Applications of γ-radonifying operators in (stochastic) analysis

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Introduction: Square functions

Burkholder–Davis–Gundy inequalities:

$$\Big\| \sup_{t\geq 0} \Big| \int_0^t \phi(s)dM(s) \Big| \Big\|_{L^p(\Omega)} \eqsim \Big\| \Big(\int_0^\infty |\phi(t)|^2 d[M]_t \Big)^{1/2} \Big\|_{L^p(\Omega)}$$

Littlewood–Paley–Stein inequalities:

$$\|f\|_{L^p(\mathbb{R}^d)} \eqsim \Big\| \Big(\int_0^\infty |t\Delta e^{t\Delta} f|^2 \frac{dt}{t} \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^d)}.$$

- How to define a square function for $\phi : \mathbb{R}^d \to X$, where X is a Banach space ?
- Can it be used to find vector-valued generalizations of results in stochastic and harmonic analysis?

Overview

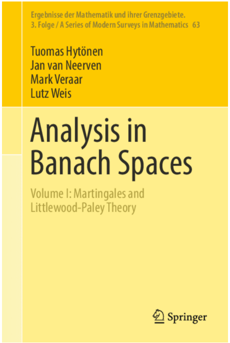
1. Introduction
   - Square functions
   - Overview
   - Preliminaries on UMD spaces
2. γ-radonifying operators
   - Definitions and basic properties
   - Function space properties
   - Embeddings under type and cotype conditions
   - R-boundedness and γ-boundedness
   - The γ-multiplier theorem
3. Applications
   - Stochastic integration in Banach spaces
   - Fourier multipliers
   - Other applications

Preliminaries on UMD spaces

In some parts of the talk we need X to have the UMD property.

- Introduced by Maurey, Pisier, Burkholder 80's
- Connections to harmonic analysis, Bourgain, Burkholder '83
- UMD implies reflexivity $X^{**} = X$
- All "classical" reflexive spaces have the UMD property.

For details on this and many of the results of the talk:



Analysis in Banach spaces Volume I: Martingales and Littlewood-Paley theory
Tuomas Hytönen, Jan van Neerven, Mark Veraar, Lutz Weis, 2016

Volume II: Probabilistic Techniques and Operator Theory
Preprint available on my webpage

γ-radonifying operators

H - separable Hilbert space with ONB (h_n)
(γ_n) - independent standard Gaussian random variables
For $T \in \mathcal{L}(H, X)$ we say $T \in \gamma_\infty(H, X)$ if

$$\|T\|_{\gamma(H,X)} := \sup_{N\geq 1} \Big\| \sum_{n=1}^{N} \gamma_n T h_n \Big\|_{L^2(\Omega;X)} < \infty.$$

We write $T \in \gamma(H, X)$ if additionally $\sum_{n\geq 1} \gamma_n T h_n$ converges in $L^2(\Omega; X)$.
The norm does not depend on the choice of the ONB.

Proposition (Operator ideal)

$\gamma(H, X)$ and $\gamma_\infty(H, X)$ are Banach spaces. Moreover, for $S \in \mathcal{L}(H)$, $R \in \mathcal{L}(X)$ and $T \in \gamma(H, X)$, $\|RTS\|_{\gamma(H,X)} \leq \|R\| \, \|T\|_{\gamma(H,X)} \, \|S\|$.

Gel'fand 1955, Segal 1956, Gross 1962, 1967, Kallianpur 1971, Linde–Pietsch 1974, Figiel–Tomczak-Jaegermann 1979

Standard lemma for the proof of the right-ideal property.

Lemma

For all $m \times n$ matrices $A = (a_{ij})_{i,j=1}^{m,n}$ and all $(x_j)_{j=1}^n$ in X,
$\Big\| \sum_{i=1}^m \gamma_i \sum_{j=1}^n a_{ij} x_j \Big\|_{L^p(\Omega;X)} \leq \|A\| \Big\| \sum_{j=1}^n \gamma_j x_j \Big\|_{L^p(\Omega;X)}$.

Proof.

