Quasidense multifunctions

by

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Abstract

Quasidensity is a concept that can be applied to subsets of $E \times E^*$, where E is a nonzero real Banach space. Every closed quasidense monotone set is maximally monotone, but there exist maximally monotone sets that are not quasidense. The graph of the subdifferential of a proper, convex lower semicontinuous function on E is quasidense. The graphs of certain subdifferentials of certain nonconvex functions are also quasidense. (This follows from joint work with Xianfu Wang.) The closed monotone quasidense sets have a number of very desirable properties, including a sum theorem and a parallel sum theorem, and so quasidensity satisfies the ideal calculus rules. We give five conditions equivalent to the statement that a closed monotone set be quasidense, but quasidensity seems to be the only concept of the six that extends easily to nonmonotone sets. There are also generalizations to general Banach spaces of the Brezis–Browder theorem on linear relations, but we will not discuss these in this talk.

3 links

There are links to three papers at the end of this talk. These contain complete proofs, and references to many papers by other authors.

Plan of talk

Symmetric linear maps and the associated quadratic form q_L.

The parallelogram law and a result of Burachik, Svaiter and Penot.

r_L and quasidensity.

A sufficient condition for maximal monotonicity.

The tail.

The quasidensity of a coincidence set in terms of the conjugate.

The "Fitzpatrick function" of a monotone multifunction.

The quasidensity of a convex subdifferential.

The quasidensity of a nonconvex subdifferential.

The convexity of $\overline{D(S)}$ and $\overline{R(S)}$.

A negative alignment criterion for quasidensity and type (ANA).

The sum theorem, the Fitzpatrick extension and the parallel sum theorem.

Two fuzzy criteria for quasidensity, and strong maximality.

Type (FPV) and type (FP).

Let B be a nonzero real Banach space. A linear map $L\colon B \to B^*$ is *symmetric* if, $\forall\ b, c \in B$, $\langle b, Lc\rangle = \big\langle c, Lb\big\rangle$. The quadratic form q_L on B is then defined by

$$q_L(b) := \tfrac{1}{2}\big\langle b, Lb\big\rangle.$$

Let E be a nonzero Banach space under the norm

$$\big\|(x, x^*)\big\| := \sqrt{\|x\|^2 + \|x^*\|^2}.$$

Let $\big(E \times E^*, \|\cdot\|\big)^* = \big(E^* \times E^{**}, \|\cdot\|\big)$, with $\big\|(y^*, y^{**})\big\| := \sqrt{\|y^*\|^2 + \|y^{**}\|^2}$ and $\big\langle (x, x^*), (y^*, y^{**})\big\rangle := \langle x, y^*\rangle + \langle x^*, y^{**}\rangle$. Define $L\colon\ E \times E^* \to \big(E \times E^*, \|\cdot\|\big)^*$ by

$$L(y, y^*) := (y^*, \widehat{y}) \qquad \big((y, y^*) \in E \times E^*\big),$$

where $\widehat{y}$ is the canonical image of y in E^{**}. Clearly, L is linear and $\|L\| \le 1$. Since

$$\big\langle (x, x^*), L(y, y^*)\big\rangle = \langle x, y^*\rangle + \langle x^*, \widehat{y}\rangle = \langle y, x^*\rangle + \langle y^*, \widehat{x}\rangle = \big\langle (y, y^*), L(x, x^*)\big\rangle,$$

the map L is symmetric and

$$q_L(x, x^*) = \langle x, x^*\rangle.$$

- $\forall\ (x^*, x^{**}) \in E^* \times E^{**}$, let

$$\widetilde{L}(x^*, x^{**}) := \big(x^{**}, \widehat{x^*}\big),$$

where $\widehat{x^*}$ is the canonical image of x^* in E^{***}. Arguing as above, $\widetilde{L}$ is symmetric, $\|\widetilde{L}\| \le 1$, and

$$q_{\widetilde{L}}(x^*, x^{**}) = \langle x^*, x^{**}\rangle.$$

• We have the parallelogram law:

$$b, c \in E \times E^* \quad \Longrightarrow \quad \tfrac{1}{2} q_L(b-c) + \tfrac{1}{2} q_L(b+c) = q_L(b) + q_L(c).$$

The proof of this is identical with the usual Hilbert space proof.

• Of course, a similar result is true for the function $q_{\widetilde{L}}$ on $E^* \times E^{**}$.

• Many of the results stated in this talk for $E \times E^*$ (or $E^* \times E^{**}$) are in fact true in the more general context of SN spaces as defined below: we have stated them for $E \times E^*$ (or $E^* \times E^{**}$) since we are not striving for the greatest generality in this talk.

(B, L) is a *symmetric nonexpansion space (SN space)* if B is a nonzero real Banach space and $L\colon B \to B^*$ is a symmetric linear map from B into B^* such that $\|L\| \le 1$.

There are three references at the end of this talk.

• The second reference deals with SN spaces, as defined above.

• The first reference, which depends heavily on the second one, deals with the situation discussed in this talk.

• We will discuss the third reference a little later.

