_1x_2)^2 \right\} - 1 = -1 \end{aligned}$$

$q(x) > -1$ for every $x \in F$

EXAMPLE.
$F := \{(x, y) \in \mathbb{R}^2 : x^2 + \exp(-x^2) - y \leq 0\}$
is convex and closed.

F does not have f-asymptotes.

$q\,(x, y) := y - x^2$

$q\,(x, y) \geq \exp(-x^2) > 0$ for every $(x, y) \in F$

$$\begin{aligned} 0 &\leq \inf_{(x,y)\in F} q\,(x, y) \leq \inf_{x\in\mathbb{R}} q\left(x, x^2 + \exp(-x^2)\right) \\ &= \inf_{x\in\mathbb{R}} \exp(-x^2) = 0 \end{aligned}$$

PROPOSITION.
Let F be a qFW-set in $\mathbb{R}^n$ and q be a quadratic function bounded below on F such that its restriction to F has a nonempty convex sublevel set.
Then q attains its infimum on F.

MOTZKIN DECOMPOSABLE SETS

F nonempty closed convex set in $\mathbb{R}^n$

F is called Motzkin decomposable if there exists a compact convex set C and a closed convex cone D such that $F = C + D$.

EXAMPLE.
$D := \{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, xy - z^2 \geq 0\}$
is a closed convex cone.

$q(x, y, z) := x^2 + (z - 1)^2$

$q\left(\frac{1}{k}, \frac{(k+1)^2}{k}, 1 + \frac{1}{k}\right) = \frac{2}{k^2} \longrightarrow 0$
$\inf_{x \in D} q\,(x, y, z) = 0$
$q\,(x, y, z) > 0$ for every $(x, y, z) \in D$

EXAMPLE.

$$F := \left\{ (x,y,z) \in \mathbb{R}^3 : z \geq \left(x^2 + y^2\right)^{\frac{1}{2}} \right\}$$

is convex and closed.

$$q(x,y,z) := (x-1)^2 - y + z$$

$$q\left(1, k, \left(1+k^2\right)^{\frac{1}{2}}\right) = \left(1+k^2\right)^{\frac{1}{2}} - k \longrightarrow 0$$

$$\begin{aligned} (x,y,z) &\in F \\ &\Longrightarrow z \geq \left(x^2+y^2\right)^{\frac{1}{2}} \geq y \\ &\Longrightarrow q(x,y,z) \geq 0 \end{aligned}$$

$$\begin{aligned} (x,y,z) &\in F \\ &\Longrightarrow \text{either } x \neq 1 \text{ or } z \geq \left(1+y^2\right)^{\frac{1}{2}} > y \\ &\Longrightarrow q(x,y,z) > 0 \end{aligned}$$

THEOREM.
Let F be a Motzkin decomposable closed convex set. Then the following statements are equivalent:
(i) F is FW.
(ii) F is qFW.
(iii) 0^+F is polyhedral.

PROOF (sketch):

(i) $\Longrightarrow$ (ii) is obvious.

(ii) $\Longrightarrow$ (iii) uses Mirkil's Theorem:
THEOREM (H. Mirkil, 1957).
If a closed convex cone has all its 2-dimensional projections closed, then it is polyhedral.

(iii) $\Longrightarrow$ (i) is based in the following facts:

1) If $F = C + 0^+F$, with C compact and convex, and

$$q(x) := \frac{1}{2}x^\top Ax + b^\top x,$$

then

$$\inf_{x\in F} q(x) = \inf_{y\in C}\left\{q(y) + \inf_{z\in 0^+F}\left\{y^\top Az + q(z)\right\}\right\}.$$

2) Let D be a polyhedral convex cone and define

$$f(c) := \inf_{x\in D}\left\{c^\top x + \frac{1}{2}x^\top Gx\right\}.$$

We assume that $x^\top Gx \geq 0$ for every $x \in D$.

Then one has:

$$\mathrm{dom}(f) = \left\{c : c^\top x \geq 0 \ \forall x \in D \text{ s.t. } x^\top Gx = 0\right\}.$$

The right hand side of this equality is a polyhedral convex cone.

