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The Inverse Function Theorems of Lawrence M. Graves

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The last message from Jon

Jon Borwein jon.borwein@gmail.com via umich.edu
6/24/16
to Asen,

Hi, here is a question I need your help with.

Let T be tangent to ellipse E at f, show that for p in as neighbourhood of f,

$$||P_E(p) - P_T(p)|| = o(||p - f||).$$

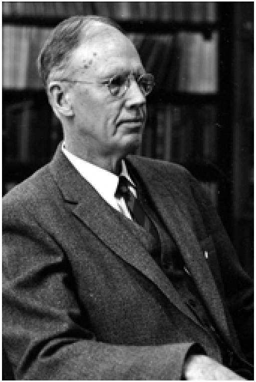
That is; P_T is the linearisation of P_E at f and

$$||P_E(p) - P_T(p)||/||p - f|| \to 0, \quad \text{as } p \to f.$$

Since projection onto a line L is linear this will let us show that the D-R operator .....

The theorems

- The Hildebrand-Graves theorem (1927)
- The (Lyusternik-) Graves theorem (1950)
- The Bartle-Graves theorem (1952)



Lawrence Murry Graves (1896–1973)

Hildebrand–Graves inverse function theorem (1927)

Lipschitz modulus

$$\mathrm{lip}(f;\bar{x}) := \limsup_{\substack{x',x\to\bar{x},\\ x\neq x'}} \frac{\|f(x')-f(x)\|}{\|x'-x\|}.$$

Theorem (Hildebrand–Graves, TAMS 29: 127–153).

Let X be a Banach space and consider a function $f : X \to X$ and a linear bounded mapping $A : X \to X$ which is invertible. Suppose that

$$\mathrm{lip}(f-A;\bar{x}) \cdot \|A^{-1}\| < 1.$$

Then f is strongly regular at $\bar{x}$ for $f(\bar{x})$.

Strong regularity: A mapping $F : X \rightrightarrows X$ is said to be strongly regular at $\bar{x}$ for $\bar{y}$ when $(\bar{x},\bar{y}) \in \mathrm{gph}\, F$ and F^{-1} has a single-valued localization around $\bar{y}$ for $\bar{x}$ which is Lipschitz continuous.

The H-G IFT implies the classical (Dini) IFT

$$f \text{ is strictly differentiable at } \bar{x} \quad \Longleftrightarrow \quad \operatorname{lip}(f - Df(\bar{x}); \bar{x}) = 0.$$

The classical (Dini) IFT

Let $f : \boldsymbol{R}^n \to \boldsymbol{R}^n$ be strictly differentiable at $\bar{x}$. Then f is strongly regular at $\bar{x}$ if and only if the derivative $Df(\bar{x})$ is nonsingular.

Clarke's IFT (1976)

Clarke's generalized Jacobian $\partial f(x)$

Theorem (F. Clarke, Pac. J. Math. 64:97–102).

Consider a function $f : \boldsymbol{R}^n \to \boldsymbol{R}^n$ which is Lipschitz continuous around $\bar{x}$ and suppose that all matrices in $\partial f(\bar{x})$ are nonsingular. Then f is strongly regular at $\bar{x}$.

Robinson's inverse function theorem (1980)

Theorem (S. M. Robinson, MOR 5:43–62).

Let X be a Banach spaces and consider a function $f : X \to X$ which is strictly differentiable at $\bar{x}$ and any set-valued mapping $F : X \rightrightarrows X$. Let $\bar{y} \in f(\bar{x}) + F(\bar{x})$. Then $f + F$ is strongly regular at $\bar{x}$ for $\bar{y}$ if and only if the mapping

$$y \mapsto (f(\bar{x}) + Df(\bar{x})(\cdot - \bar{x}) + F(\cdot))^{-1}(y)$$

has the same property.

Izmailov IFT (2014) = Clarke + Robinson

Theorem (A. Izmailov, MP (A) 147:581–590).

Let $f : \boldsymbol{R}^n \to \boldsymbol{R}^n$ be Lipschitz continuous around $\bar{x}$, let $F : \boldsymbol{R}^n \rightrightarrows \boldsymbol{R}^n$, and let $\bar{y} \in f(\bar{x}) + F(\bar{x})$. Suppose that for every $A \in \partial f(\bar{x})$ the mapping $f(\bar{x}) + A(\cdot - \bar{x}) + F(\cdot))$ is strongly regular at $\bar{x}$ for $\bar{y}$. Then $(f + F)$ has the same property.