— Quasidense multifunctions—

As above, let E be a nonzero Banach space and, $\forall(x,x^*)\in E\times E^*$,

$$L(x,x^*) := (x^*,\widehat{x}) \quad \text{and} \quad q_L(x,x^*) = \langle x,x^*\rangle.$$

Let $\emptyset \neq A \subset B$. As usual, A is *monotone* if

$$(x,x^*),(y,y^*)\in A \quad \Longrightarrow \quad \langle x-y,x^*-y^*\rangle \geq 0.$$

This is equivalent to the statement that

$$b,c\in A \Longrightarrow q_L(b-c)\geq 0.$$

General notation

- Let X be a vector space and $f\colon\ X \mapsto\,]-\infty,\infty\,]$. Then $\operatorname{dom} f := \{x\in X\colon\ f(x)\in\mathbb{R}\}$.
- f is *proper* if $\operatorname{dom} f\neq\emptyset$.
- $\mathcal{PC}(X)$ is the set of all proper convex functions $f\colon\ X \mapsto\,]-\infty,\infty\,]$.
- If X is a Banach space, $\mathcal{PCLSC}(X) := \{f\in\mathcal{PC}(X)\colon\ f \text{ is lower semicontinuous}\}$.
- If $f,g\colon X\to[-\infty,\infty]$, then $\big\{X|f=g\big\}$ is the "equality set" $\{x\in X|f(x)=g(x)\}$.

Notation for E, L and q_L as above:

- $\mathcal{PC}_{q_L}(E\times E^*) := \{f\in\mathcal{PC}(E\times E^*)\colon\ f\geq q_L \text{ on } E\times E^*\}$.
- $\mathcal{PCLSC}_{q_L}(E\times E^*) := \{f\in\mathcal{PCLSC}(E\times E^*)\colon\ f\geq q_L \text{ on } E\times E^*\}$.

Surprise result

Let $f \in \mathcal{PC}_{q_L}(E \times E^*)$ *and* $f(x, x^*) = \langle x, x^* \rangle$. *Let* $f^*: E^* \times E^{**} \to]-\infty, \infty]$ *be the usual conjugate function of* f. *Then* $f^*(x^*, \widehat{x}) = \langle x, x^* \rangle$.

Proof. This proof uses a differentiability argument, and fits in the space on this slide if one uses the q_L notation. This result is absolutely fundamental, and is used many times in the analysis that follows but, since we are not giving the details of many proofs, we will not mention it any more in this talk. Details can be found in the material on the web. □

Notation for E, L and q_L as above:

- $\mathcal{PC}_{q_L}(E \times E^*) := \{f \in \mathcal{PC}(E \times E^*)\colon\ f \ge q_L \text{ on } E \times E^*\}$.
- $\mathcal{PCLSC}_{q_L}(E \times E^*) := \{f \in \mathcal{PCLSC}(E \times E^*)\colon\ f \ge q_L \text{ on } E \times E^*\}$.

A result of Burachik, Svaiter and Penot

If $f \in \mathcal{PC}_{q_L}(E \times E^*)$ *and* $\big\{E \times E^* | f = q_L\big\} \ne \emptyset$ *then* $\big\{E \times E^* | f = q_L\big\}$ *is monotone.*

Proof. Let $b, c \in E \times E^*$, $f(b) = q_L(b)$ and $f(c) = q_L(c)$. Then, from the parallelogram law, the quadraticity of q_L, and the convexity of f,

$$\tfrac{1}{2}q_L(b-c) = q_L(b) + q_L(c) - \tfrac{1}{2}q_L(b+c) = q_L(b) + q_L(c) - 2q_L\big(\tfrac{1}{2}(b+c)\big)$$
$$\ge f(b) + f(c) - 2f\big(\tfrac{1}{2}(b+c)\big) \ge 0. \qquad \square$$

Density

Let X be a Banach space and $A \subset X$. Then A is *dense* if

$$\forall\, b \in X, \quad \inf_{a \in A} \tfrac{1}{2}\|a-b\|^2 = 0.$$

Obviously, a closed dense subset of X is identical with X. In the special case that $X = E \times E^*$, we introduce a weakening of density, quasidensity, and we will show that the closed, quasidense subsets of $E \times E^*$ are very significant.

Definition of the function r_L

$\forall\ (x, x^*) \in E \times E^*$, we define

$$r_L(x, x^*) = \tfrac{1}{2}\|x\|^2 + \langle x, x^*\rangle + \tfrac{1}{2}\|x^*\|^2 \geq 0.$$

Equivalently, $\forall\ b \in E \times E^*$,

$$r_L(b) := \tfrac{1}{2}\|b\|^2 + q_L(b).$$

Definition of quasidensity

Let $A \subset E \times E^*$. Then A is *quasidense* if, $\forall\ (w, w^*) \in E \times E^*$ and $\varepsilon > 0$, $\exists\ (s, s^*) \in A$ such that

$$\tfrac{1}{2}\|s - w\|^2 + \langle s - w, s^* - w^*\rangle + \tfrac{1}{2}\|s^* - w^*\|^2 < \varepsilon.$$

Equivalently:

$$\forall\ b \in E \times E^*, \quad \inf_{a \in A} r_L(a - b) = 0.$$

Multifunction notation

- If $S\colon\ E \rightrightarrows E^*$ let $G(S) := \{(x, x^*) \in E \times E^*\colon\ x^* \in Sx\}$. We always assume that $G(S) \neq \emptyset$.
- If $S\colon\ E \rightrightarrows E^*$ we say that S is *closed* if $G(S)$ is a closed subset of $E \times E^*$, and S is *quasidense* if $G(S)$ is a quasidense subset of $E \times E^*$.
- If $S\colon\ E \rightrightarrows E^*$ let $D(S) := \{x \in E\colon\ Sx \neq \emptyset\} \neq \emptyset$ and $R(S) := \bigcup_{x \in E} Sx \neq \emptyset$.

