Parallel, Block-Iterative, Primal-Dual Monotone Operator Splitting

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Dedicated to the memory of Jon Borwein

Workshop on Splitting Algorithms and Modern Operator Theory
Oaxaca, Mexico, September 19, 2017

Monotone operators

- $\mathcal{H}$ a real Hilbert space.
- $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone: for every $(x, x^*) \in \mathcal{H}^2$,

$$(x, x^*) \in \operatorname{gra} A \quad \Leftrightarrow \quad (\forall (y, y^*) \in \operatorname{gra} A) \quad \langle x - y \mid x^* - y^* \rangle \geqslant 0$$

- The resolvent of A, $J_A = (\mathrm{Id} + A)^{-1}\colon \mathcal{H} \to \mathcal{H}$, is firmly nonexpansive and $\operatorname{Fix} J_A = \operatorname{zer} A = \{x \in \mathcal{H} \mid 0 \in Ax\}$
- Minty's parametrization:

$$(\forall x \in \mathcal{H}) \quad (J_A x, x - J_A x) = (J_A x, J_{A^{-1}} x) \in \operatorname{gra} A$$

Solving monotone inclusions

- $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ maximally monotone,
- Problem: solve: $0 \in Ax$
- Conceptual solution methods when A is simple:
  - The proximal point algorithm (implicit):

$$x_{n+1} = (\mathrm{Id} + \gamma_n A)^{-1} x_n = J_{\gamma_n A} x_n, \quad \text{where} \quad \gamma_n > 0.$$

  - If $A\colon \mathcal{H} \to \mathcal{H}$ is β-cocoercive ($A^{-1} - \beta\mathrm{Id}$ is monotone), the explicit iteration

$$x_{n+1} = x_n - \gamma_n A x_n, \quad \text{where} \quad 0 < \gamma_n < 2\beta.$$

- For "real" problems **splitting** is required.

Splitting structured problems: 3 basic methods

$A, B\colon \mathcal{H} \to 2^{\mathcal{H}}$ maximally monotone, solve $0 \in A\overline{x} + B\overline{x}$.

- Douglas-Rachford splitting (1979)

$$\left\lfloor \begin{array}{l} y_n = J_{\gamma B} x_n \\ z_n = J_{\gamma A}(2y_n - x_n) \\ x_{n+1} = x_n + z_n - y_n \end{array} \right.$$

- $B\colon \mathcal{H} \to \mathcal{H}$ β-cocoercive: forward-backward splitting (1979+)

$$\left\lfloor \begin{array}{l} 0 < \gamma_n < 2\beta \\ y_n = x_n - \gamma_n B x_n \\ x_{n+1} = J_{\gamma_n A} y_n \end{array} \right.$$

- $B\colon \mathcal{H} \to \mathcal{H}$ μ-Lipschitzian: forward-backward-forward splitting (2000)

$$\left\lfloor \begin{array}{l} 0 < \gamma_n < 1/\mu \\ y_n = x_n - \gamma_n B x_n \\ z_n = J_{\gamma_n A} y_n \\ r_n = z_n - \gamma_n B z_n \\ x_{n+1} = x_n - y_n + r_n \end{array} \right.$$

- Spingarn's method (1983) for $0 \in A_1\overline{x} + \cdots + A_n\overline{x}$.

Splitting algorithms (1979-2000)

find $\overline{x} \in \mathcal{H}$ such that

$$z^* \in A\overline{x} + B\overline{x}$$

where:

- $z^* \in \mathcal{H}$, $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone
- $B\colon \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone

Splitting algorithms (Briceño-Arias-PLC, 2011)

find $\overline{x} \in \mathcal{H}$ such that

$$z^* \in A\overline{x} + L^* B(L\overline{x} - r)$$

where:

- $z^* \in \mathcal{H}$, $A \colon \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone
- $B \colon \mathcal{G} \to 2^{\mathcal{G}}$ is maximally monotone, $r \in \mathcal{G}$, $L \in \mathscr{B}(\mathcal{H}, \mathcal{G})$

Splitting algorithms (Briceño-Arias-PLC, 2011)

find $\overline{x} \in \mathcal{H}$ such that

$$z^* \in A\overline{x} + \sum_{k=1}^{K} L_k^* B_k(L_k\overline{x} - r_k)$$

where:

- $z^* \in \mathcal{H}$, $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone
- $B_k\colon \mathcal{G}_k \to 2^{\mathcal{G}_k}$ is maximally monotone, $r_k \in \mathcal{G}_k$, $L_k \in \mathscr{B}(\mathcal{H}, \mathcal{G}_k)$

Splitting algorithms (PLC-Pesquet, 2012)

find $\overline{x} \in \mathcal{H}$ such that

$$z^* \in A\overline{x} + \sum_{k=1}^{K} L_k^*(B_k \,\Box\, D_k)(L_k\overline{x} - r_k)$$

where:

- $z^* \in \mathcal{H}$, $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone
- $B_k\colon \mathcal{G}_k \to 2^{\mathcal{G}_k}$ is maximally monotone, $r_k \in \mathcal{G}_k$, $L_k \in \mathscr{B}(\mathcal{H}, \mathcal{G}_k)$
- $D_k\colon \mathcal{G}_k \to 2^{\mathcal{G}_k}$ is maximally monotone, D_k^{-1} is ν_k-Lipschitzian, $B_k \,\Box\, D_k = (B_k^{-1} + D_k^{-1})^{-1}$

Splitting algorithms (PLC-Pesquet, 2012)

find $\overline{x} \in \mathcal{H}$ such that

$$z^* \in A\overline{x} + \sum_{k=1}^{K} L_k^*(B_k \,\square\, D_k)(L_k - r_k\overline{x}) + C\overline{x}$$

where:

- $z^* \in \mathcal{H}$, $A\colon \mathcal{H} \to 2^{\mathcal{H}}$ is maximally monotone
- $B_k\colon \mathcal{G}_k \to 2^{\mathcal{G}_k}$ is maximally monotone, $r_k \in \mathcal{G}_k$, $L_k \in \mathscr{B}(\mathcal{H}, \mathcal{G}_k)$
- $D_k\colon \mathcal{G}_k \to 2^{\mathcal{G}_k}$ is maximally monotone, D_k^{-1} is ν_k-Lipschitzian, $B_k \,\square\, D_k = (B_k^{-1} + D_k^{-1})^{-1}$
- $C\colon \mathcal{H} \to \mathcal{H}$ is monotone and μ-Lipschtizian

Splitting algorithms (PLC, 2013)

find $\overline{x}_1 \in \mathcal{H}_1, \ldots, \overline{x}_m \in \mathcal{H}_m$ such that

$$\begin{cases} z_1^* & \in A_1\overline{x}_1 + \displaystyle\sum_{k=1}^{K} L_{k1}^*\Bigg((B_k \,\square\, D_k)\Bigg(\sum_{i=1}^{m} L_{ki}\overline{x}_i - r_k\Bigg)\Bigg) + C_1\overline{x}_1 \\ & \vdots \\ z_m^* & \in A_m\overline{x}_m + \displaystyle\sum_{k=1}^{K} L_{km}^*\Bigg((B_k \,\square\, D_k)\Bigg(\sum_{i=1}^{m} L_{ki}\overline{x}_i - r_k\Bigg)\Bigg) + C_m\overline{x}_m \end{cases}$$

where:

- $z_i^* \in \mathcal{H}_i$, $A_i \colon \mathcal{H}_i \to 2^{\mathcal{H}_i}$ is maximally monotone
- $B_k \colon \mathcal{G}_k \to 2^{\mathcal{G}_k}$ is maximally monotone, $r_k \in \mathcal{G}_k$, $L_{ki} \in \mathscr{B}(\mathcal{H}_i, \mathcal{G}_k)$
- $D_k \colon \mathcal{G}_k \to 2^{\mathcal{G}_k}$ is maximally monotone, D_k^{-1} is ν_k-Lipschitzian, $B_k \,\square\, D_k = (B_k^{-1} + D_k^{-1})^{-1}$
- $C_i \colon \mathcal{H}_i \to \mathcal{H}_i$ is monotone and μ_i-Lipschtizian

Splitting algorithms (PLC, 2013)

