Numerical Computation of Martingale Optimal Transport on Real Line

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joint work with Jan Obłój

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Outline

Introduction

Numerical Counterpart

- Numerical Computation: Primal Problem
- Numerical Computation: Dual Problem (partial results)

Objective

We aim to solve the martingale optimal transport (MOT) problem:

$$\begin{aligned} \mathsf{P}(\mu,\nu) &:= \sup_{\pi\in\mathcal{M}(\mu,\nu)} \mathbb{E}_\pi\left[c(X,Y)\right] \\ \mathsf{D}(\mu,\nu) &:= \inf_{(\varphi,\psi,h)\in\mathcal{D}} \left\{\int \varphi d\mu + \int \psi d\nu\right\} \end{aligned}$$

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- The first scheme considers the approximation of marginal distributions, *i.e.* $\mathrm{P}(\mu,\nu) \rightsquigarrow \mathrm{P}(\mu',\nu')$;
- The second one consists of solving $\mathrm{D}(\mu,\nu) = \inf_\psi J(\psi)$.

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Primal problem

Let $X(x,y) := x$ and $Y(x,y) := y$ for all $(x,y) \in \mathbb{R}^2$. For (suitable) probability measures μ and ν, define

$$\mathsf{P}(\mu,\nu) \quad := \quad \sup_{\pi \in \mathcal{M}(\mu,\nu)} \mathbb{E}_\pi \left[c(X,Y) \right],$$

where

$$\mathcal{M}(\mu,\nu) \quad := \quad \left\{ \pi : \quad X \overset{\pi}{\sim} \mu,\ Y \overset{\pi}{\sim} \nu \text{ and } (X,Y) \text{ is } \pi - \text{martingale} \right\}.$$

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$\mathcal{M}(\mu,\nu) \neq \emptyset$ iff $\int |x| d\mu$, $\int |y| d\nu < +\infty$ and $\int \psi d\mu \leq \int \psi d\nu$ for all convex ψ.
Such a pair (μ,ν) is called PCOC.

Dual problem

Let Λ be the space of Lipschitz functions on $\mathbb{R}$, and $\mathcal{D} \subset \Lambda \times \Lambda \times \mathbb{L}^0(\mathbb{R})$ be the collection of triplets (φ, ψ, h) s.t.

$$\underbrace{\varphi(x) + \psi(y)}_{\text{static trading}} + \underbrace{h(x)(y-x)}_{\text{dynamic trading}} \geq c(x,y), \quad \text{for all } (x,y) \in \mathbb{R}^2.$$

Define

$$\mathsf{D}(\mu, \nu) := \inf_{(\varphi,\psi,h)\in\mathcal{D}} \left[\int \varphi d\mu + \int \psi d\nu \right].$$

Duality

Theorem (Beiglböeck, Henry-Labordère and Penkner)

Let $c : \mathbb{R}^2 \to \mathbb{R}$ be u.s.c. and dominated from above by some affine function, i.e.

$$\sup_{(x,y)\in\mathbb{R}^2} \frac{c(x,y)}{1+|x|+|y|} < +\infty.$$

Then

(i) *there exists* $\pi^* \in \mathcal{M}(\mu,\nu)$ *s.t.* $\mathsf{P}(\mu,\nu) = \mathbb{E}_{\pi^*}[c]$;

(ii) *the duality* $\mathsf{P}(\mu,\nu) = \mathsf{D}(\mu,\nu)$ *holds.*

Questions

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- Existence and characterization of the dual optimizer (φ^*, ψ^*, h^*). Monge-Ampère equation?
- Numerical computation of $\mathsf{P}(\mu, \nu) = \mathsf{D}(\mu, \nu)$.

A heuristic idea

$\mathsf{P}(\mu, \nu)$ reduces to be a linear optimization problem if $\mathrm{supp}(\mu)$ and $\mathrm{supp}(\nu)$ are finite.

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- Construct π' by means of π, T and S;
- $|\mathbb{E}_\pi[c] - \mathbb{E}_{\pi'}[c]|$ is "small" if (μ, ν) is "close" to (μ', ν');
- In general $\pi' \notin \mathcal{M}(\mu', \nu')$.

A relaxed optimization problem

For $\varepsilon \in \mathbb{R}_+$, define

$$\mathcal{M}_\varepsilon(\mu,\nu) \quad := \quad \left\{ \pi : \quad X \overset{\pi}{\sim} \mu,\ Y \overset{\pi}{\sim} \nu \text{ and } \sup_{\|h\|_\infty \leq 1} \mathbb{E}_\pi\left[h(X)(Y-X)\right] \ \leq \ \varepsilon \right\},$$

and consider the corresponding optimization problem

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$$\mathsf{P}_\varepsilon(\mu,\nu) \quad := \quad \sup_{\pi \in \mathcal{M}_\varepsilon(\mu,\nu)} \mathbb{E}_\pi \left[c(X,Y) \right].$$

Denote $\mathcal{W}_1^{\oplus}\big((\mu,\nu),(\mu',\nu')\big) \quad := \quad \mathcal{W}_1(\mu,\mu') \ + \ \mathcal{W}_1(\nu,\nu')$.