Definition of quasidensity (restated)

Let $A \subset E \times E^*$. A is quasidense if, $\forall\ (w, w^*) \in E \times E^*$ and $\varepsilon > 0$, $\exists\ (s, s^*) \in A$ such that

$$\tfrac{1}{2}\|s - w\|^2 + \langle s - w, s^* - w^* \rangle + \tfrac{1}{2}\|s^* - w^*\|^2 < \varepsilon.$$

Sufficient condition for maximally monotonicity

Let A be a closed quasidense monotone subset of $E \times E^*$. Then A is maximally monotone.

Proof. Suppose that $(w, w^*) \in E \times E^*$ and $A \cup \{(w, w^*)\}$ is monotone. Then, for all $(s, s^*) \in A$, $\langle s - w, s^* - w^* \rangle \geq 0$, and so

$$\tfrac{1}{2}\|(s, s^*) - (w, w^*)\|^2 \leq \tfrac{1}{2}\|s - w\|^2 + \langle s - w, s^* - w^* \rangle + \tfrac{1}{2}\|s^* - w^*\|^2.$$

It follows from the quasidensity of A that

$$\inf_{(s,s^*)\in A} \tfrac{1}{2}\|(s, s^*) - (w, w^*)\|^2 \leq \inf_{(s,s^*)\in A} \left[\tfrac{1}{2}\|s - w\|^2 + \langle s - w, s^* - w^* \rangle + \tfrac{1}{2}\|s^* - w^*\|^2\right] = 0.$$

Since A is closed,

$$(w, w^*) \in A. \qquad \square$$

- As we will see on the next slide, the converse of the above result is false. There are maximally monotone subsets of $E \times E^*$ that are not quasidense.
- This proof is based partly on a suggestion of a very learned person.

The tail...

Let $E = \ell^1$, and define T: $\ell^1 \to \ell^\infty = E^*$ by

$$(Tx)_n = \sum_{k=n}^{\infty} x_k.$$

T is the "tail" operator. Then T is maximally monotone but not quasidense.

Proof. Since T is continuous, linear and monotone, T is maximally monotone. Let

$$e^* := (1, 1, \ldots) \in {\ell_1}^* = \ell_\infty.$$

Let $x \in \ell_1$, and write $\sigma = \langle x, e^* \rangle = \sum_{n \geq 1} x_n$. Clearly, $\|x\| \geq \sigma$. Since $Tx \in c_0$, we also have $\|Tx - e^*\| = \sup_n |(Tx)_n - 1| \geq \lim_n |(Tx)_n - 1| = 1$. Thus

$$\begin{aligned}\langle x, Tx \rangle &= \sum_{n \geq 1} x_n \sum_{k \geq n} x_k = \sum_{n \geq 1} x_n^2 + \sum_{n \geq 1} \sum_{k > n} x_n x_k \\ &\geq \tfrac{1}{2} \sum_{n \geq 1} x_n^2 + \sum_{n \geq 1} \sum_{k > n} x_n x_k = \tfrac{1}{2}\sigma^2.\end{aligned}$$

It follows that

$$\begin{aligned}r_L\big((x, Tx) - (0, e^*)\big) &= \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|Tx - e^*\|^2 + \langle x, Tx - e^* \rangle \\ &\geq \tfrac{1}{2}\sigma^2 + \tfrac{1}{2} + \langle x, Tx \rangle - \sigma \geq \tfrac{1}{2}\sigma^2 + \tfrac{1}{2} + \tfrac{1}{2}\sigma^2 - \sigma \\ &= \sigma^2 + \tfrac{1}{2} - \sigma \geq \tfrac{1}{4}.\end{aligned}$$

Consequently, $G(T)$ is not quasidense, *i.e.*, T is not quasidense. □

Criterion for $\{E \times E^* | f = q_L\}$ to be quasidense

Let $f \in \mathcal{PCLSC}_{q_L}(E \times E^*)$. Then:

$$\{E \times E^* | f = q_L\} \text{ is quasidense} \iff f^* \geq q_{\widetilde{L}} \text{ on } (E \times E^*)^* = E^* \times E^{**}.$$

Proof. One can prove that both conditions above are equivalent to

$$\forall\ c \in E \times E^*,\ \inf_{b \in E \times E^*} \big[(f - q_L)(b) + r_L(b - c)\big] \leq 0.$$

The proof of the equivalence with the left hand condition uses a completeness argument and is based on a result of Voisei and Zălinescu. A function $f \in \mathcal{PCLSC}_{q_L}(E \times E^*)$ such that $f^* \geq q_{\widetilde{L}}$ on $(E \times E^*)^* = E^* \times E^{**}$ is known as a "strong representative function". The proof of the equivalence with the right hand condition uses Rockafellar's version of the Fenchel duality theorem. For more details, see the material on the web. □

- I wish I did not know this result!

• We have shown how $f \in \mathcal{PC}_{q_L}(E \times E^*)$ leads to the monotone set, $\{E \times E^* | f = q_L\}$.
• We now consider the converse problem: given a monotone multifunction, S, we show how to obtain a convex function, φ_S, on $E \times E^*$. We use multifunctions rather than subsets of $E \times E^*$ purely as a matter of notational convenience.