- $\mathcal{K} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m \oplus \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_p$
- $\boldsymbol{M}\colon \mathcal{K} \to 2^{\mathcal{K}}\colon (x_1, \ldots, x_m, v_1^*, \ldots, v_p^*) \mapsto$ $(-z_1^* + A_1 x_1) \times \cdots \times (-z_m^* + A_m x_m) \times (r_1 + B_1^{-1} v_1^*) \times \cdots \times (r_p + B_p^{-1} v_p^*)$
- $\boldsymbol{Q}\colon \mathcal{K} \to \mathcal{K}\colon (x_1, \ldots, x_m, v_1^*, \ldots, v_p^*) \mapsto$ $\big(C_1 x_1 + \sum_{k=1}^K L_{k1}^* v_k^*, \ldots, C_m x_m + \sum_{k=1}^K L_{km}^* v_k^*, -\sum_{i=1}^m L_{1i} x_i + D_1^{-1} v_1^*, \ldots, \sum_{i=1}^m L_{Ki} x_i + D_K^{-1} v_K^*\big)$
- $\boldsymbol{M}$ and $\boldsymbol{Q}$ are maximally monotone, $\boldsymbol{Q}$ is Lipschitzian, the zeros of $\boldsymbol{M} + \boldsymbol{Q}$ are primal-dual solutions
- Solve $\mathbf{0} \in \boldsymbol{M}\boldsymbol{x} + \boldsymbol{Q}\boldsymbol{x}$, where $\boldsymbol{x} = (x_1, \ldots, x_m, v_1^*, \ldots, v_p^*)$ via Tseng's forward-backward-forward splitting algorithm

$$
\begin{array}{l}
\boldsymbol{y}_n = \boldsymbol{x}_n - \boldsymbol{Q}\boldsymbol{x}_n \\
\boldsymbol{p}_n = (\mathrm{Id} + \boldsymbol{M})^{-1} \boldsymbol{y}_n \\
\boldsymbol{q}_n = \boldsymbol{p}_n - \boldsymbol{Q}\boldsymbol{p}_n \\
\boldsymbol{x}_{n+1} = \boldsymbol{x}_n - \boldsymbol{y}_n + \boldsymbol{q}_n
\end{array}
$$

in $\mathcal{K}$ to get...

Splitting algorithms (PLC, 2013)

$$
\begin{array}{l}
\text{For } n = 0, 1, \ldots \\
\left\lfloor
\begin{array}{l}
\varepsilon \leqslant \gamma_n \leqslant (1-\varepsilon)/\left(\max\left\{\max\limits_{1\leqslant i\leqslant m} \mu_i, \max\limits_{1\leqslant k\leqslant K} \nu_k\right\} + \sqrt{\sum_{k=1}^K \sum_{i=1}^m \|L_{ki}\|^2}\right) \\
\text{For } i = 1, \ldots, m \\
\left\lfloor
\begin{array}{l}
s_{1,i,n} = x_{i,n} - \gamma_n\left(C_i x_{i,n} + \sum_{k=1}^K L_{ki}^* v_{k,n}^*\right) \\
p_{1,i,n} = J_{\gamma_n A_i}(s_{1,i,n} + \gamma_n z_i)
\end{array}
\right. \\
\text{For } k = 1, \ldots, K \\
\left\lfloor
\begin{array}{l}
s_{2,k,n} = v_{k,n}^* - \gamma_n\left(D_k^{-1} v_{k,n}^* - \sum_{i=1}^m L_{ki} x_{i,n}\right) \\
p_{2,k,n} = s_{2,k,n} - \gamma_n\left(r_k + J_{\gamma_n^{-1} B_k}(\gamma_n^{-1} s_{2,k,n} - r_k)\right) \\
q_{2,k,n} = p_{2,k,n} - \gamma_n\left(D_k^{-1} p_{2,k,n} - \sum_{i=1}^m L_{ki} p_{1,i,n}\right) \\
v_{k,n+1}^* = v_{k,n}^* - s_{2,k,n} + q_{2,k,n}
\end{array}
\right. \\
\text{For } i = 1, \ldots, m \\
\left\lfloor
\begin{array}{l}
q_{1,i,n} = p_{1,i,n} - \gamma_n\left(C_i p_{1,i,n} + \sum_{k=1}^K L_{ki}^* p_{2,k,n}\right) \\
x_{i,n+1} = x_{i,n} - s_{1,i,n} + q_{1,i,n}
\end{array}
\right.
\end{array}
\right.
\end{array}
$$