We may assume $\|A\| = 1$ and $m = n$. Define the $2n \times 2n$-matrix B by

$$B = \begin{pmatrix} A & (I - AA^\star)^{1/2} \\ (I - A^\star A)^{1/2} & -A^\star \end{pmatrix}.$$

Then B is unitary and hence $G_j = \sum_{i=1}^{2n} \gamma_i b_{ij}$, $1 \leq j \leq n$ is a sequence of independent standard Gaussian random variables and thus

$$\mathbb{E}\Big\| \sum_{i=1}^{n} \gamma_i \sum_{j=1}^{n} a_{ij} x_j \Big\|^p \leq \mathbb{E}\Big\| \sum_{i=1}^{2n} \gamma_i \sum_{j=1}^{n} b_{ij} x_j \Big\|^p = \mathbb{E}\Big\| \sum_{j=1}^{n} G_j x_j \Big\|^p = \mathbb{E}\Big\| \sum_{j=1}^{n} \gamma_j x_j \Big\|^p.$$

□

Connection to Gaussian measures

Proposition

$T \in \gamma(H, X)$ if and only if TT^* is a Gaussian covariance operator. In this case TT^* is the covariance operator of $\sum_{n\geq 1} \gamma_n T h_n$.

From Hoffmann-Jørgenson and Kwapień 1974 one obtains:

Proposition (Convergence for free)

Let H be infinite dimensional. Then $\gamma_\infty(H, X) = \gamma(H, X)$ if and only if X does not contain a copy of c_0.

Example

$T : \ell^2 \to c_0$ given by $Th_n = \frac{1}{(\log(n+2))^{1/2}} e_n$ satisfies $T \in \gamma_\infty(H, X)$ but $T \notin \gamma(H, X)$. The reason is that $\sum_{n\geq 1} \gamma_n T h_n$ is not Radon.

Generalized square functions Kalton–Weis 2003-2016

For suitable $\phi : \mathbb{R}^d \to X$ define $T_\phi : L^2(\mathbb{R}^d) \to X$ by

$$T_\phi h = \int_{\mathbb{R}^d} \phi(t)h(t)dt.$$

Write $\phi \in \gamma(\mathbb{R}^d; X)$ whenever $T_\phi \in \gamma(L^2(\mathbb{R}^d), X)$ and set

$$\|\phi\|_{\gamma(\mathbb{R}^d;X)} = \|T_\phi\|_{\gamma(L^2(\mathbb{R}^d),X)}.$$

Example

If $\phi = \sum_{n=1}^N \mathbf{1}_{A_n} x_n$ with $(A_n)_{n=1}^N$ disjoint and $x_1, \ldots, x_N \in X$, then

$$\|\phi\|_{\gamma(\mathbb{R}^d;X)} = \Big\| \sum_{n=1}^N \gamma_n \lambda(A_n)^{1/2} x_n \Big\|_{L^2(\Omega;X)}.$$

$\gamma(\mathbb{R}^d; X) = L^2(\mathbb{R}^d; X)$ if and only if $X \approx$ Hilbert space (Kwapień 1972).

Function space properties

The space $\gamma(\mathbb{R}^d; X)$ behaves as a function space:

- $\|\phi\|_{\gamma(A;X)} = \|\phi \mathbf{1}_A\|_{\gamma(\mathbb{R}^d;X)}$
- Versions of dominated convergence and Fatou's lemma hold
- Versions of Fubini's theorem hold.
- Hölder inequality $\|\langle \phi, \psi\rangle\|_{L^1(\mathbb{R}^d)} \leq \|\phi\|_{\gamma(\mathbb{R}^d;X)}\|\psi\|_{\gamma(\mathbb{R}^d;X^*)}$.
- A converse holds if and only if X is K-convex (Pisier 1982).
- $\|\phi\|_{\gamma(\mathbb{R}^d;L^p)} \eqsim \Big\| \Big(\int_{\mathbb{R}^d} |\phi(t)|^2 dt \Big)^{1/2} \Big\|_{L^p}$

Extension property: Given $T \in \mathcal{L}(L^2(\mathbb{R}^d))$ there is a tensor extension $\widetilde{T} \in \mathcal{L}(\gamma(\mathbb{R}^d; X))$. Examples: Fourier transform, singular integrals, etc.

Extensions on $L^p(\mathbb{R}^d; X)$ are usually more difficult to obtain:
e.g. $\mathcal{F} \in \mathcal{L}(L^2(\mathbb{R}^d; X))$ if and only if $X \eqsim$ Hilbert space.
Hilbert transform $\in \mathcal{L}(L^p(\mathbb{R}^d; X))$ if and only if X is a UMD space.

Embeddings under type and cotype conditions

X has type $p \in [1, 2]$ if

$$\Big\| \sum_{n=1}^{N} r_n x_n \Big\|_{L^p(\Omega;X)} \leq \tau \Big(\sum_{n=1}^{N} \|x_n\|^p \Big)^{1/p}.$$

Cotype $q \in [2, \infty]$: converse estimate with p replaced by q.
Introduced around 1970. See survey paper Maurey 2003.

- $L^s(\Omega)$ with $s \in [1, \infty)$ has type $\min\{s, 2\}$ and cotype $\max\{s, 2\}$.