The Fitzpatrick function of a monotone multifunction

Let $S\colon\ E \rightrightarrows E^*$ be monotone. We define $\varphi_S\colon\ E \times E^* \mapsto \,]{-\infty}, \infty]$ by

$$\varphi_S(x, x^*) := \sup_{(a,a^*) \in G(S)} \big[\langle x, a^*\rangle + \langle a, x^*\rangle - \langle a, a^*\rangle\big].$$

Nice property of φ_S

Let $S\colon\ E \rightrightarrows E^*$ be maximally monotone. Then

$$\varphi_S \in \mathcal{PCLSC}_{q_L}(E \times E^*) \quad \text{and} \quad \big\{E \times E^* | \varphi_S = q_L\big\} = G(S).$$

Criterion for $\big\{E \times E^* | f = q_L\big\}$ to be quasidense (restated)

Let $f \in \mathcal{PCLSC}_{q_L}(E \times E^*)$. Then:

$$\big\{E \times E^* | f = q_L\big\} \text{ is quasidense} \iff f^* \geq q_{\widetilde{L}} \text{ on } (E \times E^*)^* = E^* \times E^{**}.$$

Theorem on the quasidensity of a monotone multifunction

Let $S\colon\ E \rightrightarrows E^*$ be closed and monotone. Then S is quasidense if, and only if, S is maximally monotone and $\varphi_S{}^* \geq q_{\widetilde{L}}$ on $E^* \times E^{**}$.

Proof. Immediate from the two results above with $f := \varphi_S$. □

Criterion for $\{E \times E^* | f = q_L\}$ to be quasidense (restated)

Let $f \in \mathcal{PCLSC}_{q_L}(E \times E^*)$. Then:

$$\{E \times E^* | f = q_L\} \text{ is quasidense } \iff f^* \geq q_{\widetilde{L}} \text{ on } (E \times E^*)^* = E^* \times E^{**}.$$

Theorem on subdifferentials

Let $k \in \mathcal{PCLSC}(E)$. Then $G(\partial k)$ is quasidense.

Proof. Define $f \in \mathcal{PCLSC}(E \times E^*)$ by $f(x, x^*) := k(x) + k^*(x^*)$. From the Fenchel–Young inequality,

$$f(x, x^*) \geq \langle x, x^* \rangle = q_L(x, x^*),$$

so $f \in \mathcal{PCLSC}_{q_L}(E \times E^*)$. By direct computation, $\forall\ (x^*, x^{**}) \in E^* \times E^{**}$,

$$f^*(x^*, x^{**}) := k^*(x^*) + k^{**}(x^{**}).$$

From the Fenchel–Young inequality again,

$$f^*(x^*, x^{**}) \geq \langle x^*, x^{**} \rangle = q_{\widetilde{L}}(x^*, x^{**}).$$

From the criterion above, $\{E \times E^* | f = q_L\}$ is quasidense in $E \times E^*$.
But this set is exactly $G(\partial k)$. **Note that we do not use $\varphi_{\partial k}$.** □

Comment. Since $G(\partial k)$ is closed, this result is a strict generalization of Rockafellar's theorem on the maximal monotonicity of subdifferentials. There is a direct proof of this result (using the Brøndsted–Rockafellar theorem and Rockafellar's formula for the subdifferential of a sum), which is as short as the shortest explicit proof of Rockafellar's original result. See 2.9 – 2.14 of the first reference at the end of the talk.

A brief digression to non convex subdifferentials and non monotone sets (joint work with Xianfu Wang)

Ubiquitous subdifferentials

An ubiquitous subdifferential, ∂_u, is a rule that associates with each proper lower semicontinuous function $f\colon E \to]-\infty, \infty]$ a multifunction $\partial_u f : E \rightrightarrows E^*$ such that,

- $0 \in \partial_u f(x)$ if f attains a strict global minimum at x.
- $\partial_u(f + h)(x) \subseteq \partial_u f(x) + \partial h(x)$ whenever h is a continuous convex real function on E (here ∂h is the subdifferential of h of convex analysis).

Comment. The abstract subdifferential introduced by Thibault and Zagrodny gives an ubiquitous subdifferential. This implies that a number of other subdifferentials that have been introduced over the years also give ubiquitous subdifferentials. In particular, the Clarke-Rockafellar subdifferential is an ubiquitous subdifferential. The second condition above is motivated by the "separation principle" of Jules and Lassonde.

The quasidensity of ubiquitous subdifferentials

Let ∂_u be an ubiquitous subdifferential and $k\colon E \to \mathbb{R}$ be proper, lower semicontinuous and bounded below by a continuous affine functional. Then

$$G(\partial_u k) \text{ is quasidense.}$$

Comment. Of course, $G(\partial_u k)$ is not necessarily monotone if k is not convex. See the third reference at the end of the talk.

For the rest of this talk, we return to the monotone case.

Sufficient conditions for quasidensity

Let $S\colon E \rightrightarrows E^*$ be maximally monotone.

- If $R(S) = E^*$ then S is quasidense. This result is due to Fitzpatrick and Phelps.
- If E is reflexive then S is quasidense.