Some limitations of the state-of-the-art

We present a new framework that circumvents simultaneously the limitations of current methods, which require:

- inversions of linear operators or knowledge of bounds on norms of all the L_{ki}
- the proximal parameters must be the same for all the monotone operators
- activation of the resolvents of all the monotone operators: impossible in huge-scale problems
- synchronicity: all resolvent operator evaluations must be computed and used during the current iteration

and, in general,

- converge only weakly

Asynchronous, block-iterative splitting

- For every $i \in I$ (finite), $\mathcal{H}_i$ a Hilbert space, $A_i \colon \mathcal{H}_i \to 2^{\mathcal{H}_i}$ maximally monotone, $z_i^* \in \mathcal{H}_i$
- For every $k \in K$ (finite), $\mathcal{G}_k$ a Hilbert space, $B_k \colon \mathcal{G}_k \to 2^{\mathcal{G}_k}$ maximally monotone, $r_k \in \mathcal{G}_k$, $\mathrm{L}_{ki} \in \mathscr{B}(\mathcal{H}_i, \mathcal{G}_k)$
- **Initial problem:** find $(\overline{x}_i)_{i\in I} \in \boldsymbol{\mathcal{H}} = \bigoplus_{i\in I} \mathcal{H}_i$ such that

$$(\forall i \in I) \quad z_i^* \in A_i\overline{x}_i + \sum_{k\in K} L_{ki}^*\Bigg(B_k\Bigg(\sum_{j\in I} L_{kj}\overline{x}_j - r_k\Bigg)\Bigg)$$

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- **Dual problem:** find $(\overline{v}_k^*)_{k\in K} \in \boldsymbol{\mathcal{G}} = \bigoplus_{k\in K} \mathcal{G}_k$ such that

$$(\forall k \in K) \quad -r_k \in -\sum_{i\in I} L_{ki}\Bigg(A_i^{-1}\bigg(z_i^* - \sum_{l\in K} L_{li}^*\overline{v}_l^*\bigg)\Bigg) + B_k^{-1}\overline{v}_k^*$$

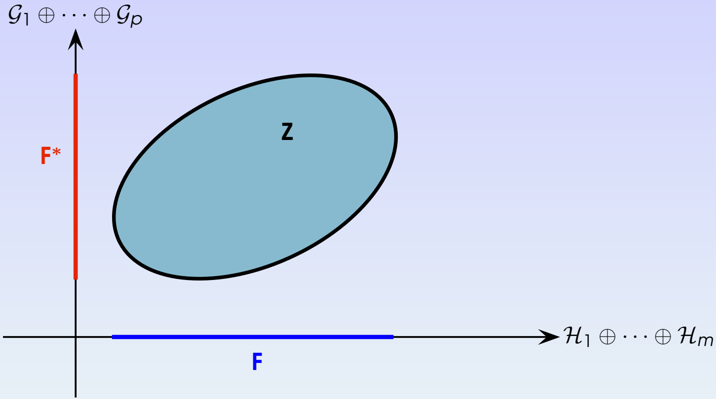
Asynchronous, block-iterative splitting

- **Solutions set:** the associated Kuhn-Tucker set

$$\boldsymbol{Z} = \Big\{ \big((\overline{x}_i)_{i\in I}, (\overline{v}_k^*)_{k\in K}\big) \;\Big|\; \overline{x}_i \in \mathcal{H}_i \;\text{ and }\; z_i^* - \sum_{k\in K} L_{ki}^*\overline{v}_k^* \in A_i\overline{x}_i, \quad \overline{v}_k^* \in \mathcal{G}_k \;\text{ and }\; \sum_{i\in I} L_{ki}\overline{x}_i - r_k \in B_k^{-1}\overline{v}_k^* \Big\}$$

- $\boldsymbol{Z}$ is a closed convex set
- The projection of $\boldsymbol{Z}$ onto $\boldsymbol{\mathcal{H}}$ is the set $\boldsymbol{F}$ of primal solutions
- The projection of $\boldsymbol{Z}$ onto $\boldsymbol{\mathcal{G}}$ is the set $\boldsymbol{F}^*$ of dual solutions

The Kuhn-Tucker set



With proper CQ, this framework includes..

