On the behavior of the Douglas–Rachford algorithm in possibly nonconvex settings

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Based on joint works with

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Outline

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1. Introduction
2. Behavior of DR algorithm in possibly inconsistent case
3. Finite convergence
4. A Lyapunov-type approach to convergence theory
5. Local linear convergence

Fundamental feasibility problem

Unless stated otherwise,

$$X : \text{a real Hilbert space},$$
$$A, B : \text{closed (possibly nonconvex) subsets of } X.$$

The fundamental feasibility problem asks to

$$\text{find } x \in A \cap B.$$

In the inconsistent case, i.e., $A \cap B = \varnothing$, it can be naturally formulated as finding a best approximation pair relative to A and B:

$$\text{find } (a, b) \in A \times B \text{ such that } \|a - b\| = \inf \|A - B\|.$$

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- is typically approached by projection methods which combine projectors and their variants in a suitable way to generate a sequence converging to a solution of the problem.

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Projectors and relaxed projectors

Let C be a nonempty closed set in X and $\lambda \in \mathbb{R}_+$. The projector onto C is

$$P_C \colon X \rightrightarrows C \colon x \mapsto P_C x := \operatorname{argmin}_{c \in C} \|x - c\|.$$

The relaxed projector for C with parameter λ is defined by

$$P_C^\lambda := (1-\lambda)\operatorname{Id} + \lambda P_C.$$

- $P_C^0 = \operatorname{Id}$, $P_C^1 = P_C$,
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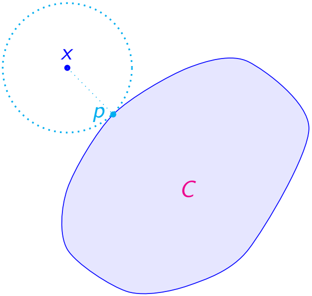

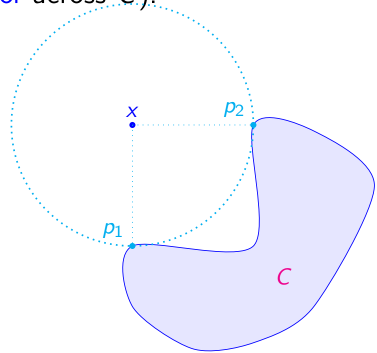

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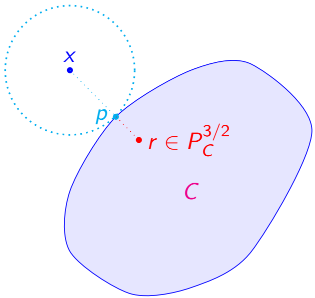

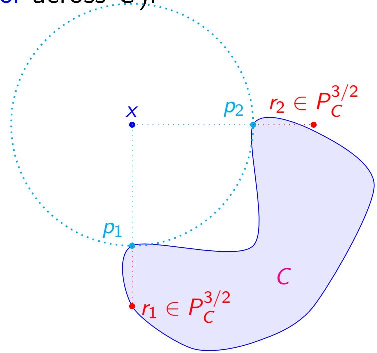

Douglas–Rachford (DR) algorithm

Use the DR operator

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to generate a DR sequence $(z_n)_{n\in\mathbb{N}}$ by

$$(\forall n \in \mathbb{N}) \quad z_{n+1} \in T_{A,B} z_n, \quad \text{where } z_0 \in X.$$

- $T_{A,B}$ is single-valued when A and B are convex.
- $z \in \operatorname{Fix} T_{A,B} := \{z \in X \mid z \in T_{A,B} z\} \quad \Rightarrow \quad (\exists a \in P_A z)\ a \in A \cap B.$

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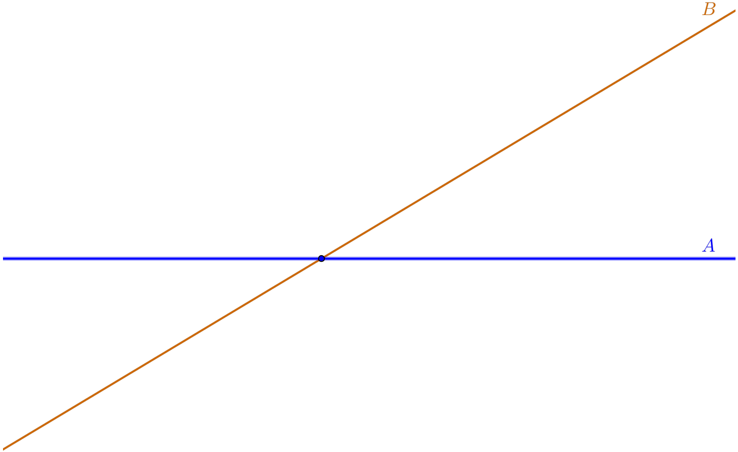

While $(z_n)_{n\in\mathbb{N}}$ "spirals" towards the origin, the "shadow sequence" $(P_A z_n)_{n\in\mathbb{N}}$ occasionally gets very close to the origin!