- If X and Y are nonempty sets, define $\pi_1\colon X \times Y \mapsto X$ and $\pi_2\colon X \times Y \mapsto Y$ by

$$\pi_1(x, y) := x \quad \text{and} \quad \pi_2(x, y) := y.$$

Theorem on domain and range

Let $S\colon E \rightrightarrows E^*$ be closed, monotone and quasidense. Then

$$\overline{D(S)} = \overline{\pi_1(\operatorname{dom}\varphi_S)} \quad \textit{and} \quad \overline{R(S)} = \overline{\pi_2(\operatorname{dom}\varphi_S)}.$$

Consequently,

$$\overline{D(S)} \quad \textit{and} \quad \overline{R(S)} \quad \textit{are convex.}$$

Comments. Gossez gave an example of a maximally monotone multifunction for which $\overline{R(S)}$ is not convex.

An example of a maximally monotone multifunction for which $\overline{D(S)}$ is not convex would lead to a counterexample for the *sum problem*! We will explain the sum problem later on.

A negative alignment criterion for quasidensity

Let $S\colon E \rightrightarrows E^*$ be closed and monotone. Then

$$S \text{ is quasidense}$$

$$\Updownarrow$$

$$\forall\ (w,w^*) \in E \times E^*,\ \exists\ \tau \ge 0 \text{ and a sequence } \{(s_n,s_n^*)\}_{n\ge 1} \text{ in } G(S) \text{ such that}$$

$$\lim_{n\to\infty} \|s_n - w\| = \tau, \quad \lim_{n\to\infty} \|s_n^* - w^*\| = \tau \quad \text{and} \quad \lim_{n\to\infty} \langle s_n - w, s_n^* - w^*\rangle = -\tau^2.$$

Comments. ($\Uparrow$) is obvious. ($\Downarrow$) is more delicate because the boundedness of the sequence is not obvious. For more details, see the material on the web.

Definition of type (ANA)

Let $S\colon\ E \rightrightarrows E^*$ be maximally monotone. Then S is of type (ANA) if, whenever $(w,w^*) \in E \times E^* \setminus G(S)$, there exists $(s,s^*) \in G(S)$ such that $s \neq w$, $s^* \neq w^*$, and

$$\frac{\langle s-w, s^*-w^*\rangle}{\|s-w\|\|s^*-w^*\|} \text{ is as near as we please to } -1.$$

Theorem on type (ANA)

Let $S\colon\ E \rightrightarrows E^*$ be closed, monotone and quasidense. Then S is maximally monotone of type (ANA).

Proof. Immediate from the properties of τ on the previous slide. □

- "(ANA)" stands for "Almost negative alignment".
- "(ANA)" was a property originally proved for subifferentials.
- We do not have an example of a maximally monotone multifunction that is not of type (ANA).
- It was proved by Bauschke–S that if $T\colon E \to E^*$ is linear and monotone then T is maximally monotone of type (ANA). In particular, the tail operator (though not quasidense) is of type (ANA).

The next result, due to S–Zălinescu, follows from the Attouch–Brezis theorem.

A bivariate version of the Fenchel duality theorem

Let X and Y be nonzero Banach spaces and $f, g \in \mathcal{PCLSC}(X \times Y)$. Let

$$\bigcup_{\lambda>0} \lambda\big[\pi_1 \operatorname{dom} f - \pi_1 \operatorname{dom} g\big] \text{ be a closed subspace of } X.$$

$\forall\ (x, y) \in X \times Y$, let

$$h(x, y) := \inf\big\{f(x, y - \eta) + g(x, \eta)\colon\ \eta \in Y\big\} > -\infty.$$

Then $h \in \mathcal{PC}(X \times Y)$ and, $\forall\ (x^*, y^*) \in X^* \times Y^*$,

$$h^*(x^*, y^*) = \min\big\{f^*(x^* - \xi^*, y^*) + g(\xi^*, y^*)\colon\ \xi^* \in X^*\big\}.$$

- If $S, T\colon E \rightrightarrows E^*$ then, $\forall x \in E$, $(S + T)x := \big\{x^* + y^*\colon\ x^* \in Sx,\ y^* \in Tx\big\}$.
- $S + T$ is known as the "Minkowski sum" of S and T.
- If $S, T\colon E \rightrightarrows E^*$ then $S \parallel T := (S^{-1} + T^{-1})^{-1}$.
- $S \parallel T$ is known as the "parallel sum" of S and T.

Sum theorem for closed, monotone, quasidense multifunctions

Let $S, T\colon\ E \rightrightarrows E^*$ be closed, monotone and quasidense and

$$D(S) \cap \operatorname{int} D(T) \neq \emptyset.$$

Then

$$S + T \text{ is closed, monotone and quasidense.}$$

Comments. This result follows from the bivariate version of Fenchel duality, and is in stark contrast to the situation for maximally monotone multifunctions. This can also be deduced from a result of Voisei and Zălinescu. Is is apparently still unknown whether $S + T$ is maximally monotone when S and T are maximally monotone and $D(S) \cap \operatorname{int} D(T) \neq \emptyset$. This is what is commonly called the *sum problem*.

Theorem on the quasidensity of a monotone multifunction (restated)

Let $S\colon\ E \rightrightarrows E^*$ be closed and monotone. Then S is quasidense if, and only if, S is maximally monotone and $\varphi_S{}^* \ge q_{\widetilde{L}}$ *on* $E^* \times E^{**}$.