- Let $\boldsymbol{F}$ be the set of solutions to the problem

$$\underset{(x_i)_{i\in I}\in\boldsymbol{\mathcal{H}}}{\text{minimize}} \sum_{i\in I}\big(f_i(x_i)-\langle x_i \mid z_i^*\rangle\big)+\sum_{k\in K} g_k\Bigg(\sum_{i\in I} L_{ki}x_i - r_k\Bigg)$$

where $f_i \in \Gamma_0(\mathcal{H}_i)$, $g_k \in \Gamma_0(\mathcal{G}_k)$, $L_{ki} \in \mathscr{B}(\mathcal{H}_i,\mathcal{G}_k)$

- Let $\boldsymbol{F}^*$ be the set of solutions to the dual problem

$$\underset{(v_k^*)_{k\in K}\in\bigoplus_{k\in K}\mathcal{G}_k}{\text{minimize}} \sum_{i\in I} f_i^*\Bigg(z_i^* - \sum_{k\in K} L_{ki}^* v_k^*\Bigg)+\sum_{k\in K}\big(g_k^*(v_k^*)+\langle v_k^* \mid r_k\rangle\big)$$

- Associated Kuhn-Tucker set

$$\boldsymbol{Z}=\Bigg\{\big((\overline{x}_i)_{i\in I},(\overline{v}_k^*)_{k\in K}\big)\;\Bigg|\;\overline{x}_i\in\mathcal{H}_i \text{ and } z_i^*-\sum_{k\in K}L_{ki}^*\overline{v}_k^*\in\partial f_i(\overline{x}_i), \quad \overline{v}_k^*\in\mathcal{G}_k \text{ and } \sum_{i\in I}L_{ki}\overline{x}_i-r_k\in\partial g_k^*(\overline{v}_k^*)\Bigg\}$$

Methodology for 2 operators

- $0 \in A\overline{x} + L^*\big(BL\overline{x}\big)$ and $0 \in -L\big(A^{-1}\big(-L^*\overline{v}^*\big)\big) + B^{-1}\overline{v}^*$

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- Take $(\overline{x}, \overline{v}^*) \in \boldsymbol{Z}$. Then $-L^*\overline{v}^* \in A\overline{x}$ and $L\overline{x} \in B^{-1}\overline{v}^*$, i.e.,

$$\big(\overline{x}, -L^*\overline{v}^*\big) \in \operatorname{gra} A \quad \text{and} \quad \big(L\overline{x}, \overline{v}^*\big) \in \operatorname{gra} B$$

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- Suppose that at iteration n

$$(a_n, a_n^*) \in \operatorname{gra} A \quad \text{and} \quad (b_n, b_n^*) \in \operatorname{gra} B$$

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- Suppose that at iteration n

$$(a_n, a_n^*) \in \operatorname{gra} A \quad \text{and} \quad (b_n, b_n^*) \in \operatorname{gra} B$$

- By monotonicity of A and B,

$$\langle a_n - \overline{x} \mid a_n^* + L^*\overline{v}^*\rangle + \langle b_n - L\overline{x} \mid b_n^* - \overline{v}^*\rangle \geqslant 0$$

i.e.,

$$\langle (\overline{x}, \overline{v}) \mid (a_n^* + L^* b_n^*, b_n - La_n)\rangle \leqslant \langle a_n \mid a_n^*\rangle + \langle b_n \mid b_n^*\rangle$$

Methodology for 2 operators

- $0 \in A\overline{x} + L^*\big(BL\overline{x}\big)$ and $0 \in -L\big(A^{-1}\big(-L^*\overline{v}^*\big)\big) + B^{-1}\overline{v}^*$
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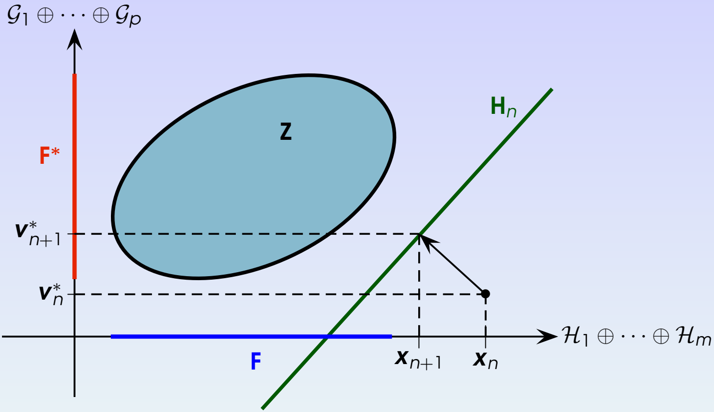
i.e.,

$$\langle (\overline{x}, \overline{v}) \mid (a_n^* + L^* b_n^*, b_n - La_n) \rangle \leqslant \langle a_n \mid a_n^* \rangle + \langle b_n \mid b_n^* \rangle$$