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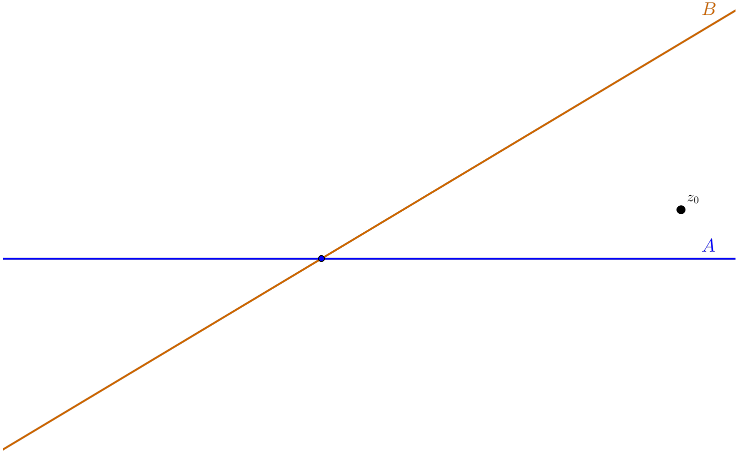

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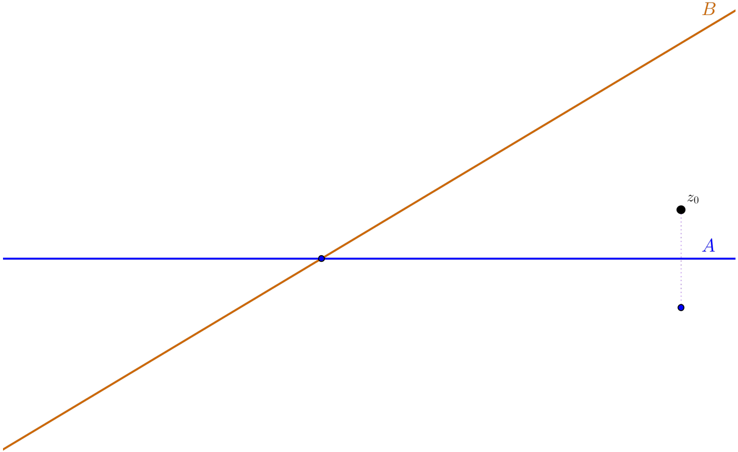

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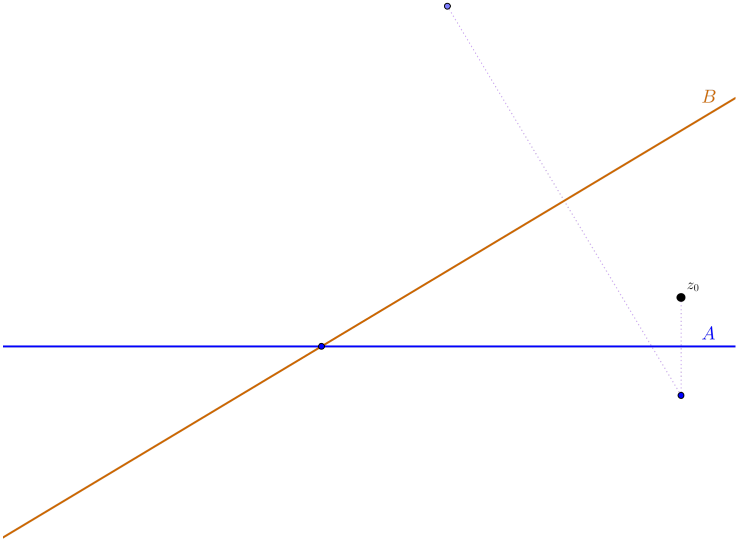

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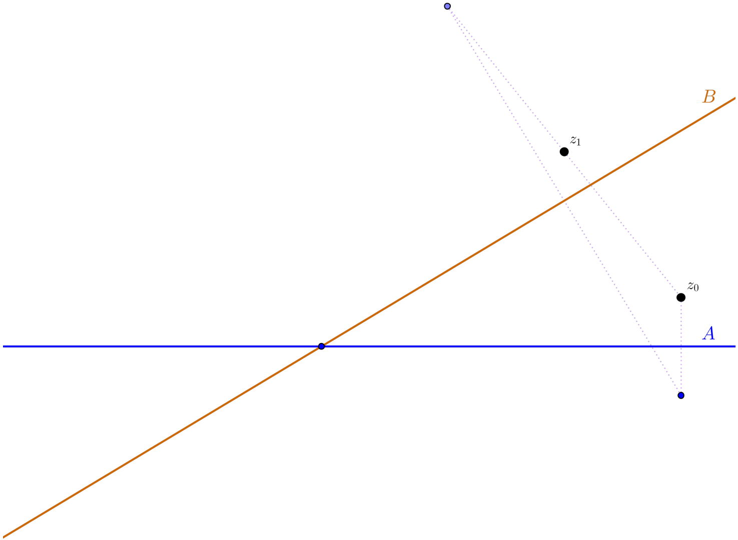

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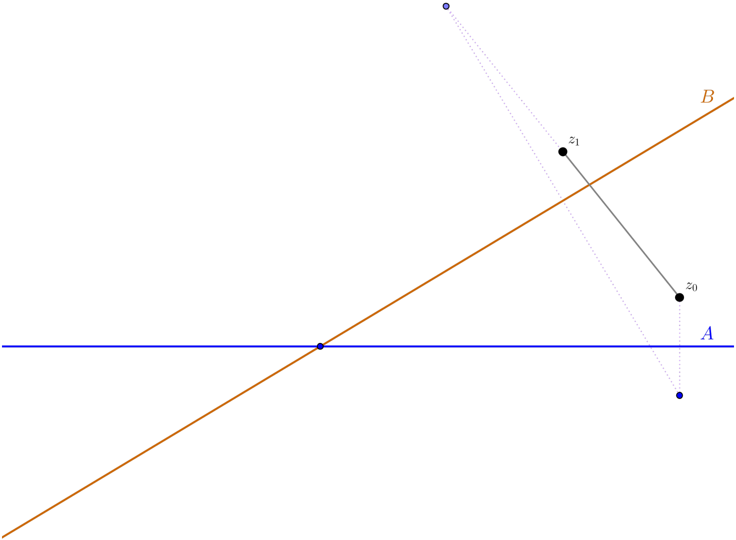