A result of Burachik, Svaiter and Penot (restated)

If $f \in \mathcal{PC}_{q_L}(E \times E^*)$ *and* $\big\{E \times E^* | f = q_L\big\} \ne \emptyset$ *then* $\big\{E \times E^* | f = q_L\big\}$ *is monotone.*

The Fitzpatrick extension

Let $S\colon\ E \rightrightarrows E^*$ be closed, monotone and quasidense. The Fitzpatrick extension of S is the multifunction $S^{\mathbb{F}}\colon\ E^* \rightrightarrows E^{**}$ defined by

$$x^{**} \in S^{\mathbb{F}}(x^*) \iff \varphi_S{}^*(x^*, x^{**}) = q_{\widetilde{L}}(x^*, x^{**}).$$

• If we apply the Burachik–Svaiter–Penot result above to $\big(E^* \times E^{**}, \widetilde{L}\big)$ with $f := \varphi_S{}^*$, we see that if $S\colon\ E \rightrightarrows E^*$ is closed, monotone and quasidense then the multifunction $S^{\mathbb{F}}\colon\ E^* \rightrightarrows E^{**}$ is monotone.

• The reason that we use the word "extension" is that

$$\widehat{x} \in S^{\mathbb{F}}(x^*) \iff x^* \in S(x).$$

• In fact, $S^{\mathbb{F}}$ is maximally monotone, but this seems quite hard to prove. We do not know if $S^{\mathbb{F}}$ is necessarily quasidense. The following special case is true: if $k \in \mathcal{PCLSC}(E)$ and $S := \partial k$ then $S^{\mathbb{F}} = \partial(k^*)$, and so $S^{\mathbb{F}}$ is also quasidense. The proof of the statement that $(\partial k)^{\mathbb{F}} = \partial(k^*)$ is nontrivial.

Parallel sum theorem for closed, monotone, quasidense multifunctions

Let $S, T\colon\ E \rightrightarrows E^*$ *be closed, monotone and quasidense and*

$$R(S) \cap \operatorname{int} R(T) \neq \emptyset.$$

Then

the multifunction $y \mapsto (S^{\mathbb{F}} + T^{\mathbb{F}})^{-1}(\widehat{y})$ *is closed, monotone and quasidense.*

Comment. Again, this result follows from the bivariate version of Fenchel duality. Under certain additional conditions, one can also prove that

$$S \parallel T \text{ is closed, monotone and quasidense.}$$

Definition of quasidensity (reminder)

Let $A \subset E \times E^*$. Then A is quasidense if $\forall\ (w, w^*) \in E \times E^*$ and $\eta > 0$, $\exists\ (s, s^*) \in A$ such that

$$\tfrac{1}{2}\|s - w\|^2 + \tfrac{1}{2}\|s^* - w^*\|^2 + \langle s - w, s^* - w^* \rangle < \eta.$$

First fuzzy equivalence

Let $A \subset E \times E^*$ be closed and monotone. Then A is quasidense $\iff \forall\ w \in E$, nonempty $w(E^*, E)$–compact convex subsets $\widetilde{W}$ of E^* and $\eta > 0$, $\exists\ (s, s^*) \in A$ such that

$$\tfrac{1}{2}\|s - w\|^2 + \tfrac{1}{2}\mathrm{dist}(s^*, \widetilde{W})^2 + \max\big\langle s - w, s^* - \widetilde{W}\big\rangle < \eta.$$

Second fuzzy equivalence

Let $A \subset E \times E^*$ be closed and monotone. Then A is quasidense $\iff \forall$ nonempty $w(E, E^*)$–compact convex subsets W of E, $w^* \in E^*$ and $\eta > 0$, $\exists\ (s, s^*) \in A$ such that

$$\tfrac{1}{2}\mathrm{dist}(s, W)^2 + \tfrac{1}{2}\|s^* - w^*\|^2 + \max\langle s - W, s^* - w^* \rangle < \eta.$$

Proof. The "$\Longrightarrow$" implications follow from the sum and parallel sum theorems. The "$\Longleftarrow$" implications follow by taking $\widetilde{W}$ and W to be singletons. $\square$

Strong maximality

Let $S\colon E \rightrightarrows E^*$ be monotone. We say that S is *strongly maximal* if:

(a) Whenever $\widetilde{K}$ is a nonempty $w(E^*,E)$–compact convex subset of E^*, $w \in E$ and,

$$\forall\ (s,s^*) \in G(S),\ \exists\ w^* \in \widetilde{K} \text{ such that } \langle s-w, s^*-w^*\rangle \geq 0$$

then $S(w) \cap \widetilde{K} \neq \emptyset$.

(b) Whenever K is a nonempty $w(E,E^*)$–compact convex subset of E, $w^* \in E^*$ and,

$$\forall\ (s,s^*) \in G(S),\ \exists\ w \in K \text{ such that } \langle s-w, s^*-w^*\rangle \geq 0$$

then $w^* \in S(K)$.

Strong maximality theorem

Let $S\colon E \rightrightarrows E^*$ be closed, monotone and quasidense. Then S is strongly maximal.

Comments. Of course, if either K or $\widetilde{K}$ is a singleton, then both (a) and (b) become exactly the statement of maximal monotonicity. The strong maximality theorem follows from the two fuzzy equivalences on the previous slide.

- Strong maximality was another property originally proved for subifferentials.
- We do not have an example of a maximally monotone multifunction that is not strongly maximal.
- It was proved by Bauschke–S that if $T\colon E \to E^*$ is linear and monotone then T is strongly maximal. In particular, the tail operator (though not quasidense) is strongly maximal.