- This places $\boldsymbol{Z}$ in the closed affine half-space with outer normal $(a_n^* + L^* b_n^*, b_n - La_n)$!

Fejér monotone scheme in the general case



- Choose suitable points in the graphs of $(A_i)_{i\in I}$ and $(B_k)_{k\in K}$ to construct a half-space $\mathbf{H}_n$ containing $\mathbf{Z}$
- Algorithm: $(\boldsymbol{x}_{n+1}, \boldsymbol{v}^*_{n+1}) = P_{\mathbf{H}_n}(\boldsymbol{x}_n, \boldsymbol{v}^*_n) \rightharpoonup (\boldsymbol{x}, \boldsymbol{v}^*) \in \mathbf{Z} \subset \mathbf{F} \times \mathbf{F}^*$

Main novelties

- **Block iterations:** At iteration n, we require calculation of new points in the graphs of only some the operators $(A_i)_{i\in I_n\subset I}$ and $(B_k)_{k\in K_n\subset K}$. The control sequences $(I_n)_{n\in\mathbb{N}}$ and $(K_n)_{n\in\mathbb{N}}$ dictate how frequently the various operators are used.
- **Asynchronicity:** A new point $(a_{i,n}, a^*_{i,n}) \in \operatorname{gra} A_i$ being incorporated into the calculations at iteration n may be based on data $x_{i,c_i(n)}$ and $(v^*_{k,c_i(n)})_{k\in K}$ available at some possibly earlier iteration $c_i(n) \leqslant n$. Therefore, the calculation of $(a_{i,n}, a^*_{i,n})$ could have been initiated at iteration $c_i(n)$, with its results becoming available only at iteration n. Likewise, for $(b_{k,n}, b^*_{k,n}) \in \operatorname{gra} B_k$.

Also:

- No knowledge of the $\|L_{ki}\|$s is required
- No linear operator inversion is required
- No bounds required on the proximal parameters

Asynchronous block-iterative proximal splitting I

for $n = 0, 1, \ldots$

- for every $i \in I_n$
  - $l^*_{i,n} = \sum_{k\in K} L^*_{ki} v^*_{k,c_i(n)}$
  - $(a_{i,n}, a^*_{i,n}) = \Big(J_{\gamma_{i,c_i(n)} A_i}\big(x_{i,c_i(n)} + \gamma_{i,c_i(n)}(z_i - l^*_{i,n})\big), \gamma^{-1}_{i,c_i(n)}(x_{i,c_i(n)} - a_{i,n}) - l^*_{i,n}\Big)$
- for every $i \in I \smallsetminus I_n$
  - $(a_{i,n}, a^*_{i,n}) = (a_{i,n-1}, a^*_{i,n-1})$
- for every $k \in K_n$
  - $l_{k,n} = \sum_{i\in I} L_{ki} x_{i,d_k(n)}$
  - $(b_{k,n}, b^*_{k,n}) = \Big(r_k + J_{\mu_{k,d_k(n)} B_k}\big(l_{k,n} + \mu_{k,d_k(n)} v^*_{k,d_k(n)} - r_k\big), v^*_{k,d_k(n)} + \mu^{-1}_{k,d_k(n)}(l_{k,n} - b_{k,n})\Big)$
- for every $k \in K \smallsetminus K_n$
  - $(b_{k,n}, b^*_{k,n}) = (b_{k,n-1}, b^*_{k,n-1})$
- $\big((t^*_{i,n})_{i\in I}, (t_{k,n})_{k\in K}\big) = \big((a^*_{i,n} + \sum_{k\in K} L^*_{ki} b^*_{k,n})_{i\in I}, (b_{k,n} - \sum_{i\in I} L_{ki} a_{i,n})_{k\in K}\big)$
- $\tau_n = \sum_{i\in I} \|t^*_{i,n}\|^2 + \sum_{k\in K} \|t_{k,n}\|^2$
- if $\tau_n > 0$
  - $$\theta_n = \frac{\lambda_n}{\tau_n} \max\Big\{0, \sum_{i\in I} \big(\langle x_{i,n} \mid t^*_{i,n}\rangle - \langle a_{i,n} \mid a^*_{i,n}\rangle\big) + \sum_{k\in K} \big(\langle t_{k,n} \mid v^*_{k,n}\rangle - \langle b_{k,n} \mid b^*_{k,n}\rangle\big)\Big\}$$
- else $\theta_n = 0$
- for every $i \in I$
  - $x_{i,n+1} = x_{i,n} - \theta_n t^*_{i,n}$
- for every $k \in K$
  - $v^*_{k,n+1} = v^*_{k,n} - \theta_n t_{k,n}$