A stability result

Proposition

Let (μ', ν') be another PCOC. If c is Lipschitz, then one has $C > 0$ s.t.

$$\mathsf{P}_\varepsilon(\mu,\nu) \quad \leq \quad \mathsf{P}_{\varepsilon+d}(\mu',\nu') \ + \ Cd, \quad \text{with } d \ := \ \mathcal{W}_1^{\oplus}\big((\mu,\nu),(\mu',\nu')\big).$$

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Corollary

Let $\big((\mu_n,\nu_n)\big)_{n\geq 1}$ be a sequence of PCOCs converging to (μ,ν) under $\mathcal{W}_1^{\oplus}$. Set $d_n := \mathcal{W}_1^{\oplus}\big((\mu_n,\nu_n),(\mu,\nu)\big)$, then one has $C > 0$ s.t.

$$\mathsf{P}(\mu,\nu) \;\leq\; \mathsf{P}_{d_n}(\mu_n,\nu_n) + Cd_n \;\leq\; \mathsf{P}_{2d_n}(\mu,\nu) + 2Cd_n.$$

Stability: continuation...

Proposition

(i) *Let c be u.s.c. Then* $\lim_{\varepsilon\to 0} \mathsf{P}_\varepsilon(\mu,\nu) = \mathsf{P}(\mu,\nu)$.

(ii) *Let* (μ,ν) *be boundedly supported, with* $a = \inf(\operatorname{supp}(\mu))$ *and* $b = \sup(\operatorname{supp}(\mu))$. *Assume further* $c \in \mathcal{C}^2(\mathbb{R}^2)$ *and*

$$\int_{[a,b]} \left(\frac{1}{x-a} + \frac{1}{b-x}\right) d\mu \quad < \quad +\infty.$$

Then one has $C > 0$ *s.t.* $0 \leq \mathsf{P}_\varepsilon(\mu,\nu) - \mathsf{P}(\mu,\nu) \leq C\varepsilon$ *and*

$$\left|\mathsf{P}(\mu,\nu) - \mathsf{P}_{d_n}(\mu_n,\nu_n)\right| \quad \leq \quad C d_n.$$

An explicit construction

Define

$$\mu_n(\{k/n\}) := \int_{[(k-1)/n,k/n)} (nx+1-k)d\mu + \int_{[k/n,(k+1)/n)} (1+k-nx)d\mu,$$
$$\nu_n(\{k/n\}) := \int_{[(k-1)/n,k/n)} (nx+1-k)d\nu + \int_{[k/n,(k+1)/n)} (1+k-nx)d\nu.$$

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Lemma

(i) *μ_n and ν_n are probability measures.*

(ii) *(μ_n, ν_n) are PCOCs and $d_n \leq 2/n$.*

A linear optimization problem

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$$\begin{aligned}
\mathrm{P}_{d_n}(\mu_n, \nu_n) \;=\; & \sup_{p=(p_{i,j})_{i,j\in\mathbb{Z}}} \sum_{i,j\in\mathbb{Z}} p_{i,j} c(i/n, j/n) \\
\text{s.t. } & \sum_{i,j\in\mathbb{Z}} p_{i,j} \;=\; 1 \text{ and } p_{i,j} \;\geq\; 0, \text{ for all } i,j \in \mathbb{Z}, \\
& \sum_{j\in\mathbb{Z}} p_{k,j} \;=\; \alpha_k \text{ and } \sum_{i\in\mathbb{Z}} p_{i,k} \;=\; \beta_k, \text{ for all } k \in \mathbb{Z}, \\
& \sum_{j\in\mathbb{Z}} p_{k,j} j/n \;\leq\; (\geq) \left(\sum_{j\in\mathbb{Z}} p_{k,j}\right)\left(k/n \pm d_n\right), \text{ for all } k \in \mathbb{Z}.
\end{aligned}$$

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$\mu = \mathcal{U}([-1,1])$, $\nu = \mathcal{U}([-2,2])$, $c(x,y) = |x-y|$. Then $\mathsf{P}(\mu,\nu) = 1$.

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- $\alpha_{-n} = \alpha_n = 1/4n$, $\alpha_k = 1/2n$, for $-n < k < n$;
- $\beta_{-2n} = \beta_{2n} = 1/8n$, $\beta_k = 1/4n$, for $-2n < k < 2n$;
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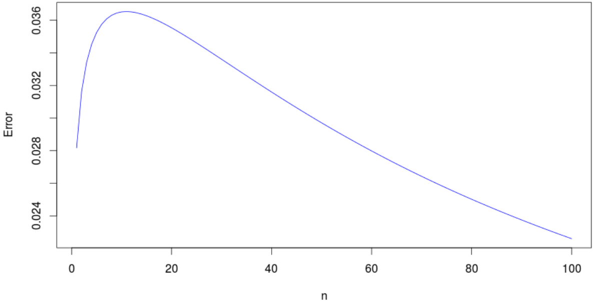

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- Generalize the numerical solver for optimal transport;
- Estimate the convergence rate $\mathsf{P}_\varepsilon(\mu,\nu) - \mathsf{P}(\mu,\nu)$ under more general conditions;
- This approach can apply for the multi-dimensional case.