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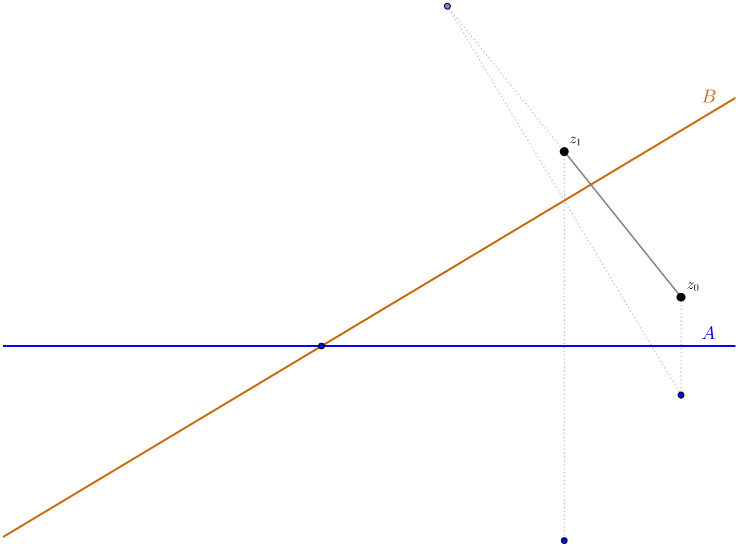

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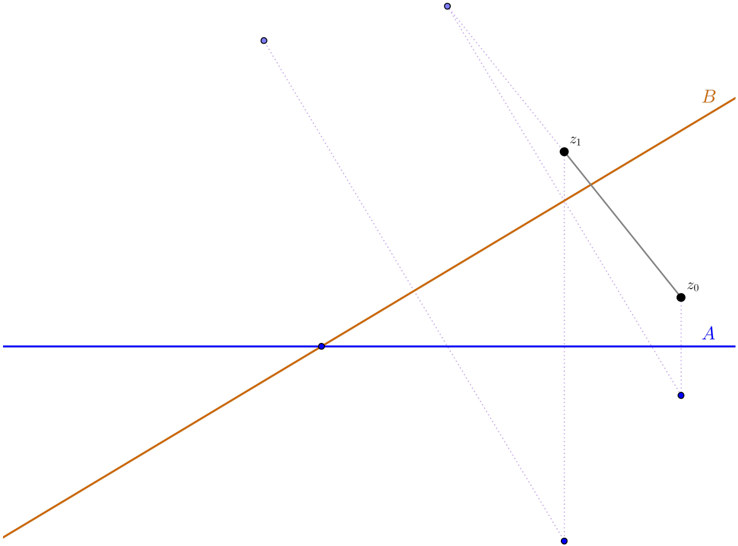

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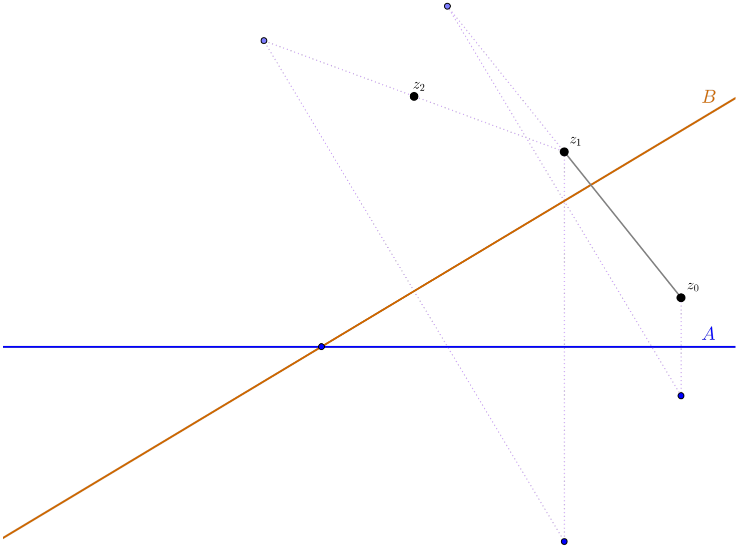

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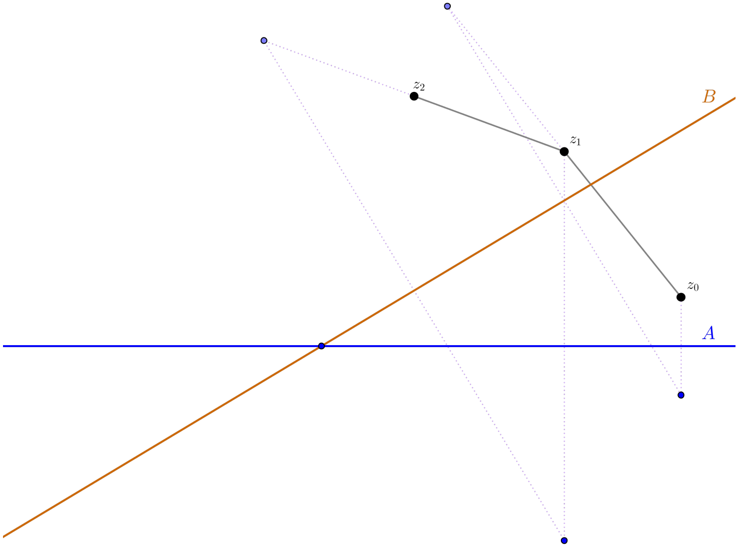