Convergence

- $(I_n)_{n\in\mathbb{N}}$ is a sequence of nonempty subsets of I, and $(K_n)_{n\in\mathbb{N}}$ is a sequence of nonempty subsets of K such that $I_0 = I$, $K_0 = K$, and

$$(\forall n \in \mathbb{N}) \quad \left(\bigcup_{j=n}^{n+M-1} I_j = I \quad \text{and} \quad \bigcup_{j=n}^{n+M-1} K_j = K \right). \tag{1}$$

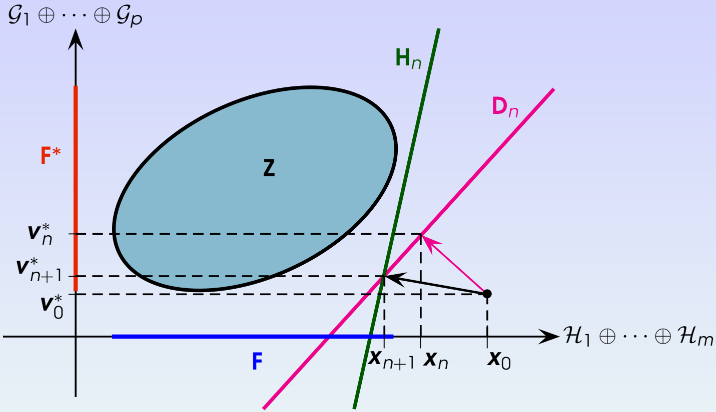
- $(c_i(n))_{n\in\mathbb{N}}$ and $(d_k(n))_{n\in\mathbb{N}}$ are sequences in $\mathbb{N}$ such that

$$(\forall i \in I) \;\; n - D \leqslant c_i(n) \leqslant n \quad \text{and} \quad (\forall k \in K) \;\; n - D \leqslant d_k(n) \leqslant n$$

- $\varepsilon \in \,]0,1[$ and $(\gamma_{i,n})_{n\in\mathbb{N}}$ and $(\mu_{k,n})_{n\in\mathbb{N}}$ are sequences in $[\varepsilon, 1/\varepsilon]$.

Set $x_n = (x_{i,n})_{i\in I}$ and $v_n^* = (v_{k,n}^*)_{k\in K}$. Then $(x_n)_{n\in\mathbb{N}}$ converges weakly to a point $\overline{x} \in \boldsymbol{F}$, $(v_n^*)_{n\in\mathbb{N}}$ converges weakly to a point $\overline{v} \in \boldsymbol{F}^*$, and $(\overline{x}, \overline{v}^*) \in \boldsymbol{Z}$.

Asynchronous block-iterative proximal splitting II



- Construct $\mathbf{H}_n$ as before
- The half-space $\mathbf{D}_n$ satisfies $(\boldsymbol{x}_n, \boldsymbol{v}_n^*) = P_{\mathbf{D}_n}(\boldsymbol{x}_0, \boldsymbol{v}_0^*)$
- Algorithm: $(\boldsymbol{x}_{n+1}, \boldsymbol{v}_{n+1}^*) = P_{\mathbf{H}_n \cap \mathbf{D}_n}(\boldsymbol{x}_0, \boldsymbol{v}_0^*) \to P_{\mathbf{Z}}(\boldsymbol{x}_0, \boldsymbol{v}_0^*) \in \mathbf{F} \times \mathbf{F}^*$

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