Proof and extension to Banach spaces: AD and R. Cibulka, MP (A) 156: 257–270, 2016.

Lyusternik-Graves theorem (1934-1950)

Theorem.

Let X, Y be Banach spaces and onsider a function $f : X \to Y$ and a point $\bar{x} \in \operatorname{int}\operatorname{dom} f$ along with a bounded linear mapping $A : X \to Y$ which is surjective, such that

$$\operatorname{lip}(f - A; \bar{x}) \cdot \|A^{-1}\|^{-} < 1,$$

where the inner "norm" of A is defined as

$$\|A^{-1}\|^{-} := \sup_{\|y\| \le 1} \inf_{x \in A^{-1}(y)} \|x\|.$$

Then f is metrically regular at $\bar{x}$ for $f(\bar{x})$.

Metric Regularity

A mapping $F : X \rightrightarrows Y$ is said to be metrically regular at $\bar{x}$ for $\bar{y}$ when $\bar{y} \in F(\bar{x})$, $\operatorname{gph} F$ is locally closed at $(\bar{x}, \bar{y})$ and there is a constant $\tau \geq 0$ together with neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$d\big(x, F^{-1}(y)\big) \leq \tau d\big(y, F(x)\big) \quad \text{for every} \quad (x, y) \in U \times V.$$

The infimum of all constants $\tau \geq 0$ for which this inequality holds is the regularity modulus of F at $\bar{x}$ for $\bar{y}$ denoted by $\operatorname{reg}(F; \bar{x} \,|\, \bar{y})$.

Euivalent to the Aubin property of the inverse:

$$F^{-1}(x) \cap V \subset F^{-1}(x') + \tau\rho(x, x')\boldsymbol{B}$$

Extended [Robinson] (Lyusternik)-Graves theorem

Theorem.

Let X be a complete metric space, Y be a linear metric space with shift-invariant metric. Consider a mapping $F : X \rightrightarrows Y$ and a function $f : X \to Y$ such that there exist nonnegative scalars κ and μ with

$$\kappa\mu < 1, \quad \mathrm{reg}(F; \bar{x} \,|\, \bar{y}) \le \kappa \quad \text{and} \quad \mathrm{lip}(f; \bar{x}) \le \mu.$$

Then $f + F$ is [strongly] metrically regular at $\bar{x}$ for $\bar{y} + g(\bar{x})$ with

$$\mathrm{reg}(g + F; \bar{x} \,|\, \bar{y}) \le (\kappa^{-1} - \mu)^{-1}.$$

Open problem. Is there a Lyustenik-Graves theorem in nonlinear metric spaces?

Nonsmooth L-G theorems

Theorem (Pourciau, JOTA 22,311–351, 1977).

Let $f : \boldsymbol{R}^n \to \boldsymbol{R}^m$ be Lipschitz continuous around $\bar{x}$, and every $A \in \partial f(\bar{x})$ is surjective. Then f is metrically regular at $\bar{x}$ for $f(\bar{x})$.

Extension to mapping of the form $f + F$ acting in Banach spaces: R. Cibulka, AD and V. Veliov, (SICON 54: 3273–3296, 2016)

Bartle-Graves theorem (1952)

Bartle-Graves theorem (TAMS 72:400–413).

Let X and Y be Banach spaces and let $f : X \to Y$ be a function which is strictly differentiable at $\bar{x}$ and such that the derivative $Df(\bar{x})$ is surjective. Then there is a neighborhood V of $f(\bar{x})$ along with a constant $\gamma > 0$ such that f^{-1} has a continuous selection s on V with the property

$$\|s(y) - \bar{x}\| \leq \gamma \|y - f(\bar{x})\| \quad \text{for every } y \in V.$$

Extended Bartle-Graves theorem

Theorem (AD, JCA 11:81–94, 2004).