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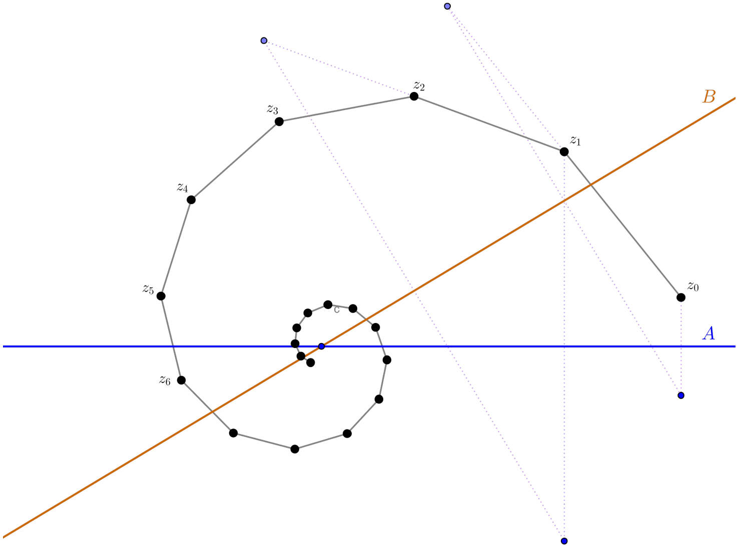

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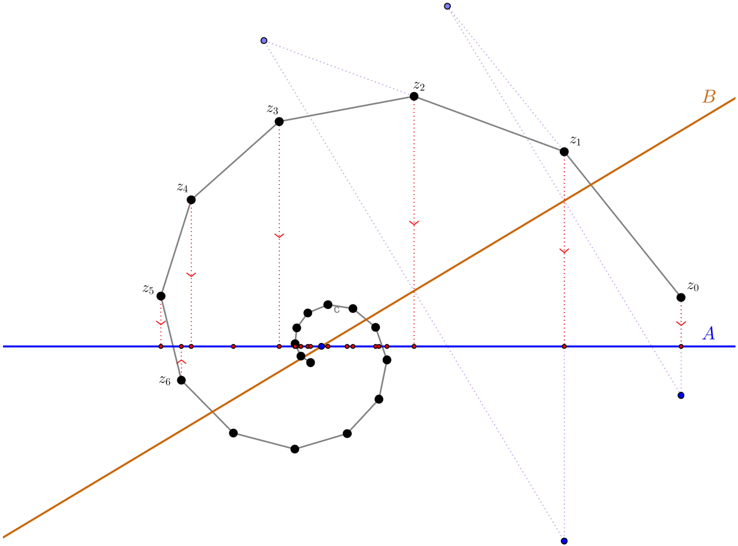

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Outline

Convex-convex case

Suppose that A and B are convex.

Fact (Lions–Mercier 1979, Svaiter 2011, Bauschke–Combettes–Luke 2004)

If $A \cap B \neq \varnothing$, then $z_n \rightharpoonup z \in \operatorname{Fix} T = (A \cap B) + N_{A-B}(0)$ and $P_A z_n \rightharpoonup P_A z \in A \cap B$; otherwise, $\|z_n\| \to +\infty$.

Now assume that $g := P_{\overline{B-A}} 0 \in B - A$, or equivalently,

$$E := A \cap (B - g) \neq \varnothing \quad \text{and} \quad F := (A + g) \cap B \neq \varnothing.$$

Fact (Bauschke–Combettes–Luke 2004)

The sequence $(P_A z_n)_{n \in \mathbb{N}}$ is bounded and its weak cluster points lie in E.

Here $N_{A-B}(0)$ is the normal cone of the set $A - B = \{a - b \mid a \in A, b \in B\}$.

Affine-convex case

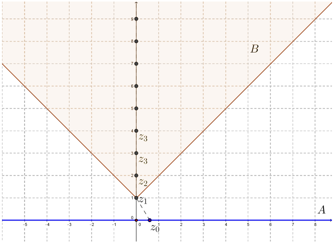
Theorem (Bauschke–D–Moursi 2016, When A is a closed affine subspace)

1. $P_A z_n \rightharpoonup a \in E = A \cap (B - g)$.
2. *No general conclusion can be drawn about the sequence* $(P_B z_n)_{n\in\mathbb{N}}$.

This is strengthened by Bauschke–Moursi 2017 for A convex.

Example ($X = \mathbb{R}^2$, $A = \mathbb{R} \times \{0\}$, $B = \text{epi}(|\cdot| + 1)$)

For $z_0 \in [-1, 1] \times \{0\}$, $z_n = (0, n) \in B$, and $\|P_B z_n\| = \|z_n\| = n \to \infty$.



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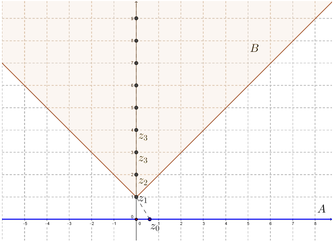

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Spingarn's method

Consider the problem to find a *least-squares solution* of $\bigcap_{j=1}^{M} C_j$, i.e., to

$$\text{find minimizers of } \sum_{j=1}^{M} d_{C_j}^2, \tag{1}$$

where $C_1, \ldots, C_M$ are nonempty closed convex (possibly nonintersecting) subsets of X with corresponding distance functions $d_{C_1}, \ldots, d_{C_M}$.

Set $\mathbf{X} := X^M$, $\mathbf{A} := \{(x, \ldots, x) \in \mathbf{X} \mid x \in X\}$, and $\mathbf{B} := C_1 \times \cdots \times C_M$. Assume that $\mathbf{g} = (g_1, \ldots, g_M) := P_{\overline{\mathbf{B}-\mathbf{A}}} 0 \in \mathbf{B} - \mathbf{A}$. Then

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Corollary

Let $(\mathbf{z}_n)_{n\in\mathbb{N}}$ *be a DR sequence for* $(\mathbf{A}, \mathbf{B})$. *Then*

$$P_{\mathbf{A}} \mathbf{z}_n \rightharpoonup \mathbf{z} = (z, \ldots, z) \in \mathbf{A} \cap (\mathbf{B} - \mathbf{g}),$$

where $z \in \bigcap_{j=1}^{M} (C_j - g_j)$ *and z is a least-squares solution of* (1).