Consequently, f is continuous relative to $\mathrm{dom}(f)$.

$$F := \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0, xy \geq 1\}$$

COROLLARY.
A Motzkin decomposable set without f-asymptotes is FW.

THEOREM.
Any nonempty intersection of finitely many Motzkin decomposable FW-sets is again a Motzkin decomposable FW-set.

PROPOSITION.
If the preimage $T^{-1}(F)$ of a Motzkin decomposable FW-set F under a linear mapping T is nonempty, it is a Motzkin decomposable FW-set too.

PROOF.

$$T^{-1}(F) = \left(T_{(KerT)^{\perp}}\right)^{-1}(F \cap R(T)) + KerT$$

q-ASYMPTOTES

A nonempty closed set in $\mathbb{R}^n$
F nonempty closed convex set in $\mathbb{R}^n$

A is said to be *asymptotic* to F if
$A \cap F = \emptyset$ and $\text{dist}(F, A) = 0$.

$Q := \{x \in \mathbb{R}^n : q(x) := \frac{1}{2}x^{\mathsf{T}}Ax + 2b^{\mathsf{T}}x + c = 0\}$
is a q-*asymptote* of F if
$F \cap Q = \emptyset$ and $\text{dist}(Q \times \{0\}, \{(x, q(x)) : x \in F\}) = 0$.

Q is a q-*asymptote* of F

$$\Longleftrightarrow$$

$Q \times \{0\}$ is asymptotic to $graph\left(q_{|F}\right)$

Q is a q-*asymptote* of $F \Rightarrow Q$ is asymptotic to F

EXAMPLE.
$F := \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$
$Q := \{(x,y) \in \mathbb{R}^2 : q(x,y) := xy + 1 = 0\}$

Q is asymptotic to F.

$$q(x,y) \geq 1 \text{ for every } (x,y) \in F$$

$$\mathrm{dist}(Q \times \{0\}, \{((x,y), q(x,y)) : (x,y) \in F\}) \geq 1$$

Q is not a q-asymptote of F.

THEOREM.
A convex set F is FW
if and only if
it has no q-asymptotes.

F, Q be closed sets, $F \cap Q = \emptyset$ and $\mathrm{dist}(F, Q) = 0$
Q' closed set.

Q' is *squeezed in between* F and Q if:
$F \cap Q' = \emptyset = Q \cap Q'$
and for every $x \in F$ and $y \in Q$ one has $[x,y] \cap Q' \neq \emptyset$.

$Q_\alpha := \{x \in \mathbb{R}^n : q(x) - \alpha = 0\}$

PROPOSITION.
Let F be a closed convex set.
Then Q_0 is a q-asymptote of F
if and only if
Q_0 is asymptotic to F and no Q_α can be squeezed in between F and Q.

PROPOSITION.
Let F be a closed convex set in $\mathbb{R}^n$.
Let $Q := \{x \in \mathbb{R}^n : q(x) = 0\}$ be a quadric.
Suppose Q degenerates to an affine subspace.
Then Q is a q-asymptote of F
if and only if
it is an f-asymptote of F.
Moreover, for any f-asymptote M of F there exists a quadric representation

$$M := \{x \in \mathbb{R}^n : q(x) = 0\},$$

and then M is also a q-asymptote of F.

PROOF.

$M := \{x \in \mathbb{R}^n : Ax - b = 0\}$ f-asymptote of F

$Q_\alpha := \{x : \|Ax - b\|^2 - \alpha = 0\}$

$Q_0 = M$

Suppose Q_α can be squeezed in between Q_0 and F.

$\alpha > 0$

$F \subset \{x : \|Ax - b\|^2 > \alpha\}$

For $x \in F$ and $y \in M$ one has

$$\begin{aligned} \alpha^{\frac{1}{2}} &< \|Ax - b\| = \|Ax - Ay\| = \|A(x - y)\| \\ &\leq \|A\|\|x - y\| \end{aligned}$$

$$\|x - y\| > \frac{\alpha^{\frac{1}{2}}}{\|A\|}$$

dist$(M, F) \geq \frac{\alpha^{\frac{1}{2}}}{\|A\|} > 0$, contradiction