Consider a mapping $F : X \rightrightarrows Y$ and any $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ and suppose that for some $c > 0$ the mapping $B_c(\bar{y}) \ni y \mapsto F^{-1}(y) \cap B_c(\bar{x})$ is closed-convex-valued. Consider also a function $f : X \to Y$ with $\bar{x} \in \operatorname{int} \operatorname{dom} f$. Let κ and μ be nonnegative constants such that

$$\kappa\mu < 1, \quad \operatorname{reg}(F; \bar{x} \,|\, \bar{y}) \le \kappa \quad \text{and} \quad \operatorname{lip}(f; \bar{x}) \le \mu.$$

Then for every $\gamma > \kappa/(1 - \kappa\mu)$ the mapping $(f + F)^{-1}$ has a continuous local selection s around $f(\bar{x}) + \bar{y}$ for $\bar{x}$ with the property

$$\|s(y) - \bar{x}\| \le \gamma \|y - \bar{y}\| \quad \text{for every } y \in V.$$

A nonsmooth Bartle-Graves theorem ?

Conjecture.

Consider a function $f : \boldsymbol{R}^n \to \boldsymbol{R}^m$ which is Lipschitz continuous around $\bar{x}$ and a convex and closed set $C \subset \boldsymbol{R}^n$ and suppose that for all matrices A in $\partial f(\bar{x})$ the mapping

$$x \mapsto f(\bar{x}) + A(x - \bar{x}) + C$$

is metrically regular at $\bar{x}$ for $\bar{y}$. Then $(f + C)^{-1}$ has a continuous local selection around $\bar{y}$ for $\bar{x}$ which is calm at $\bar{y}$.

Newton Method for Variational Inequalities

Variational inequality (VI): find $x \in C$ such that

$$f(x) + N_C(x) \ni 0,$$

where $N_C(x)$ the normal cone to C at x:

$$N_C(x) = \{w \mid \langle w, y - x \rangle \leq 0 \quad \text{for all} \quad y \in C\}$$

Newton's method for VI: at each step solve a linear VI:

$$f(x_k) + Df(x_k)(x_{k+1} - x_k) + N_C(x_{k+1}) \ni 0$$

Josephy (1979): If $f + N_C$ is strongly regular at $\bar{x}$ for 0 then Then there exists a neighborhood O of $\bar{x}$ such that for every $x_0 \in O$ the method generates a unique in O sequence and this sequence is superlinearly convergent to $\bar{x}$.

Strong Regularity for Newton's Method

Newton method for a parameterized VI

$$x_0 = a, \quad f(x_k) + Df(x_k)(x_{k+1} - x_k) + N_C(x_{k+1}) \ni p$$

Consider the mapping

$$\boldsymbol{R}^n \times \boldsymbol{R}^n \ni (a, p) \mapsto \Xi(a, p) = \Big\{ \{x_k\} \in l_\infty(\boldsymbol{R}^n) \,\Big|\, x_0 = a,$$

$$f(x_k) + Df(x_k)(x_{k+1} - x_k) + N_C(x_{k+1}) \ni p, \quad k = 1, 2, \ldots \Big\}$$

Theorem (with RTR (2010) and Aragon et al. (2011)).

Let $f(\bar{x}) + N_C(\bar{x}) \ni 0$; then $\{\bar{x}\} \in \Xi(\bar{x}, 0)$. The mapping Ξ has a Lipschitz continuous single-valued localization around $(\bar{x}, 0)$ for $\{\bar{x}\}$ each value of which is a superlinearly convergent sequence to a solution $x(p)$ of $f(x) + N_C(x) \ni p$ if and only if $f + N_C$ is strongly regular at $\bar{x}$ for 0.

Open problem

Conjecture.

Let f be Lipschitz continuous around $\bar{x}$ for 0 and for each $A \in \partial f(\bar{x})$ the mapping

$$x \mapsto f(\bar{x}) + A(x - \bar{x}) + N_C(x)$$

is strongly regular at $\bar{x}$ for 0. Then the mapping $\boldsymbol{R}^n \times \boldsymbol{R}^n \ni (a, p) \mapsto$ the set of all sequence $\{x_k\} \in l_\infty(\boldsymbol{R}^n)$ such that $x_0 = a$, and

$$f(x_k) + A(x_{k+1} - x_k) + N_C(x_{k+1}) \ni p$$

for some $A \in \partial f(x_k) \quad k = 1, 2, \ldots,$ has a Lipschitz continuous single-valued localization around $(\bar{x}, 0)$ for $\{\bar{x}\}$ each value of which is a superlinearly convergent sequence to a solution $x(p)$ of $f(x) + N_C(x) \ni p$.

Muchas Gracias!