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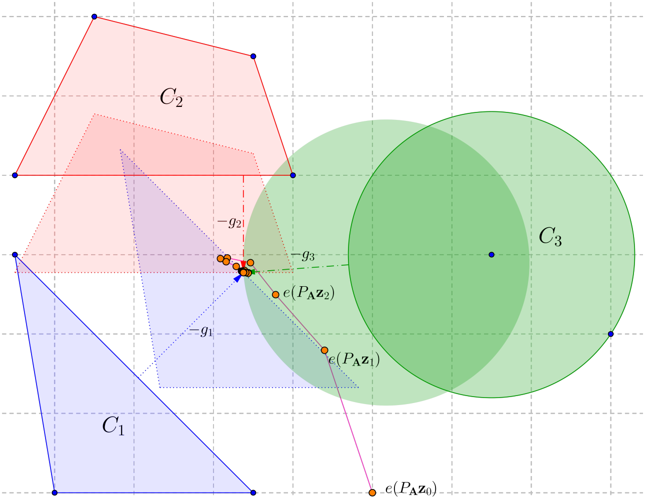
Find a point in the generalized intersection



Three closed convex sets are shown along with their translations forming the generalized intersection. The first few terms of $(e(P_{\mathbf{A}}\mathbf{z}_n))_{n\in\mathbb{N}}$ (yellow points) are also depicted. Here $e : \mathbf{A} \to \mathbb{R}^2 : (x, x, x) \mapsto x$.

Outline

Presence of Slater's condition ($A \cap \operatorname{int} B \neq \varnothing$)

From now on, X is finite-dimensional.

Lemma

If A and B are convex and $0 \in \operatorname{int}(A - B)$, then $z_n \to z \in A \cap B$; the convergence is finite provided that $z \in A \cap \operatorname{int} B$.

Theorem (Bauschke–D–Noll–Phan 2016, Bauschke–D 2017)

Suppose that $A \cap \operatorname{int} B \neq \varnothing$. Then the DR algorithm converges finitely to a point in $A \cap B$ in each of the following cases:

1. *A is an affine subspace and B is a polyhedron.*
2. *$A \in \{X \times \{0\}, X \times \mathbb{R}_+, X \times \mathbb{R}_-\}$ and $B = \operatorname{epi} f$, where $f\colon X \to \left]-\infty, +\infty\right]$ is convex, l.s.c., and proper.*
3. *A is a hyperplane/halfspace and B is a finite intersection of closed balls B_j such that $(\forall x \in A \cap \operatorname{bdry} B)(\exists! B_j)\ x \in \operatorname{bdry} B_j$.*

int C: the interior of C.

Absence of Slater's condition

- In the case of an affine subspace and a polyhedron, if the Slater's condition is replaced by "$A \cap \operatorname{ri} B \neq \varnothing$", then finite convergence fails in general, e.g., the case of two lines in $\mathbb{R}^2$.
- If $A \in \{X \times \{0\}, X \times \mathbb{R}_-\}$ and $B = \operatorname{epi} f$, where $\inf_X f \geq 0$ and f is differentiable at its minimizers, then $(P_A z_n)_{n\in\mathbb{N}}$ and hence $(z_n)_{n\in\mathbb{N}}$ do not converge finitely whenever $z_0 = (x_0, \rho_0) \in B$ with $x_0 \notin \operatorname{argmin} f$.

Theorem (Bauschke–D 2017)

Suppose that A is a hyperplane/halfspace and that $A \cap B \neq \varnothing$. Then the DR sequence converges finitely to a point in $z \in \operatorname{Fix} T_{A,B}$ with $P_A z \in A \cap B$ in each of the following cases:

1. *B is a halfspace of X.*
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When one set is finite

Suppose that B is a finite subset of X and let $(z_n)_{n\in\mathbb{N}}$ be a DR sequence for (A, B).

Theorem (Bauschke–D 2017)

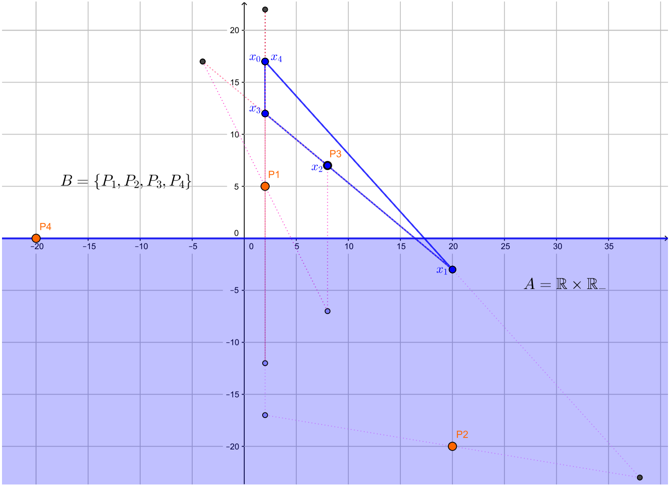
If A is an affine subspace/a halfspace, $A \cap B \neq \varnothing$, and the sequence $(z_n)_{n\in\mathbb{N}}$ is asymptotically regular, i.e., $z_n - z_{n+1} \to 0$, then $(z_n)_{n\in\mathbb{N}}$ converges in finitely many steps to a point $z \in \operatorname{Fix} T_{A,B}$ with $P_A z \in A \cap B$.