Type (FPV)

Let $S\colon\ E \rightrightarrows E^*$ be monotone. We say that S is *of type (FPV)* provided that the following holds: if U is an open convex subset of E, $U \cap D(S) \neq \emptyset$, $(w, w^*) \in U \times E^*$ and

$$\langle s - w, s^* - w^* \rangle \geq 0 \text{ whenever } (s, s^*) \in G(S) \text{ and } s \in U$$

then $(w, w^*) \in G(S)$.

• If we take $U = E$, we can see that every monotone multifunction of type (FPV) is maximally monotone.

• This concept was originally introduced by Fitzpatrick, Phelps and the Veronas. Their term for it was "maximal monotone locally".

Type (FPV) theorem

Let $S\colon\ E \rightrightarrows E^*$ be closed, monotone and quasidense. Then S is of type (FPV).

Comments. This follows easily from the sum theorem.

• It was proved by Fitzpatrick and Phelps that if $S\colon\ E \rightrightarrows E^*$ is maximally monotone and $D(S) = E$ then S is of type (FPV). In particular, the tail operator (though not quasidense) is of type (FPV).

• An example of a maximally monotone multifunction that is not of type (FPV) would lead to a counterexample for the sum problem!

Type (FP)

Let $S\colon E \rightrightarrows E^*$ be monotone. We say that S is *of type (FP)* provided that the following holds: if U is an open convex subset of E^*, $U \cap R(S) \neq \emptyset$, $(w, w^*) \in E \times U$ and

$$\langle s - w, s^* - w^* \rangle \geq 0 \text{ whenever } (s, s^*) \in A \text{ and } s^* \in U$$

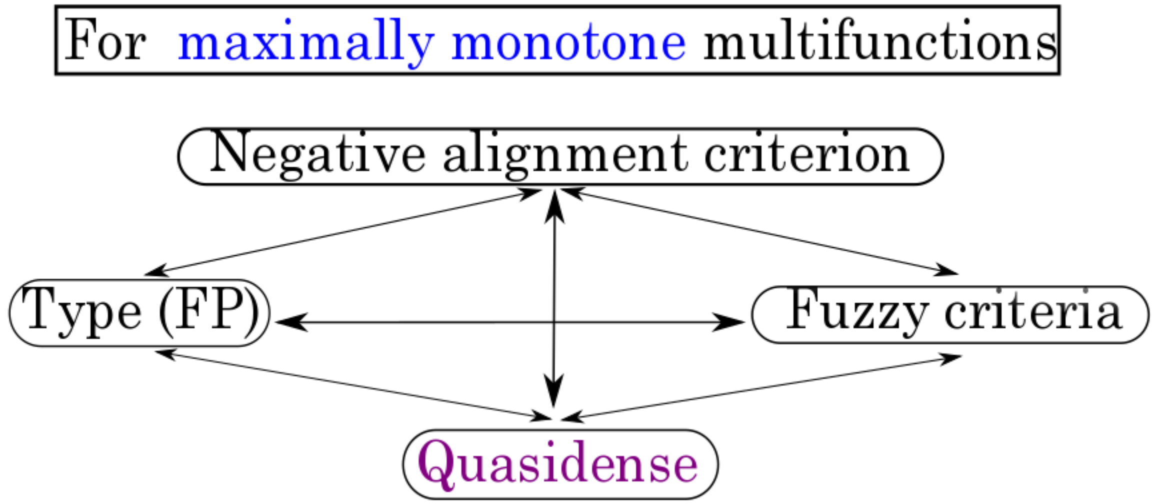
then $(w, w^*) \in G(S)$.

- If we take $U = E^*$, we can see that every monotone multifunction of type (FP) is maximally monotone.
- This concept was originally introduced by Fitzpatrick and Phelps. Their term for it was "locally maximal monotone".

Type (FP) criterion for quasidensity

Let $S\colon E \rightrightarrows E^*$ be closed and monotone. Then S is of type (FP) $\Longleftrightarrow$ S is quasidense.

Comments. "$\Longrightarrow$" follows from an adaptation of a proof of Bauschke, Borwein, Wang and Yao.
"$\Longleftarrow$" follows from the parallel sum theorem, but the proof that we have seems to be unnecessarily hard.



- The proofs of the Type (FP) and the Fuzzy results use:

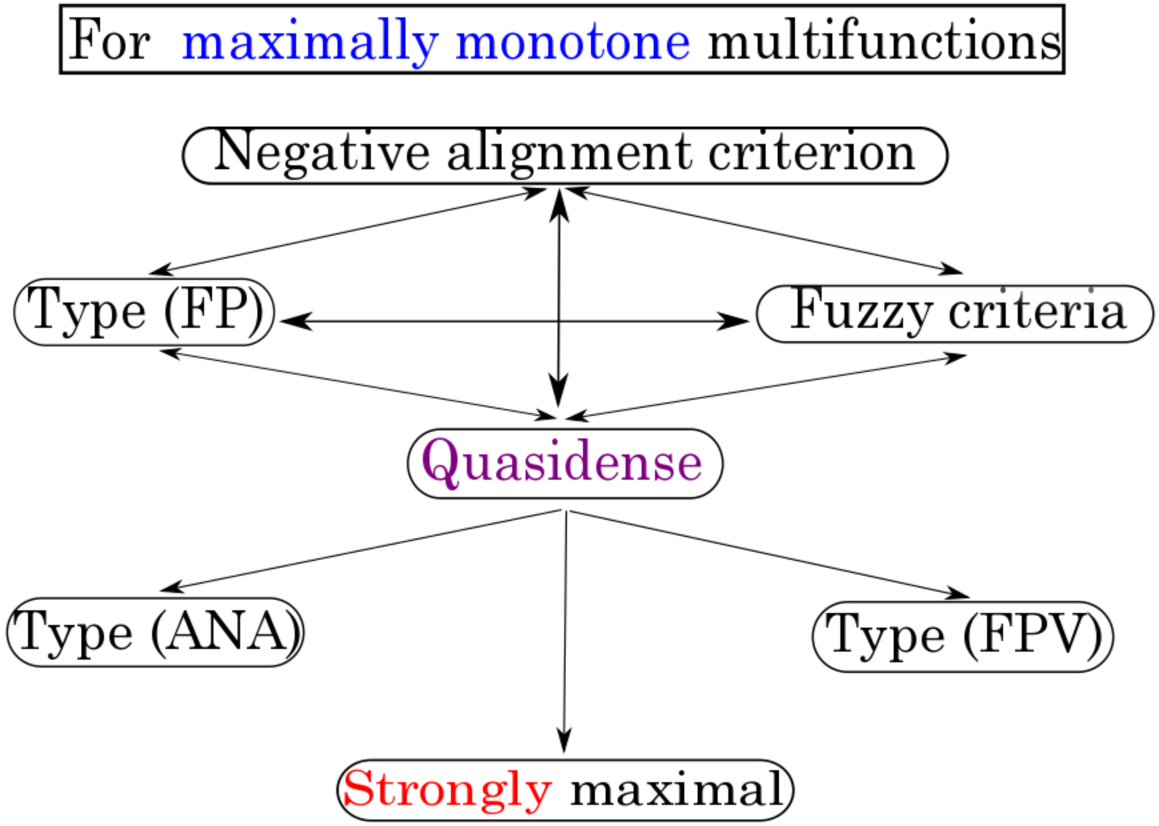
Theorem on the quasidensity of a monotone multifunction

Let $S\colon E \rightrightarrows E^*$ be closed and monotone. Then S is quasidense if, and only if, S is maximally monotone and $\varphi_S{}^* \geq q_{\widetilde{L}}$ on $E^* \times E^{**}$.

- The definitions of Type (FP) and the Fuzzy criteria do not use E^{**} in any way.

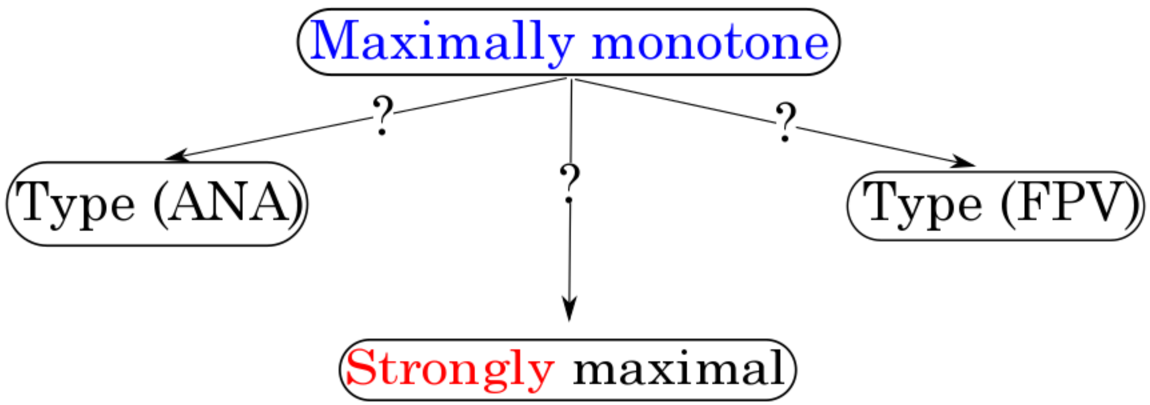
- It should be possible to find direct proofs of these results that do not use E^{**}, just like the result I referred to earlier on for subdifferentials.

- Good luck!



- The tail operator shows that the bottom three arrows cannot be reversed.

Three questions



A fourth question. Is the Fitzpatrick extension of a closed, monotone and quasidense multifunction quasidense? (It is maximally monotone.)
A problem. Find a simple proof that if $k \in \mathcal{PCLSC}(E)$ then $(\partial k)^{\mathbb{F}} = \partial(k^*)$.

Other subclasses of the maximally monotone multifunctions

- Type (D) (1971).
- Dense type (1971).
- Type (NI) (1996).
- Type (WD) (1996).
- Type (ED) (1998).
- The above subclasses share the feature that they require E^{**} for their definition.
- The above subclasses also share the feature that they all coincide with the closed, monotone quasidense multifunctions. This class does not require E^{**} for its definition.

Note that all the results in this talk follow ultimately from the Fenchel duality theorem, and do not depend on any fixed–point theorems.

Downloads

You can download files containing related materials and complete references from

<www.math.ucsb.edu/~simons/QD.html>.

Note that you must type the whole address. See the next slide.

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