An alternative formulation

Proposition

One has

$$D(\mu,\nu) = \inf_{\psi\in\Lambda} \left\{ \int v_\psi d\mu + \int \psi d\nu \right\}, \quad \textit{with } v_\psi(x) := \big(c(x,\cdot) - \psi\big)^c(x).$$

Here $\big(c(x,\cdot) - \psi\big)^c$ denotes the concave envelope of $y \mapsto c(x,y) - \psi(y)$. In addition, see Obermann, $\big(c(x,\cdot) - \psi\big)^c$ is the viscosity solution of

$$\max\Big(c(x,y) - \psi(y) - u(y), \quad u''(y) \Big) = 0.$$

First approximation

Define

$$J(\psi) \quad := \quad \int v_\psi d\mu \; + \; \int \psi d\nu,$$

Set $\mathsf{D}_L(\mu,\nu) := \inf_{\psi\in\Lambda_L} J(\psi)$, where $\Lambda_L \subset \Lambda$ consists of $L-$Lipschitz functions ψ with $\psi(0)=0$. Then

Proposition

$J : \Lambda \to \mathbb{R}$ is convex and

$$\mathsf{D}(\mu,\nu) \quad = \quad \lim_{L\to+\infty} \mathsf{D}_L(\mu,\nu).$$

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Further approximation

Let (μ, ν) be supported on a finite interval, *e.g.* $[0, 1]$. Consider the set $\Lambda_L(n) \subset \Lambda_L$ of functions ψ which are affine on $[(k-1)/n, k/n]$ for $k = 1, \cdots n$.

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Remark

Let $U_L(n) \subset \mathbb{R}^n$ be the set of vectors $(u_k)_{1 \leq k \leq n}$ s.t. $|u_k| \leq L$, then there exists a bijection between $U_L(n)$ and $\Lambda_L(n)$. Denote by $\Phi : U_L(n) \to \Lambda_L(n)$ this bijection.

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Define

$$c_n(x, y) := (1 + \lfloor ny \rfloor - ny) c\big(\lfloor nx \rfloor / n, \lfloor ny \rfloor / n\big) + (ny - \lfloor ny \rfloor) c\big(\lfloor nx \rfloor / n, (1 + \lfloor ny \rfloor)/n\big)$$

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Define similarly $J_n(\psi)$ by replacing c by c_n.

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Lemma

If c is Lipschitz, then one has $C > 0$ s.t.

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\begin{aligned}
0 \leq \inf_{\psi \in \Lambda_L(n)} J(\psi) - \mathrm{D}_L(\mu, \nu) &\leq \frac{C}{n}, \\
\left| \inf_{\psi \in \Lambda_L(n)} J_n(\psi) - \inf_{\psi \in \Lambda_L(n)} J(\psi) \right| &\leq \frac{C}{n}.
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$$\left| \inf_{\psi \in \Lambda_L(n)} J_n(\psi) - \inf_{\psi \in \Lambda_L(n)} J(\psi) \right| \leq \frac{C}{n}.$$

We have

$$\inf_{\psi \in \Lambda_L(n)} J_n(\psi) = \inf_{u \in U_L(n)} \mathcal{J}_n(u), \quad \text{with } \mathcal{J}_n := J_n \circ \Phi.$$

Notice $U_L(n) \subset \mathbb{R}^n$ is convex and compact, and the map $\mathcal{J}_n$ is convex.

More...

For any numerical solver for $\mathcal{J}_n(u)$, *e.g.* Boyd & Vandenberghe, we may compute $\inf_{u \in U_L(n)} \mathcal{J}_n(u)$ by the following gradient projection algorithm.

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Letting $(\gamma_i)_{i \geq 0} \subset \mathbb{R}_+$ be a sequence satisfying $\sum_{i \geq 0} \gamma_i = +\infty$:

1. Let $u^0 := 0$.
2. Given u^i, compute the sub-gradient $\nabla \mathcal{J}_n(u^i)$ of $\mathcal{J}_n$ at u^i.
3. Let $u^{i+1} := Proj_{U_L(n)}(u^i + \gamma_i \nabla \mathcal{J}_n(u^i))$.
4. Go back to Step 2.

Remarks

- Estimate the convergence rate $\mathsf{D}_L(\mu,\nu) - \mathsf{D}(\mu,\nu)$;

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- Is there some necessary condition for the optimizer $\operatorname{argmin} J_{\psi\in\Lambda}(\psi)$ (if exists) or $\operatorname{argmin} J_{\psi\in\Lambda_L}(\psi)$?

Thank you very much !

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