Theorem (Bauschke–D 2017)

If A is a hyperplane/halfspace and B is contained in one of two halfspaces generated by A, then either

1. *$(z_n)_{n\in\mathbb{N}}$ converges finitely to a point $z \in \operatorname{Fix} T_{A,B}$ with $P_A z \in A \cap B$, or*
2. *$A \cap B = \varnothing$ and $\|z_n\| \to +\infty$ in which case $(P_A z_n)_{n\in\mathbb{N}}$ converges finitely to a best approximation solution $a \in A$ relative to A and B.*

Without asymptotic regularity or “one-side” property



A 4-cycle of the DR algorithm for a halfspace and a finite set. Interchanging the roles of two sets gives finite convergence, as shown by Aragón Artacho–Borwein–Tam 2016.

Periodic behavior

Theorem (Bauschke–D–Lindstrom 2017)

Suppose that A is a hyperplane and that $B = \{b_1, b_2\}$, where b_1 and b_2 do not belong to the same halfspace generated by A. Let $(z_n)_{n\in\mathbb{N}}$ be a DR sequence for (A, B). Then

1. *$(z_n)_{n\in\mathbb{N}}$ does not converge.*
2. *$(z_n)_{n\in\mathbb{N}}$ cycles after certain steps regardless the starting point if and only if there exist $k_1, k_2 \in \mathbb{N} \smallsetminus \{0\}$ such that $k_1 d_A(b_1) = k_2 d_A(b_2)$.*

Outline

The case of a line and a circle

Fact (Borwein–Sims 2011)

Let $\alpha \in [0,1[$. Then the DR algorithm for

$$A = \mathbb{R} \times \{\alpha\} \quad \textit{and} \quad B = \left\{(x,\rho) \in \mathbb{R}^2 \;\middle|\; x^2 + \rho^2 = 1\right\}$$

is locally convergent around $(\pm\sqrt{1-\alpha^2}, \alpha)$.

Conjecture (BS11): The DR algorithm is actually globally convergent. This has since been resolved in the affirmative by Benoist (2015).

Idea: Consider $V\colon \mathbb{R}^2 \to]-\infty, +\infty]$ given by

$$V(x,\rho) := \tfrac{1}{2}x^2 - (1-\alpha)\ln|x| + \alpha\sqrt{1-x^2} - \alpha\ln(1+\sqrt{1-x^2}) + \tfrac{1}{2}(\rho-\alpha)^2.$$

Then V decreases along DR sequences: $V(T_{A,B}z) \le V(z)$ with equality if and only if $z \in \operatorname{Fix} T_{A,B}$.

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Finding a zero of a function

In this section,

$$f\colon X \to [-\infty, +\infty] \text{ is proper with closed graph.}$$

Consider the feasibility problem in $X \times \mathbb{R}$ with constraints

$$A = X \times \{0\} \quad \text{and} \quad B = \operatorname{gra} f := \big\{(x,\rho) \in X \times \mathbb{R} \;\big|\; f(x) = \rho\big\},$$

which can be cast as

$$\text{find } x \in X \text{ such that } f(x) = 0.$$

- B is generally not convex unless f is affine.
- **For a line and a circle:** Up to symmetry, take $f(x) = -\sqrt{1-x^2} + \alpha$.

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A Lyapunov-type approach

Definition (Lyapunov-type function)

A function $V\colon X\times\mathbb{R}\to\,]-\infty,+\infty]$ is a Lyapunov-type function for f on a nonempty convex subset D of X if it can be expressed in the form

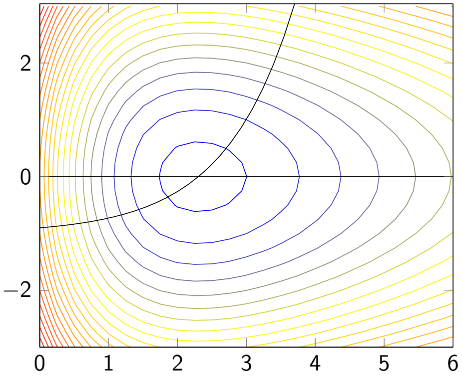
$$V(x,\rho)=F(x)+\frac{1}{2}\rho^2$$

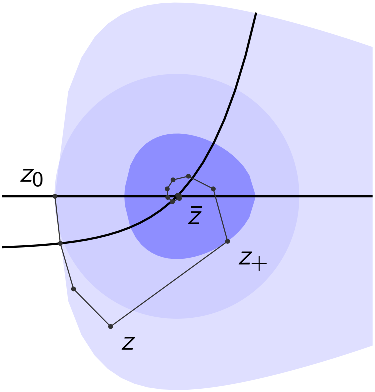
for some proper coercive convex function $F\colon D\to\,]-\infty,+\infty]$ whose subdifferential satisfies

$$(\forall x\in D)\quad \partial F(x)\supseteq\begin{cases}\left\{\frac{f(x)}{\|x^*\|^2}x^*\ \middle|\ x^*\in\partial^0 f(x)\right\} & \text{if } 0\notin\partial^0 f(x),\\ \{0\} & \text{if } f(x)=0.\end{cases}$$

$\partial^0 f:=\partial f\cup-\partial(-f)$: the symmetric (limiting) subdifferential of f.

A Lyapunov-type approach: Some intuition





A Lyapunov-type function for $f(x) = \frac{1}{10}\exp(x) - 1$, which guarantees global convergence of the DR algorithm to $\bar{z} := (\ln(10), 0)$.

Convergence theorem

Write $z_n = (x_n, \rho_n) \in X \times \mathbb{R}$. Suppose that there exists a Lyapunov-type function for f on D, that f is locally Lipschitz continuous on $D \smallsetminus f^{-1}(0)$, and that

$$(\exists n_0 \in \mathbb{N})(\forall n \geq n_0) \quad x_n \in D \quad \text{and} \quad x_{n+1} \notin (\partial^0 f)^{-1}(0) \smallsetminus f^{-1}(0).$$

Theorem (D–Tam 2017)

The DR sequence $(z_n)_{n\in\mathbb{N}}$ is bounded and asymptotically regular, and each of its cluster points $\bar{z}$ satisfy $P_A\bar{z} \in A \cap B$. Suppose, in addition, that $\overline{D} \cap f^{-1}(0) = \{\bar{x}\}$ is contained in D. Then

1. $z_n \to \bar{z}$ *with* $P_A\bar{z} \in A \cap B$.
2. $z_n \to \bar{z} = (\bar{x}, 0) \in A \cap B$ *provided that* $0 \notin \partial^0 f(\bar{x})$ *and* $f|_D$ *is continuous at* $\bar{x}$.