Theorem (Hoffmann-Jørgenson–Pisier 1976, Rosiński–Suchanecki 1980)

1. $L^2(\mathbb{R}^d; X) \hookrightarrow \gamma(\mathbb{R}^d; X)$ *if and only if X has type* 2.
2. $\gamma(\mathbb{R}^d; X) \hookrightarrow L^2(\mathbb{R}^d; X)$ *if and only if X has cotype* 2

Theorem is false if one replaces the exponent 2 by p or q.

Embeddings under type and cotype conditions

Proof of $L^2(\mathbb{R}^d;X) \hookrightarrow \gamma(\mathbb{R}^d;X)$ if X has type 2:

Proof.

Assume X has type 2. Let $\phi = \sum_{n=1}^N \mathbf{1}_{A_n} x_n$. Then

$$\begin{aligned}\|\phi\|^2_{\gamma(\mathbb{R}^d;X)} &= \mathbb{E}\Big\|\sum_{n=1}^N \gamma_n \lambda(A_n)^{1/2} x_n\Big\|^2 \\ &\leq \tau^2 \sum_{n=1}^N \lambda(A_n)\|x_n\|^2 \\ &= \tau^2 \|\phi\|^2_{L^2(\mathbb{R}^d;X)}.\end{aligned}$$

By density the result follows. □

Embeddings under type and cotype conditions

For $s \in \mathbb{R}$, $p \in [1,\infty]$, $W^{s,p}(\mathbb{R}^d;X)$ - Sobolev space of fractional order or Besov space $B^s_{p,p}(\mathbb{R}^d;X)$.

Theorem (Kalton-Neerven-V.-Weis 2008)

Let X be a Banach space, $p \in [1,2]$ and $q \in [2,\infty]$.

1. $W^{\frac{d}{p}-\frac{d}{2},p}(\mathbb{R}^d;X) \hookrightarrow \gamma(\mathbb{R}^d;X)$ *if and only if X has type p.*
2. $\gamma(\mathbb{R}^d;X) \hookrightarrow W^{\frac{d}{q}-\frac{d}{2},p}(\mathbb{R}^d;X)$ *if and only if X has cotype q.*

- Conditions are dimension dependent except if $p = 2$ or $q = 2$.
- Version for homogenous function spaces holds as well.
- Proof based on Littlewood-Paley description of the Besov norm.
- Improvement to Bessel potential spaces is an open problem. Partial results for p-convex and q-concave spaces in [V. 2013]

R-boundedness and γ-boundedness

A set $\mathcal{T} \subseteq \mathcal{L}(X, Y)$ is called γ-bounded if for all finite sequences $(T_n)_{n=1}^N$ in $\mathcal{T}$ and $(x_n)_{n=1}^N$ in X we have

$$\Big\| \sum_{n=1}^{N} \gamma_n T_n x_n \Big\|_{L^2(\Omega;Y)} \leq C \Big\| \sum_{n=1}^{N} \gamma_n x_n \Big\|_{L^2(\Omega;X)}.$$

The least admissible C is denoted by $\gamma(\mathcal{T})$.
Replacing the $(\gamma_n)_{n\geq 1}$ by Rademachers leads to R-boundedness.

- R-boundedness $\Rightarrow$ γ-boundedness $\Rightarrow$ uniform boundedness.
- Uniform boundedness implies R-boundedness if and only if X has type 2 and Y has cotype 2.

Theorem (Kwapień-V.-Weis 2016)

Let X and Y be Banach spaces. TFAE:

1. *γ-boundedness implies R-boundedness.*
2. *X has finite cotype.*

The γ-multiplier theorem

Theorem (Kalton–Weis 2003-2016)

Let $N : \mathbb{R}^d \to \mathcal{L}(X, Y)$ be strongly measurable and such that

$$\gamma\{N(t) : t \in \mathbb{R}^d\} \leq C.$$

Then for all $\phi \in \gamma(\mathbb{R}^d; X)$, we have $N\phi \in \gamma_{\infty}(\mathbb{R}^d; Y)$ and

$$\|N\phi\|_{\gamma_\infty(\mathbb{R}^d,Y)} \leq C\|\phi\|_{\gamma(\mathbb{R}^d,X)}$$

A converse holds for all (strong) Lebesgue points of N.

Open problem: $N\phi \in \gamma(\mathbb{R}^d; Y)$ in general. True in the following cases:

- $t \mapsto N(t)\mathbf{1}_A(t)x \in \gamma(\mathbb{R}^d; X)$ for all A with $\lambda(A) < \infty$ and $x \in X$.
- Y does not contain a copy of c_0.