Linear convergence

Corollary

Suppose that $\overline{D} \cap f^{-1}(0) = \{\bar{x}\} \subseteq D$. *Then*

1. *If f is continuously differentiable around* $\bar{x}$ *with* $\nabla f(\bar{x}) \neq 0$, *then* $z_n \to \bar{z} = (\bar{x}, 0) \in A \cap B$ *with R-linear rate.*
2. *If* $X = \mathbb{R}$ *and f is twice strictly differentiable at* $\bar{x}$ *with* $f'(\bar{x}) \neq 0$, *then* $z_n \to \bar{z} = (\bar{x}, 0) \in A \cap B$ *with Q-linear rate*

$$\kappa := \frac{1}{\sqrt{1 + |f'(\bar{x})|^2}}.$$

A sequence $(z_n)_{n\in\mathbb{N}}$ is said to converge to a point $\bar{z}$

- with *R-linear rate* $\kappa \in [0,1[$ if $(\exists \eta \in \mathbb{R}_+)(\forall n \in \mathbb{N})\ \|z_n - \bar{z}\| \leq \eta\kappa^n$;
- with *Q-linear rate* $\kappa \in [0,1[$ if $\limsup_{n\to\infty} \frac{\|z_{n+1}-\bar{z}\|}{\|z_n-\bar{z}\|} \leq \kappa$.

Some examples

- $X = \mathbb{R}$, $f(x) = \alpha \exp(x) - \beta$ with $(\alpha, \beta) \in \mathbb{R}^2_{++}$. One possible F is

$$F(x) := \int \frac{f(x)}{f'(x)} dx = \int \left(1 - \frac{\beta}{\alpha} \exp(-x)\right) dx = x + \frac{\beta}{\alpha} \exp(-x)$$

$\longrightarrow$ Global Q-linear convergence with rate $\kappa = 1/\sqrt{1+\beta^2}$.

- $X = \mathbb{R}$, $p \in]1, +\infty[$,

$$f(x) := \begin{cases} x^p & \text{if } x \geq 0, \\ x & \text{if } x < 0, \end{cases} \qquad \partial^0 f(x) = \begin{cases} px^{p-1} & \text{if } x \geq 0, \\ [0,1] & \text{if } x = 0, \\ 1 & \text{if } x < 0. \end{cases}$$

Note that f is nonconvex and nonsmooth at $x = 0$. Define F by

$$F(x) := \begin{cases} \frac{1}{2p} x^2 & \text{if } x \geq 0, \\ \frac{1}{2} x^2 & \text{if } x < 0, \end{cases}$$

then V is a Lyapunov-type function for f on $\mathbb{R}$.

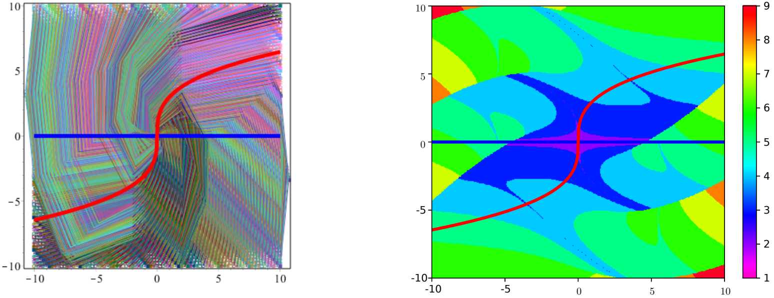
Some examples

Suppose $f = \alpha\|\cdot\|^p$ for $\alpha \in \mathbb{R} \smallsetminus \{0\}$ and $p \in]0, +\infty[$. Then whenever $x \neq 0$, we have $\partial^0 f(x) = \{\alpha p\|x\|^{p-2}x\}$ and

$$\frac{f(x)}{\|\nabla f(x)\|}\nabla f(x) = \frac{\alpha\|x\|^p}{\alpha^2 p^2\|x\|^{2p-2}}\alpha p\|x\|^{p-2}x = \frac{1}{p}x,$$

which leads to $F(x) = \frac{1}{2p}\|x\|^2$. The global convergence follows.

The same function F works for $f = \alpha|\cdot|^p\,\mathrm{sgn}(\cdot)$ on $\mathbb{R}$.



Illustrations of the DR algorithm for $f(x) = 3\sqrt[3]{x}$ on $[-10, 10] \times [-10, 10]$.

Outline

1 Introduction

2 Behavior of DR algorithm in possibly inconsistent case

3 Finite convergence

4 A Lyapunov-type approach to convergence theory

5 Local linear convergence

Generalized DR operator

Let $\lambda, \mu \in]0, 2]$ and $\alpha \in]0, 1]$. The generalized DR operator for (A, B) with parameters (λ, μ, α) is defined by

$$T_{\lambda,\mu}^{\alpha} := (1-\alpha)\,\mathrm{Id} + \alpha P_B^{\mu} P_A^{\lambda}.$$

- $T_{1,1}^{1} = P_B P_A$ is the classical alternating projection (AP) operator.
- $T_{2,2}^{1/2} = \frac{1}{2}(\mathrm{Id} + R_B R_A)$ is the classical DR operator.
- $T_{2,2\alpha}^{1/2} = (1-\alpha)P_A + \frac{\alpha}{2}(\mathrm{Id} + R_B R_A)$ is the relaxed avaraged alternating reflection (RAAR) operator.
- If B is an affine subspace of X, then

$$T_{1+\alpha,1+\alpha}^{1/(1+\alpha)} = (1-\alpha)P_B P_A + \frac{\alpha}{2}(\mathrm{Id} + R_B R_A)$$

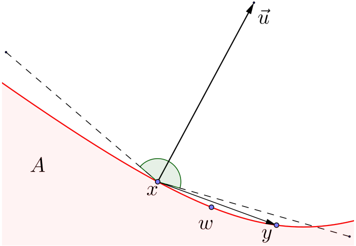
(a convex combination of the classical AP and DR operators).

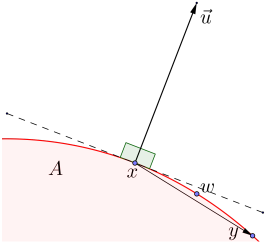
Regularity of sets

Let $\varepsilon \in \mathbb{R}_+$ and $\delta \in \mathbb{R}_{++}$. A set C is said to be (ε, δ)-regular at $w \in X$ if

$$\forall x, y \in C \cap \mathbb{B}_\delta(w), \forall u \in N_C^{\text{prox}}(x) : \quad \langle u, y - x \rangle \le \varepsilon \|u\| \cdot \|y - x\|$$

and superregular at w if $\forall \varepsilon \in \mathbb{R}_{++}, \exists \delta \in \mathbb{R}_{++}$: C is (ε, δ)-regular at w.





- Convex sets and sets with "smooth" boundary are superregular.

$N_C^{\text{prox}}(x) := \text{cone}(P_C^{-1}(x) - x) = \left\{ \lambda(z - x) \;\middle|\; z \in P_C^{-1}(x),\ \lambda \in \mathbb{R}_+ \right\}.$

Key properties

Lemma

Let $\varepsilon_1 \in [0, 1/3]$, $\varepsilon_2 \in [0, 1[$ *and set* $\gamma := 1 - \alpha + \alpha\big(1 + \frac{\lambda\varepsilon_1}{1-\varepsilon_1}\big)\big(1 + \frac{\mu\varepsilon_2}{1-\varepsilon_2}\big)$, $\beta := \frac{1-\alpha}{\alpha}$. *If* A *and* B *are* (ε_1, δ)*- and* $(\varepsilon_2, \sqrt{2}\delta)$*-regular at* $w \in A \cap B$, *then* $T^{\alpha}_{\lambda,\mu}$ *is* $(A \cap B \cap \mathbb{B}_\delta(w), \gamma, \beta)$*-quasi firmly Fejér monotone on* $\mathbb{B}_{\delta/2}(w)$ *in the sense that*

$$\forall x \in \mathbb{B}_{\delta/2}(w),\ \forall x_+ \in T^{\alpha}_{\lambda,\mu}x,\ \forall \overline{x} \in A \cap B \cap \mathbb{B}_\delta(w):$$
$$\|x_+ - \overline{x}\|^2 + \beta\|x - x_+\|^2 \le \gamma\|x - \overline{x}\|^2.$$

Lemma

Let $\varepsilon \in [0, 1/3]$. *If* A *is superregular at* w *and* $\{A, B\}$ *is strongly regular at* $w \in A \cap B$, *then there exist* $\delta \in \mathbb{R}_{++}$, $\nu \in \mathbb{R}_{++}$ *such that* $T^{\alpha}_{\lambda,\mu}$ *is* $(A \cap B, \nu)$*-quasi coercive on* $\mathbb{B}_{\delta/2}(w)$ *in the sense that*

$$\forall x \in \mathbb{B}_{\delta/2}(w),\ \forall x_+ \in T^{\alpha}_{\lambda,\mu}x: \quad \|x - x_+\| \ge \nu d_{A\cap B}(x).$$

$\{A, B\}$ is strongly regular at $w \in A \cap B$ if $N_A(w) \cap (-N_B(w)) = \{0\}$.

Local linear convergence

Let A and B be closed subsets of X with $A \cap B \neq \varnothing$. Suppose that $\{A, B\}$ is superregular and strongly regular at some point $w \in A \cap B$.

Fact (Phan 2016)

When started at a point sufficiently close to w, the DR sequence converges R-linearly to a point in $A \cap B$.

Theorem (D–Phan 2016)

Let $\lambda, \mu \in \,]0, 2]$ and $\alpha \in \,]0, 1[$. Then when started at a point sufficiently close to w, the generalized DR sequence generated by $T^{\alpha}_{\lambda,\mu}$ converges R-linearly to a point in $A \cap B$.

Some key references

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Contact

The manuscripts corresponding to this talk are

- H.H. Bauschke and MND, On the finite convergence of the Douglas–Rachford algorithm for solving (not necessarily convex) feasibility problems in Euclidean spaces, *SIAM Journal on Optimization* 27 (2017), 507–537.
- H.H. Bauschke, MND, and S.B.. Lindstrom, On the behavior of the Douglas–Rachford iteration, In preparation.
- H.H. Bauschke, MND, and W.M. Moursi, The Douglas–Rachford algorithm in the affine-convex case, *Operations Research Letters* 44 (2016), 379–382.
- H.H. Bauschke, MND, D. Noll, and H.M. Phan, On Slater's condition and finite convergence of the Douglas–Rachford algorithm for solving convex feasibility problems in Euclidean spaces, *Journal of Global Optimization* 65 (2016), 329–349.
- MND and H.M. Phan, Linear convergence of projection algorithms, (2016), arXiv:1609.00341.
- MND and M.K. Tam, A Lyapunov-type approach to convergence of the Douglas–Rachford algorithm, (2017), arXiv:1706.04846.

THANK YOU VERY MUCH!

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