Application I: Stochastic integration in Banach spaces

W - Brownian motion.

Example (Yor 1974, Rosiński–Suchanecki 1980)

$\forall p \in [1, 2)$, $\exists \phi \in L^\infty(0, 1; \ell^p)$ such that $\int_0^1 \phi dW$ does not exist.

Example (Neerven-V.-Weis 2007)

$\forall p \in [1, 2)$, $\exists \phi \in C^{\frac{1}{p}-\frac{1}{2}}(0, 1; \ell^p)$ such that $\int_0^1 \phi dW$ does not exist.

Proposition (Rosiński–Suchanecki 1980)

$\int_0^1 \phi dW$ exists if and only if $\phi \in \gamma(0, 1; X)$.

Further: Brzeźniak, Garling, McConnell, Neerven, Ondreját, V. Weis

Neerven–V.–Weis AOP07: Sharp estimates in terms of γ-norms, UMD
Neerven–V.–Weis AOP12: Sharp regularity results for SPDEs

Next we consider more general martingales:

H separable Hilbert space, X Banach space
$M \in \mathcal{M}^c_{\mathrm{loc}}(H)$: continuous local martingale with values in H.
$[M] : \mathbb{R}_+ \times \Omega \to \mathbb{R}$: quadratic variation process
$q_M : \mathbb{R}_+ \times \Omega \to \mathcal{L}(H)$: certain cross variation process

Let X be a Banach space. For $\Phi : \mathbb{R}_+ \times \Omega \to \mathcal{L}(H, X)$ of the form $\Phi = \mathbf{1}_{(a,b]}(t) h \otimes \xi$ where $\xi : \Omega \to X$ is $\mathcal{F}_a$-measurable, let

$$\int_0^t \Phi \, dM = ((Mh)_{b\wedge t} - (Mh)_{a\wedge t})\xi$$

and extend this by linearity and approximation.

Goal: characterize integrable processes and prove L^p-estimates ?

Application I: Stochastic integration in Banach spaces

Theorem (Veraar-Yaroslavtsev EJP16)

Let X be a UMD space and $M \in \mathcal{M}^c_{\rm loc}(H)$. For a progressive measurable $\Phi : \mathbb{R}_+ \times \Omega \to \mathcal{L}(H, X)$ the following are equivalent:

(1) *Φ is stochastically integrable w.r.t. M.*

(2) *$\Phi q_M^{1/2} \in \gamma(\mathbb{R}_+, [M], H; X)$ a.s.*

Moreover, for all $p \in (0, \infty)$ the following two-sided estimate holds:

$$\mathbb{E} \sup_{t\in\mathbb{R}_+} \Big\| \int_0^t \Phi dM \Big\|^p \eqsim \mathbb{E} \|\Phi q_M^{1/2}\|^p_{\gamma(L^2(\mathbb{R}_+,[M],H;X)}$$

- Result holds for cylindrical M which admit a quadratic variation.
- Reduction to Brownian case.
- UMD is necessary Garling '86.
- Results for more general martingales Dirksen–Yaroslavtsev 2017.

Application II: Fourier multipliers $p = q$

$\mathcal{F}f = \widehat{f}$ - Fourier transform of $f : \mathbb{R}^d \to X$.
Under suitable conditions on $f : \mathbb{R}^d \to X$ and $m : \mathbb{R}^d \to \mathcal{L}(X, Y)$ one can define the Fourier multiplier operator: $T_m f = \mathcal{F}^{-1}(m\widehat{f})$.

Theorem (Operator-valued Mihlin (Weis 2001))

Let X be a UMD space If $\{m(t) : t \in \mathbb{R}\}$ and $\{tm'(t) : t \in \mathbb{R}\}$ are R-bounded, then $T_m : L^p(\mathbb{R}; X) \to L^p(\mathbb{R}; X)$ is bounded.

- Widely used for parabolic PDEs.
- m scalar valued is due to Bourgain 1986.

Necessity of R-boundedness:

Theorem (Clément-Prüss 2001)

If $T_m : L^p(\mathbb{R}; X) \to L^p(\mathbb{R}; X)$ is bounded, then $\{m(t) : t \in \mathbb{R}\}$ is R-bounded.

Application II: Fourier multipliers $p < q$

Theorem (Rozendaal-V. JFAA17)

Assume X has type $> p \in [1, 2)$ and Y has cotype $< q \in (2, \infty]$. Then

$$\{|\xi|^{\frac{d}{p}-\frac{d}{q}} m(\xi) : \xi \in \mathbb{R}^d \setminus \{0\}\} \subseteq \mathcal{L}(X, Y)$$

is γ-bounded, then $T_m : L^p(\mathbb{R}^d; X) \to L^q(\mathbb{R}^d; X)$ is bounded.

- Motivation: polynomial stability theory for semigroups (RV 2017)
- Limiting case of Theorem if X is p-convex and Y is q-concave.
- Converse statements available as well.
- Scalar case considered by Hörmander 1960 for $m \in L^{r,\infty}$
- Full version of Hörmander 1960 under Fourier type assumptions.

Sketch of the proof: X type $p_0 > p$ and Y cotype $q_0 < q$

Proof.

Let $s_r = \frac{d}{r} - \frac{d}{2}$ and $K_m = \gamma(\{|\cdot|^{s_p - s_q} m\})$. Then

$$\begin{aligned}
\|T_m f\|_{L^q(\mathbb{R}^d;Y)} &\lesssim \|(-\Delta)^{-s_q} T_m f\|_{\dot{W}^{s_{q_0},q_0}(\mathbb{R}^d;Y)} \\
&\lesssim \|(-\Delta)^{-s_q} T_m f\|_{\gamma(\mathbb{R}^d;Y)} && \text{cotype } q_0 \text{ of } Y \\
&= \||\xi|^{-s_q} m\hat{f}\|_{\gamma(\mathbb{R}^d;Y)} && \gamma\text{-extension of } \mathcal{F} \\
&\leq K_m \||\xi|^{-s_p}\hat{f}\|_{\gamma(\mathbb{R}^d;Y)} && \gamma\text{-multiplier} \\
&= K_m \|(-\Delta)^{-s_p} f\|_{\gamma(\mathbb{R}^d;Y)} && \gamma\text{-extension of } \mathcal{F} \\
&\lesssim K_m \|(-\Delta)^{-s_p} f\|_{\dot{W}^{s_{p_0},p_0}(\mathbb{R}^d;Y)} && \text{type } p_0 \text{ of } X \\
&\lesssim K_m \|f\|_{L^p(\mathbb{R}^d;Y)}.
\end{aligned}$$

□

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$$= \||\xi|^{-s_q} m\hat{f}\|_{\gamma(\mathbb{R}^d;Y)} \quad \gamma\text{-extension of } \mathcal{F}$$

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&= \||\xi|^{-s_q} m \hat{f}\|_{\gamma(\mathbb{R}^d;Y)} && \gamma\text{-extension of } \mathcal{F} \\
&\leq K_m \||\xi|^{-s_p} \hat{f}\|_{\gamma(\mathbb{R}^d;Y)} && \gamma\text{-multiplier} \\
&= K_m \|(-\Delta)^{-s_p} f\|_{\gamma(\mathbb{R}^d;Y)} && \gamma\text{-extension of } \mathcal{F} \\
&\lesssim K_m \|(-\Delta)^{-s_p} f\|_{\dot{W}^{s_{p_0},p_0}(\mathbb{R}^d;Y)} && \text{type } p_0 \text{ of } X \\
&\lesssim K_m \|f\|_{L^p(\mathbb{R}^d;Y)}.
\end{aligned}
$$

□

Other applications

Functional calculus: If A is a sectorial operator with a bounded holomorphic functional calculus on X, then for many $\phi \in H^\infty(\text{Sector})$ one has Littlewood–Paley–Stein inequalities (Kalton–Weis 2016):

$$\|\phi(tA)x\|_{\gamma(\mathbb{R}_+,dt/t;X)} \eqsim \|x\|_X.$$

Useful for PDEs. Originally developed by McIntosh to solve the Kato square root problem.

Malliavin calculus: vector-valued version of the Meyer inequalities for UMD spaces X: (Pisier 1988)

$$\|DF\|_{L^p(\Omega;\gamma(H,X))} \lesssim \|(-L)^{1/2}F\|_{L^p(\Omega;X)} \lesssim \|DF\|_{L^p(\Omega;\gamma(H,X))} + \|F\|_{L^p(\Omega;X)}.$$

$D : L^p(\Omega; X) \to L^p(\Omega; \gamma(H, X))$ is the Malliavin derivative
L is the generator of the Ornstein–Uhlenbeck semigroup on $L^p(\Omega; X)$.
Higher order case: (Maas 2010)
Clark–Ocone formula (Maas–Neerven 2008)

Quote

A deep concept in mathematics is usually not an idea in its pure form, but rather takes various shapes depending on the uses it is put to. The same is true of square functions. These appear in a variety of forms, and while in spirit they are all the same, in actual practice they can be quite different. Thus the metamorphosis of square functions is all important."

—Elias M. Stein 1982

Thank you for your attention!